STAT 5703 Homework 1

Shijie He(sh3975), Yunjun Xia(yx2569), Shuyu Huang(sh3967)

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Exercise 3

(1)

Use moment generating function to get the $E(R_1^3)$.

Moment generating function for normal distribution is:

$$M_R(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

Hence,

$$\begin{aligned}\gamma &= E[R_1^3] \\ &= \frac{d^3}{dt^3} M_R(t)|_{t=0} \\ &= e^{t\mu + \frac{1}{2}\sigma^2 t^2}((\mu + \sigma^2 t)^3 + 3\sigma^2(\mu + \sigma^2 t))|_{t=0} \\ &= \mu^3 + 3\mu\sigma^2\end{aligned}$$

(2)

(a)

We know $\bar{R} \sim N(\mu, \frac{\sigma^2}{n})$. Thus, similar to the first part of this problem, $E[\hat{\gamma}] = E[\bar{R}^3] = \mu^3 + 3\mu\frac{\sigma^2}{n}$.

Thus,

$$bias(\hat{\gamma}) = E[\hat{\gamma}] - \gamma = -\frac{n-1}{n} 3\mu\sigma^2$$

(b)

We test the limit of the bias:

$$\lim_{n \to \infty} bias(\hat{\gamma}) = -3\mu\sigma^2 \neq 0$$

Thus, $\hat{\gamma}$ is not consistent.

(3)

Since $E[\hat{\gamma}] = \mu^3 + 3\mu\frac{\sigma^2}{n}$, we need an unbiased estimator $\hat{\beta}$ for $\beta = 3\mu\frac{\sigma^2}{n}$ so that $E[\hat{\gamma} - \hat{\beta}] = \mu^3$ is an unbiased estimator for μ^3.

We use unbiased estimators for μ and σ^2: $\hat{\mu} = \bar{R}$, $\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2$.

Since $\hat{\mu}$ and $\hat{\sigma^2}$ are independent, we can define $\hat{\beta} = 3\hat{\mu}\frac{\hat{\sigma^2}}{n}$.

$$
\begin{aligned}
E[\hat{\gamma}-\hat{\beta}] &= E[\hat{\gamma}]-E[\hat{\beta}] \\
&= \mu^3+3\mu\frac{\sigma^2}{n}-3\mu\frac{\sigma^2}{n} \\
&= \mu^3
\end{aligned}
$$

Thus, $\hat{\gamma}-\hat{\beta}=\hat{\gamma}-3\hat{\mu}\frac{\hat{\sigma^2}}{n}$ is an unbiased estimator for μ^3.

(4)

(a)

$$
E[\tilde{\gamma}]=\frac{1}{n}nE[R_1^3]=\mu^3+3\mu\sigma^2=\gamma
$$

$\tilde{\gamma}$ is unbiased, so the biased is 0.

(b)

$var(\tilde{\gamma})=\frac{1}{n}var(R_1^3)$. As $n\rightarrow\infty$, $var(\tilde{\gamma})\rightarrow 0$. Since $\tilde{\gamma}$ is unbiased, we can conclude that $\tilde{\gamma}$ is consistent.

(5)

Similar to part 3, we construct an unbiased estimator for γ as $U=\hat{\gamma}+\frac{n-1}{n}3\hat{\mu}\hat{\sigma}^2$.

$$
\begin{aligned}
E[U] &= E[\hat{\gamma}+\frac{n-1}{n}3\hat{\mu}\hat{\sigma}^2] \\
&= E[\hat{\gamma}]+E[\frac{n-1}{n}3\hat{\mu}\hat{\sigma}^2] \\
&= \mu^3+3\mu\frac{\sigma^2}{n}+\frac{n-1}{n}3\mu\sigma^2 \\
&= \mu^3+3\mu\sigma^2
\end{aligned}
$$

Thus, U is an unbiased estimator of γ.

Sufficient statistics are $T_1=\sum\limits_{i=1}^{n}R_i$ and $T_2=\sum\limits_{i=1}^{n}R_i^2$.

According to Rao-Blackwell Theorm,

$$
E[U|T_1,T_2]=E[\hat{\gamma}+\frac{n-1}{n}3\hat{\mu}\hat{\sigma}^2|T_1,T_2]
$$

Since $\hat{\gamma}=(\frac{1}{n}T_1)^3$, $\hat{\mu}=\frac{1}{n}T_1$ and $\hat{\sigma}^2=\frac{1}{n-1}(\sum\limits_{i=1}^{n}R_i^2-n\bar{R}^2)=\frac{1}{n-1}(T_2-\frac{1}{n}T_1^2)$, we can see that all the estimators are the functions of sufficient statistics.

Thus, we can conclude that $E[U|T_1,T_2]=U$. The estimator U itself is the minimum variance unbiased estimator.