STAT 5703 Homework 1

Shijie He(sh3975), Yunjun Xia(yx2569), Shuyu Huang(sh3967)

2/6/2020

Exercise 1

(1)

We need $P(D < Q_D(p)) = p$:

$$\int_0^{Q_D(p)} f(x;\lambda)dx = \int_0^{Q_D(p)} \lambda e^{-\lambda x} dx = -e^{-\lambda x}|_0^{Q_D(p)} = 1 - e^{-\lambda Q_D(p)} = p$$

We get:

$$Q_D(p) = -\frac{1}{\lambda}\ln(1-p)$$

(2)

Population moment:

$$\mu_1 = E[D] = \frac{1}{\lambda}$$

Empirical moment:

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=0}^{n} D_i = \overline{D_n}$$

Use the empirical moment to estimate the population moment:

$$\hat{\lambda}^{MM} = \frac{1}{\overline{D_n}}$$

Thus,

$$\hat{Q}_D(p)^{MM} = -\frac{1}{\hat{\lambda}^{MM}}\ln(1-p) = -\overline{D_n}\ln(1-p)$$

(3)

According to CLT:

$$\sqrt{n}(\overline{D_n} - \frac{1}{\lambda}) \xrightarrow[n\to\infty]{D} N(0, \frac{1}{\lambda^2})$$

Using delta method:

$$\sqrt{n}(\hat{Q}_D(p)^{MM} + \frac{\ln(1-p)}{\lambda}) \xrightarrow[n\to\infty]{D} N(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

Based on the distribution of $\sqrt{n}(\hat{Q}_D(p)^{MM}$, we get:

$$L(\mathbf{D}) = -\overline{D_n}\ln(1-p) - z_{1-\alpha/2} * \frac{\ln(1-p)\overline{D_n}}{\sqrt{n}}$$

$$U(\mathbf{D}) = -\overline{D_n}\ln(1-p) + z_{1-\alpha/2} * \frac{\ln(1-p)\overline{D_n}}{\sqrt{n}}$$

Thus, the approximate (1-α)-confidence interval is $[L(\mathbf{D}), U(\mathbf{D})]$.

(4)

Since $D_i \sim \exp(\lambda)$ and D_is are iid,

$$\sum_{i=1}^{n} D_i \sim \Gamma(n, \lambda)$$

$$\lambda\overline{D_n} = \frac{\lambda}{n}\sum_{i=1}^{n} D_i \sim \Gamma(n, n)$$

The distribution of $\lambda\overline{D_n}$ does not depend on λ, so it is a pivot.

To find the 95% confidence interval for $Q_D(0.5) = \frac{\ln 2}{\lambda}$, let a and b be the 0.025 and 0.975 quantile of the gamma distribution $\Gamma(n, n)$.

Hence, we have:

$$P(a \leqslant \lambda\overline{D_n} \leqslant b) = 0.95$$

Thus,

$$P(\frac{\overline{D_n}\ln 2}{b} \leqslant \frac{\ln 2}{\lambda} \leqslant \frac{\overline{D_n}\ln 2}{a}) = 0.95$$

.

The 95% confidence interval is $[\frac{\overline{D_n}\ln 2}{b}, \frac{\overline{D_n}\ln 2}{a}]$.