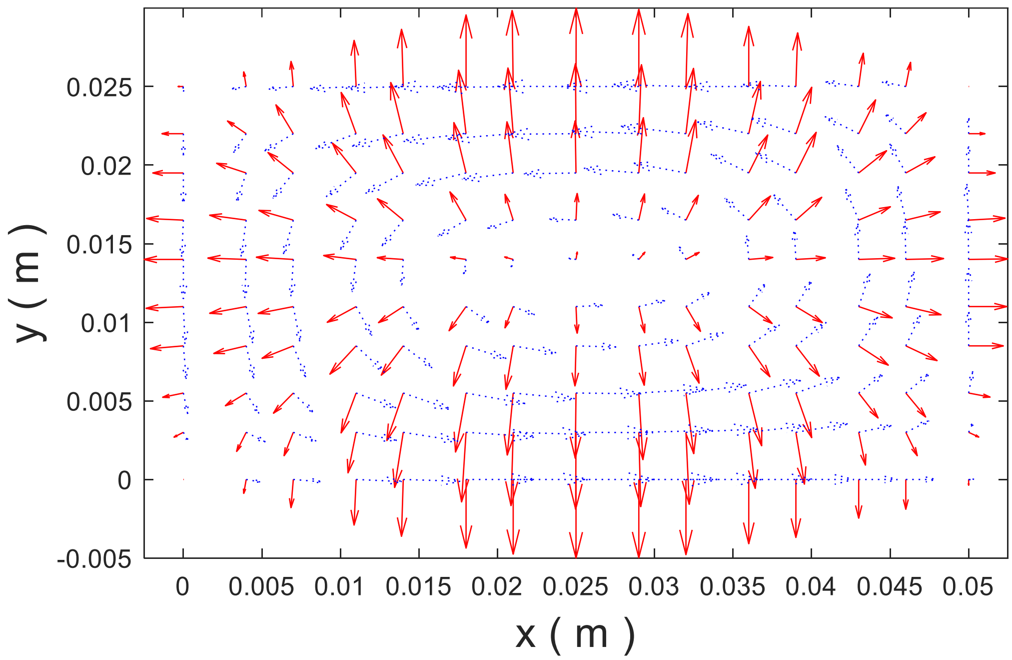

Interesting cover page!

A+
Successful implementation, good validation

Rectangular Waveguide Analysis using Finite Element Method (FEM)

[REPORT TO PROJECT II]

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Introduction

Finite Element Method (FEM) is “a numerical procedure to convert partial differential equations into a set of linear algebraic equations to obtain approximate solutions to boundary-value problems of mathematical physics” [1]. Different from finite difference method which approximates the differential operators, FEM approximates the solution with low-order basis functions in small elements. As a result, FEM gains the advantage of dealing with arbitrary shapes and inhomogeneous media. Besides, the irregular mesh allows FEM to use higher spatial resolution in desired area while use larger elements in unimportant or smooth domain, therefore, the entire computation cost could be reduced by reducing the computation nodes. In this report, the author and his teammates simulated a rectangular waveguide with FEM. Comparing with analytical result validates both the FEM code and the accuracy of FEM.

Analytical Solution to Uniform Rectangular Waveguide

The problem we simulated is a uniform rectangular waveguide as Fig. 1. The waveguide is infinitely long along the z-axis, is uniformly filled with homogeneous material having a permittivity ε and a permeability μ, and has a cross-section of $a \times b$. Such a uniform waveguide can support both Transverse Electric (TE) mode and Transverse Magnetic (TM) mode. Assuming the modes propagate in the z-direction with a propagation constant β, the TE modes and TM modes can be solved by solving either H_z or E_z component only, since all other field components can be derived from them.

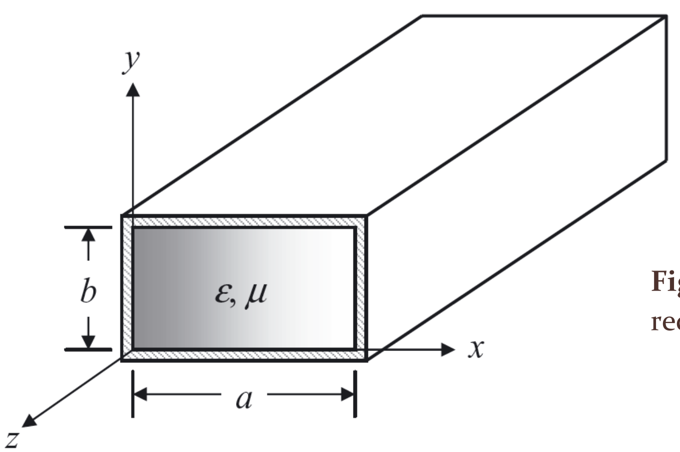


Figure 1 Uniformly filled rectangular waveguide.

TM MODE

E_z satisfies the Helmholtz equation

$$\nabla_t^2 E_z + k_c^2 E_z = 0 \quad \text{on } \Omega, \tag{1}$$

where $\nabla_t^2 = \nabla_t \cdot \nabla_t = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the 2D Laplacian, $k_c^2 = k^2 - \beta^2$ with $k = \omega\sqrt{\mu\varepsilon}$, Ω denotes the cross-section of the waveguide. The boundary condition is

$$E_z = 0 \quad \text{on } \Gamma, \tag{2}$$

where Γ denotes the conducting waveguide wall.

The general solution to above boundary-value partial differential equation is

$$E_z = E_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}, \tag{3}$$

where $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, the propagation constant

$$\beta = k_{zmn} = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{4}$$

The other field components are

$$E_x = -E_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$E_y = -E_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$H_x = E_{mn} \frac{j\omega\varepsilon}{k_{tmn}^2} \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$H_y = -E_{mn} \frac{j\omega\varepsilon}{k_{tmn}^2} \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$H_z = 0$$

TE MODE

The analysis of TE mode is similar. H_z satisfies the Helmholtz equation

$$\nabla_t^2 H_z + k_c^2 H_z = 0 \quad \text{on } \Omega. \tag{5}$$

The boundary condition is

$$\partial H_z / \partial \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{6}$$

The general solution to above boundary-value partial differential equation is

$$H_z = H_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z} \tag{7}$$

where $m = 0,1,2,3,\ldots$, $n = 0,1,2,3,\ldots$ except for $m = n = 0$. The propagation constant

$$\beta = k_{zmn} = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{8}$$

The other field components are

$$E_x = H_{mn} \frac{j\omega\mu}{k_{tmn}^2} \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$E_y = -H_{mn} \frac{j\omega\mu}{k_{tmn}^2} \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

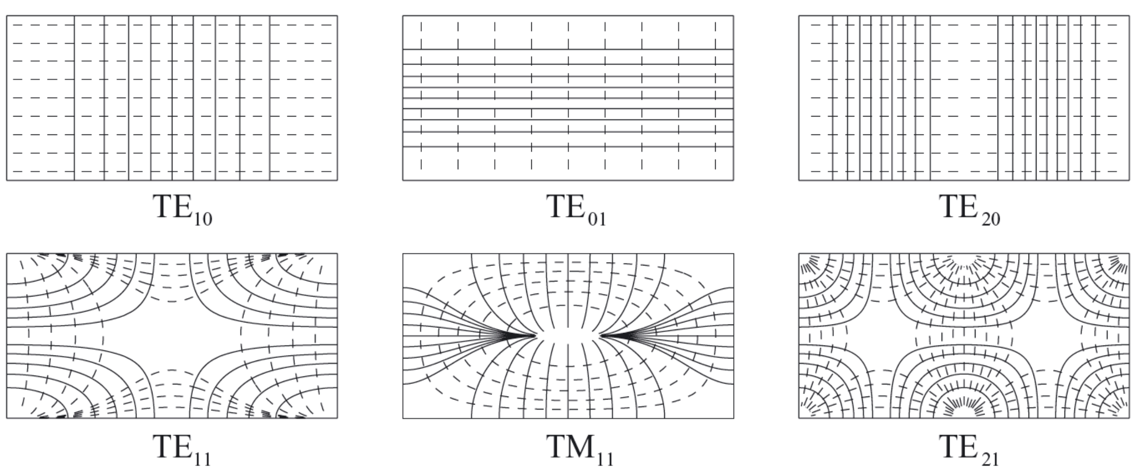
$$E_z = 0$$

$$H_x = H_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$H_y = H_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}.$$

TRANSVERSE FIELD DISTRIBUTION WHEN $a/b = 2$

For such a uniform rectangular waveguide with $a/b = 2$, the transverse field distribution was plotted in Jin's book [1]. The solid lines represent electric field lines, and the dashed lines represent the magnetic field lines.



Next, in our GUI, we will repeat this figure, plot corresponding propagation constants, and compare with analytic results.

FEM Coding Analysis

We directly used the FEM matrix formula derived in the lecture notes for such a general partial differential equation

$$-\nabla\cdot(\alpha\nabla\phi)+\beta\phi=f \quad \text{on } \Omega. \tag{9}$$

Letting $\alpha=-1$, $\beta=k_c^2$, $f=0$, for both TM mode (1) and TE mode (5), we obtained the matrix equation

$$[A]\{\phi\}=k_c^2[B]\{\phi\}. \tag{10}$$

Here, $\{\phi\}$ is $\{E_z\}$ in TM mode and $\{H_z\}$ in TE mode. The matrix elements are

$$A_{ij}=\iint_\Omega \nabla_t N_i\cdot\nabla_t N_j\, d\Omega \qquad i,j=1,2,\ldots,N$$

$$B_{ij}=\iint_\Omega N_i\cdot N_j\, d\Omega \qquad i,j=1,2,\ldots,N.$$

For TE mode, the Neumann boundary condition turns out to contribute nothing to the matrix. After applying the integration formula in the book, we directly obtained the matrix elements in small element e

$$A_{ij}^e=\frac{1}{4\Delta^e}\left(b_i^e b_j^e+c_i^e c_j^e\right) \tag{11}$$

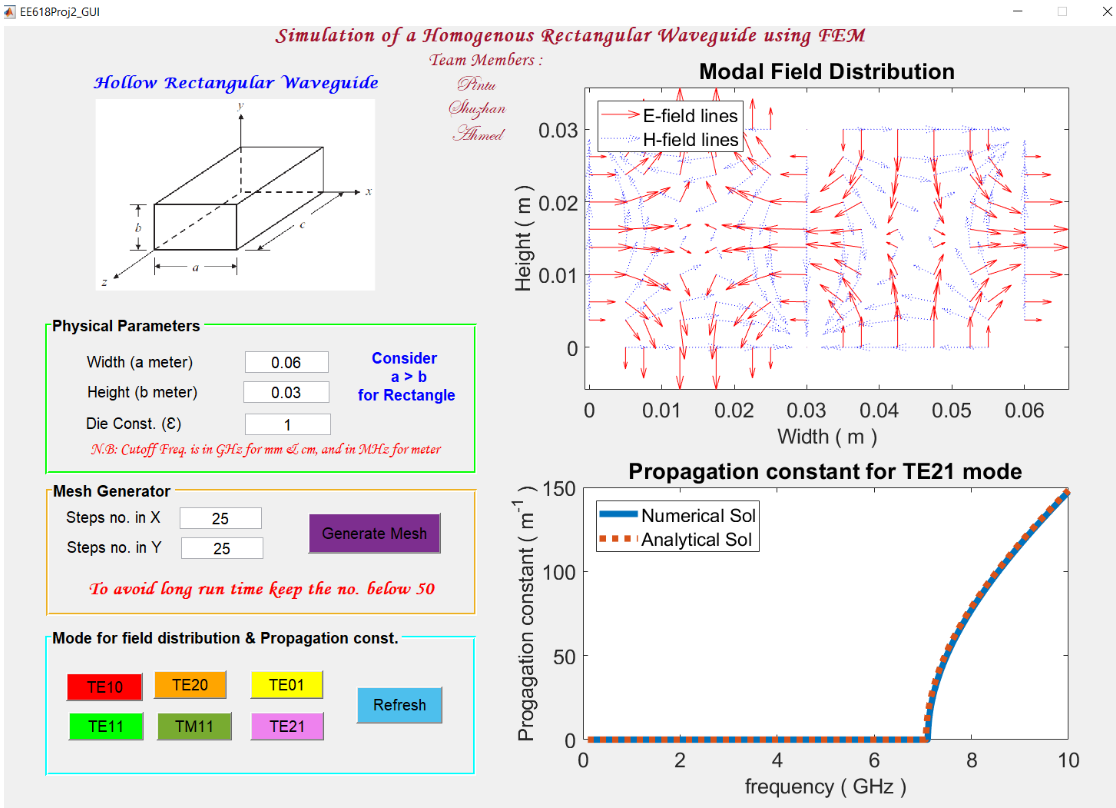
$$B_{ij}^e=\frac{\Delta^e}{12}\left(1+\delta_{ij}\right) \tag{12}$$

Then, we did assembly over all elements to get the global matrices $[A]$ and $[B]$. The matrix equation (10) was further solved with eigenvalue solver for sparse matrix.

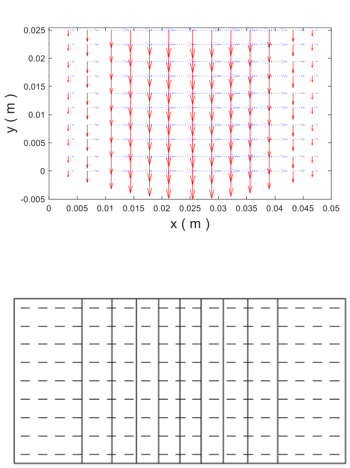
For TM mode, after assembly and getting the global matrices $[A]$ and $[B]$, we also imposed the boundary condition. Since the fixed value at the boundary is always 0, we directly removed n-th row and n-th column in the matrix, where n is the global boundary node index. Then, we solved the eigenvalue to get the propagation constant and solved the eigenvector to get the field distribution.

GUI and Simulation Results

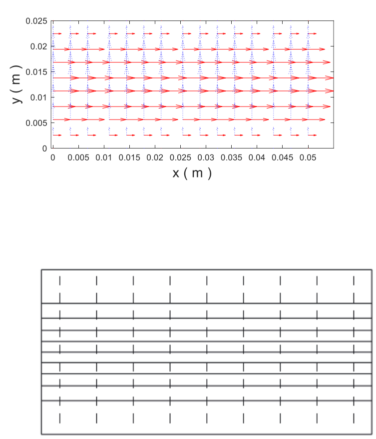
The GUI panel is designed as below. Everything is labeled clearly. When comparing with propagation constants, we used the analytical solution (4). The numerical propagation constants matched the analytical results very well.



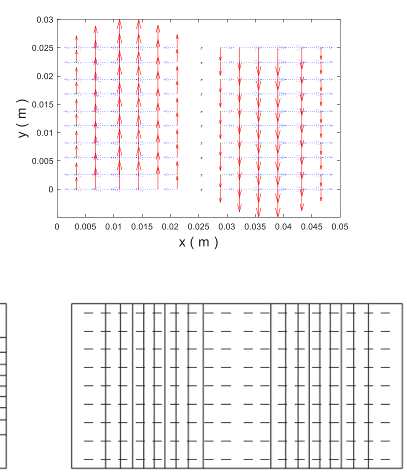
In plotting the transverse field components, we chose $k_z = 1$, calculated corresponding field components to form E-field vector (E_x, E_y) and H-field vector (H_x, H_y), then, normalized the field distribution with respect to maximum total electric field. The first 6 modes and their comparison with the figures in the book are



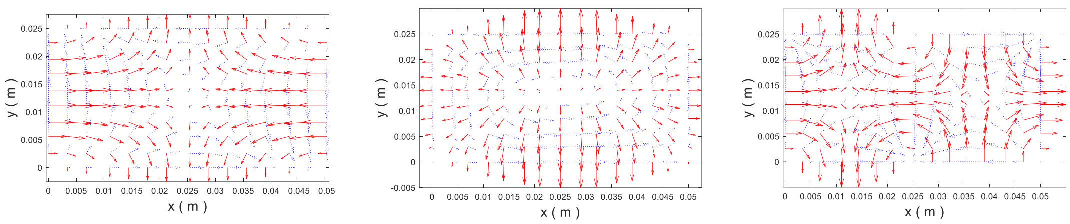
TE_{10}

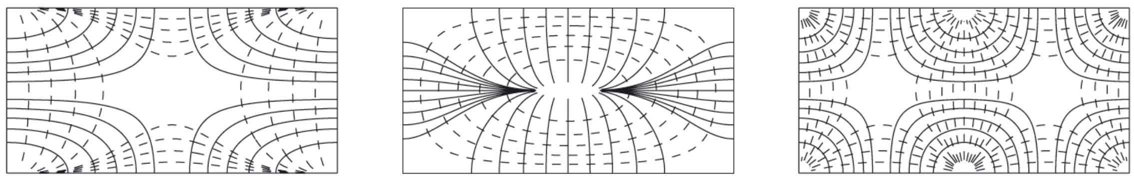


TE_{01}



TE_{20}





TE$_{11}$ TM$_{11}$ TE$_{21}$

The results prove that FEM is very accurate in capturing the first few dominate modes with about 20 by 20 nodes.

References

[1] J.-M. Jin, *Theory and computation of electromagnetic fields*: John Wiley & Sons, 2011.