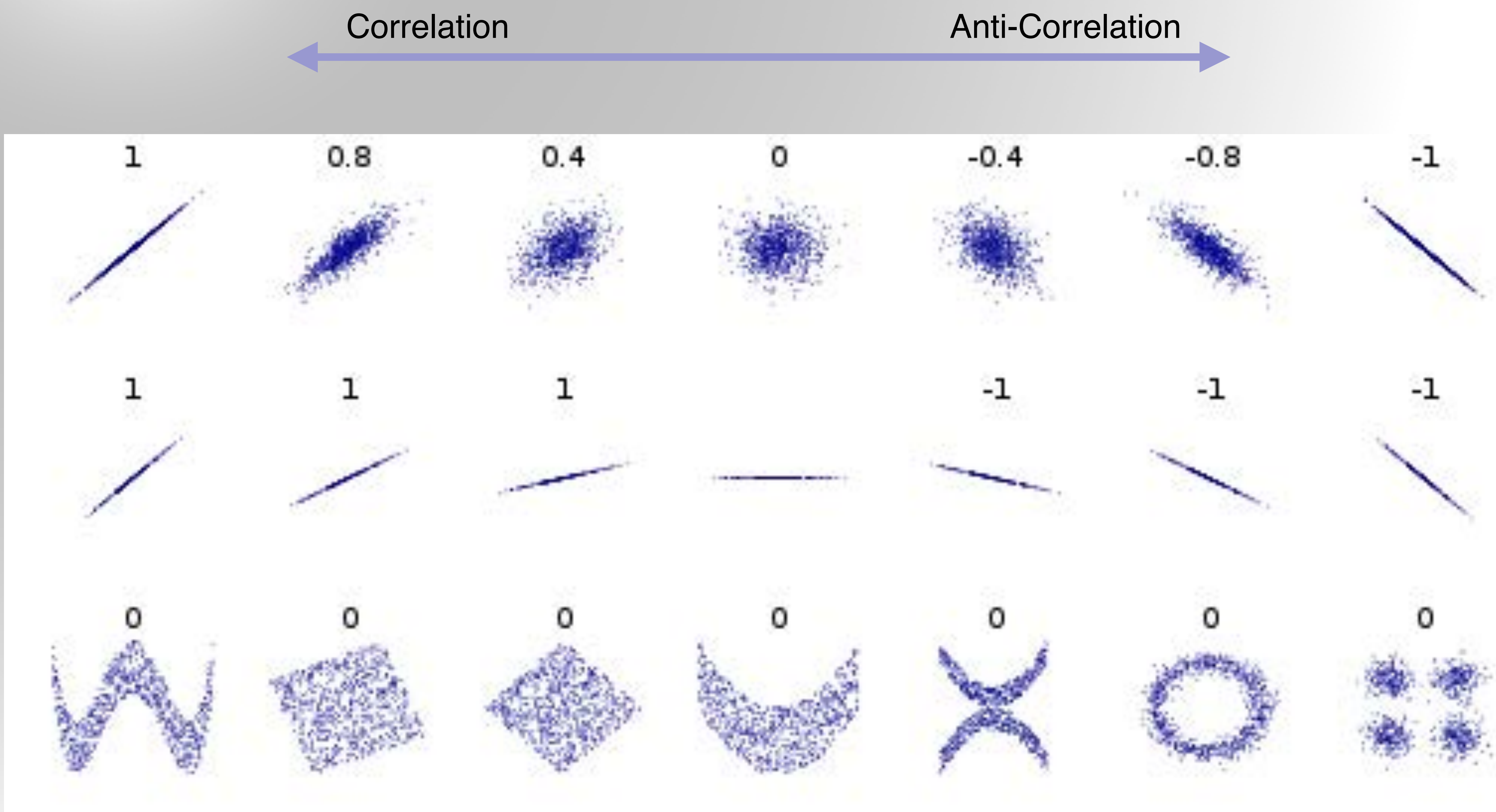
ENM 531: Data-driven modeling and probabilistic scientific computing

Lecture #3: Probability and Statistics primer

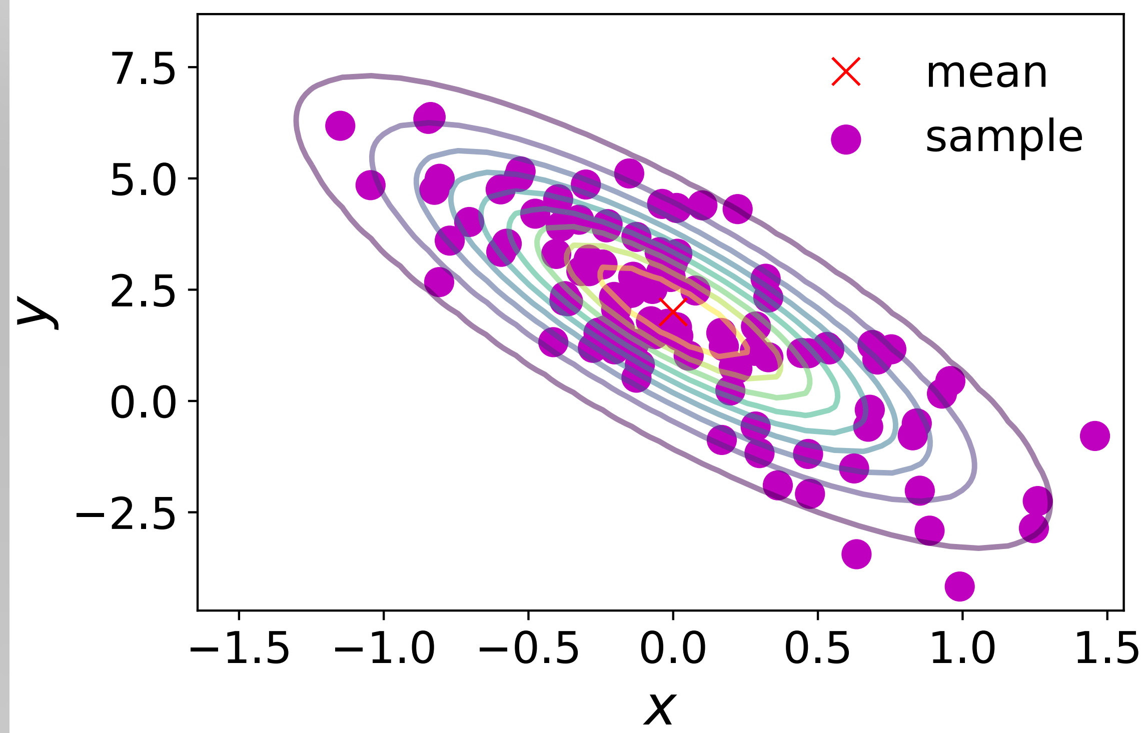
Paris Perdikaris
January 24, 2019

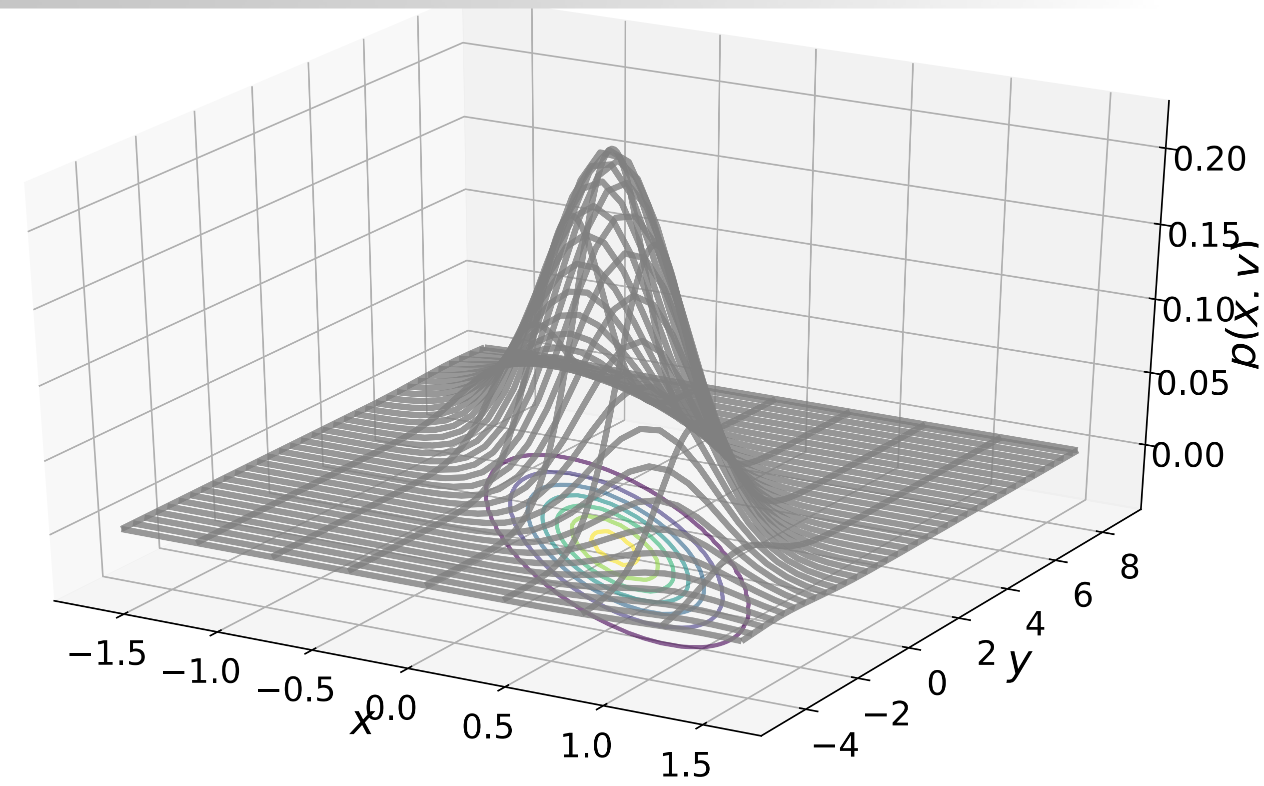
Penn
University of Pennsylvania

Correlation and linear dependence



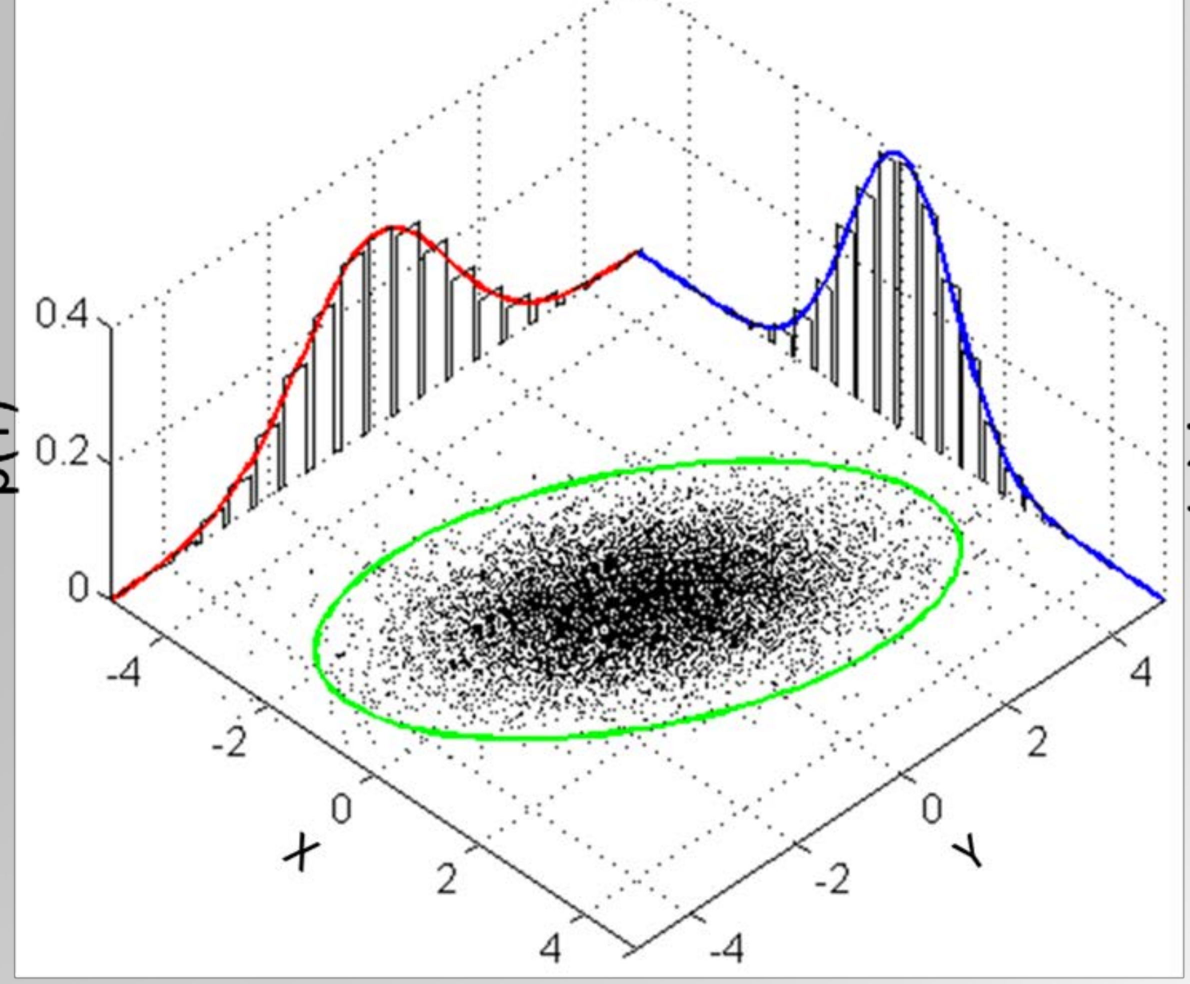
The multivariate Gaussian





$$p(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{D}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\big(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\big)$$

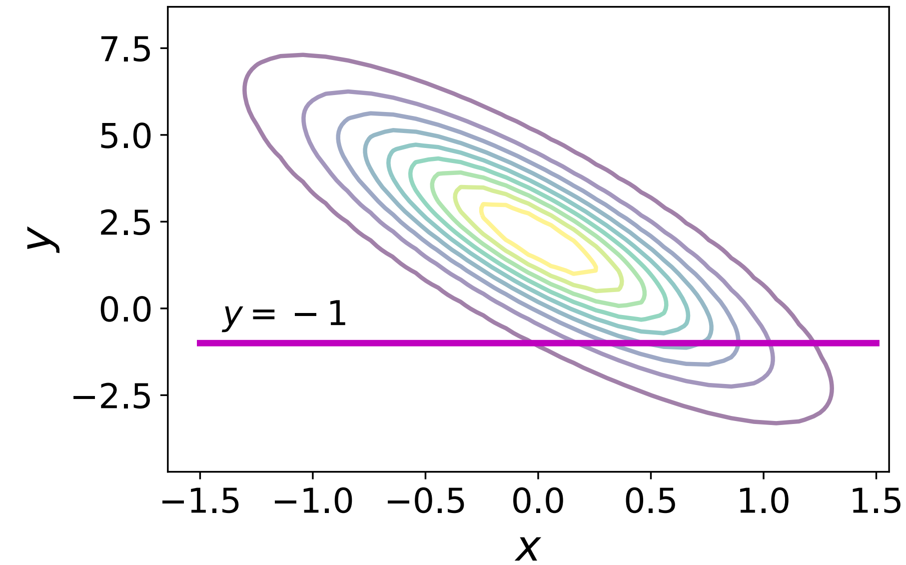
The multivariate Gaussian



Notation	$\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
Parameters	$\boldsymbol{\mu} \in \mathbf{R}^k$ — location $\boldsymbol{\Sigma} \in \mathbf{R}^{k \times k}$ — covariance (positive semi-definite matrix)
Support	$\boldsymbol{x} \in \boldsymbol{\mu} + \mathrm{span}(\boldsymbol{\Sigma}) \subseteq \mathbf{R}^k$
PDF	$\det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}}\, e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$, exists only when $\boldsymbol{\Sigma}$ is positive-definite
Mean	$\boldsymbol{\mu}$
Mode	$\boldsymbol{\mu}$
Variance	$\boldsymbol{\Sigma}$

Marginals and conditionals of a Gaussian

$$p(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{N}\left(\begin{bmatrix}\boldsymbol{\mu}_x \\ \boldsymbol{\mu}_y\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_{yy}\end{bmatrix}\right)$$



Conditional distribution

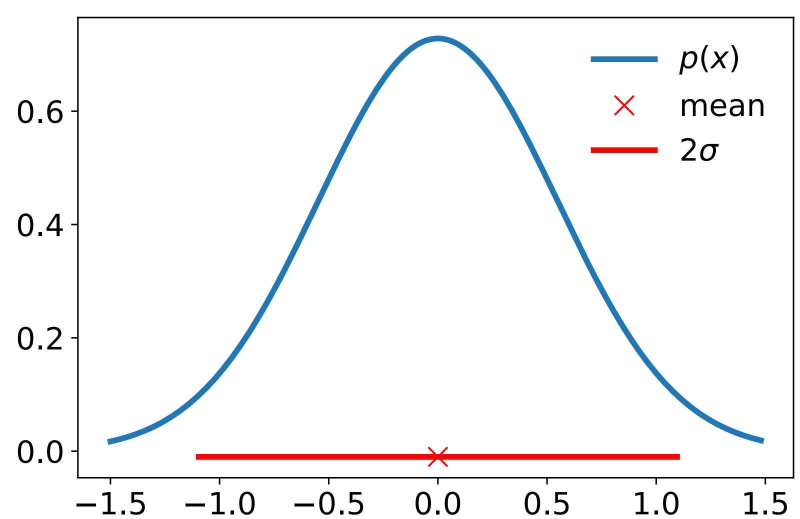
$$p(\boldsymbol{x} \mid \boldsymbol{y}) = \mathcal{N}\left(\boldsymbol{\mu}_{x \mid y}, \boldsymbol{\Sigma}_{x \mid y}\right)$$

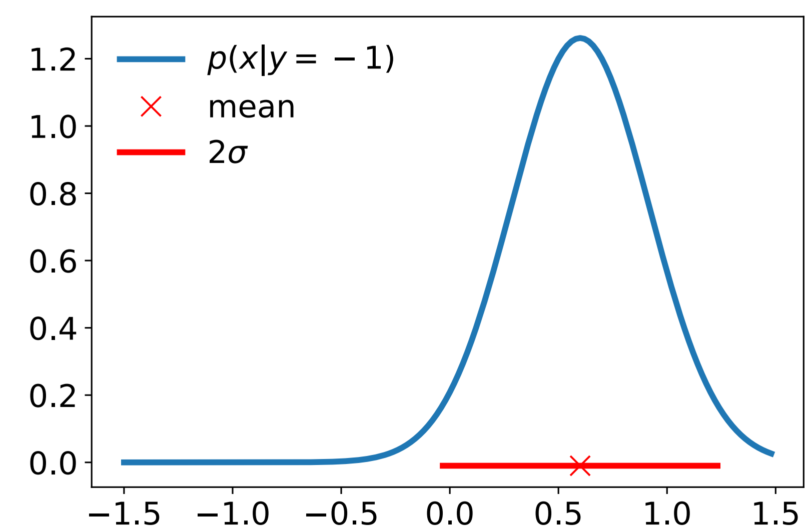
$$\boldsymbol{\mu}_{x \mid y} = \boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_y)$$

$$\boldsymbol{\Sigma}_{x \mid y} = \boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}\,.$$

Marginal distribution

$$p(\boldsymbol{x}) = \int p(\boldsymbol{x}, \boldsymbol{y})d\boldsymbol{y} = \mathcal{N}\left(\boldsymbol{x} \mid \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx}\right)$$





These are unique properties that make the Gaussian distribution very simple and attractive to compute with! It is essentially our main building block for computing under uncertainty.

Transformations