Simple Digital Filters

- Later in the course we shall review various methods of designing frequency-selective filters satisfying prescribed specifications
- We now describe several low-order FIR and IIR digital filters with reasonable selective frequency responses that often are satisfactory in a number of applications

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Simple FIR Digital Filters

- FIR digital filters considered here have integer-valued impulse response coefficients
- These filters are employed in a number of practical applications, primarily because of their simplicity, which makes them amenable to inexpensive hardware implementations

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Simple FIR Digital Filters

Lowpass FIR Digital Filters

- The simplest lowpass FIR digital filter is the 2-point moving-average filter given by

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

- The above transfer function has a zero at $z = -1$ and a pole at $z = 0$
- Note that here the pole vector has a unity magnitude for all values of ω

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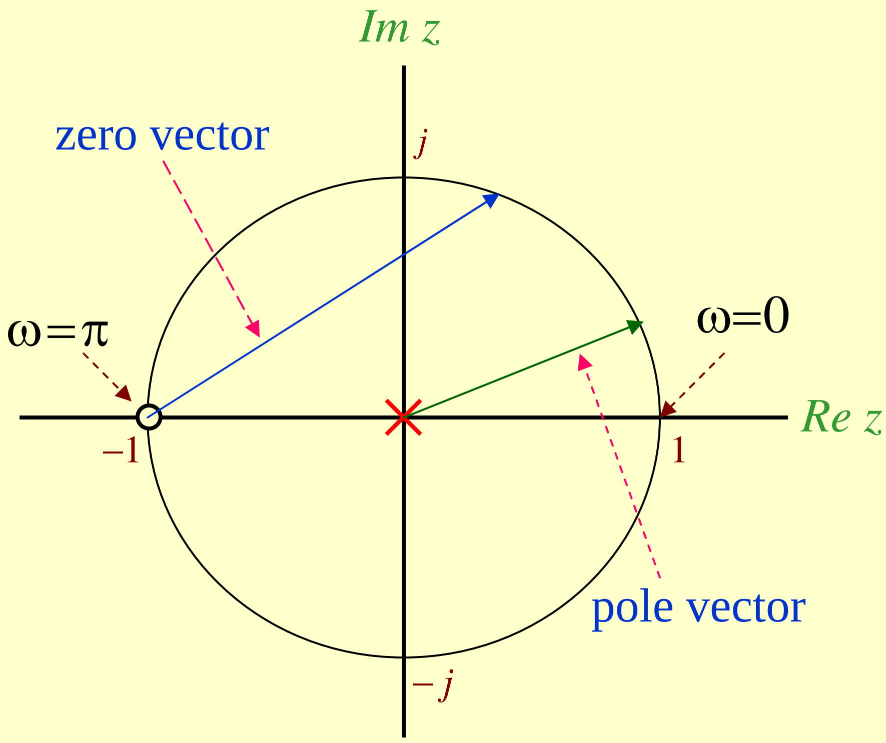
Simple FIR Digital Filters



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Simple FIR Digital Filters

- On the other hand, as ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
- Hence, the magnitude response $|H_0(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$

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- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$, i.e.,

$$\left|H_0(e^{j0})\right| = 1, \quad \left|H_0(e^{j\pi})\right| = 0$$

- The frequency response of the above filter is given by

$$H_0(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

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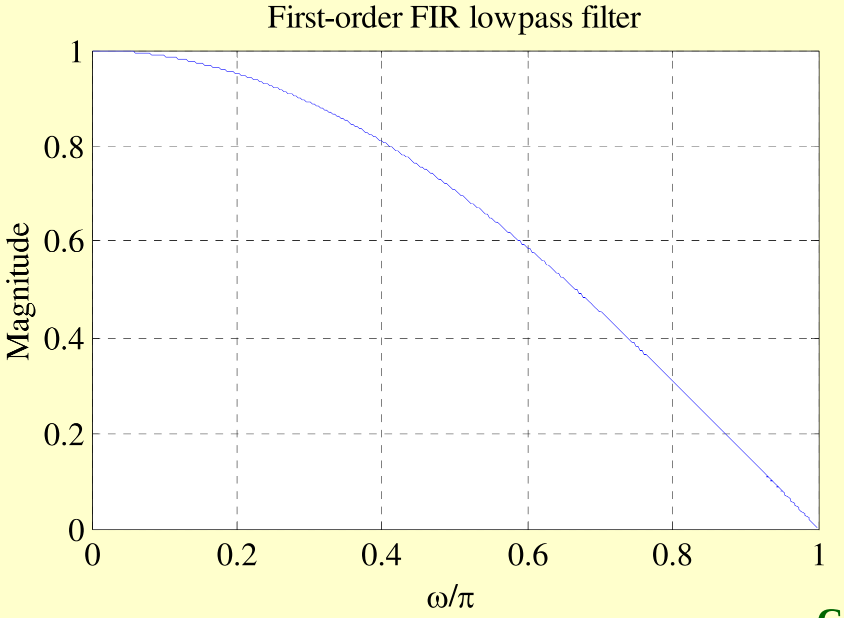
Simple FIR Digital Filters

- The magnitude response $|H_0(e^{j\omega})| = \cos(\omega/2)$ can be seen to be a monotonically decreasing function of ω



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Simple FIR Digital Filters

- The frequency $\omega = \omega_c$ at which

$$\left|H_0(e^{j\omega_c})\right| = \frac{1}{\sqrt{2}}\left|H_0(e^{j0})\right|$$

is of practical interest since here the gain $\mathcal{G}(\omega_c)$ in dB is given by

$$\mathcal{G}(\omega_c) = 20\log_{10}\left|H(e^{j\omega_c})\right|$$
$$= 20\log_{10}\left|H(e^{j0})\right| - 20\log_{10}\sqrt{2} \cong -3\text{ dB}$$

since the dc gain $\mathcal{G}(0) = 20\log_{10}\left|H(e^{j0})\right| = 0$

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- Thus, the gain $\mathcal{G}(\omega)$ at $\omega = \omega_c$ is approximately 3 dB less than the gain at $\omega = 0$
- As a result, ω_c is called the **3-dB cutoff frequency**
- To determine the value of ω_c we set

$$|H_0(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = \frac{1}{2}$$

which yields $\omega_c = \pi/2$

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- The 3-dB cutoff frequency ω_c can be considered as the passband edge frequency
- As a result, for the filter $H_0(z)$ the passband width is approximately $\pi/2$
- The stopband is from $\pi/2$ to π
- Note: $H_0(z)$ has a zero at $z = -1$ or $\omega = \pi$, which is in the stopband of the filter

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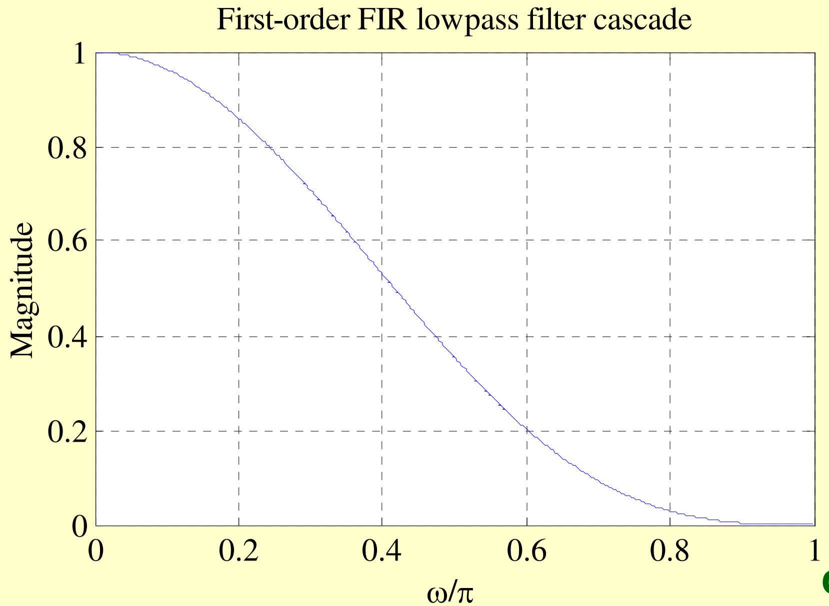
Simple FIR Digital Filters

- A cascade of the simple FIR filter

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1})$$

results in an improved lowpass frequency response as illustrated below for a cascade of 3 sections



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Simple FIR Digital Filters

- The 3-dB cutoff frequency of a cascade of M sections is given by

$$\omega_c = 2\cos^{-1}(2^{-1/2M})$$

- For $M = 3$, the above yields $\omega_c = 0.302\pi$
- Thus, the cascade of first-order sections yields a sharper magnitude response but at the expense of a decrease in the width of the passband

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Simple FIR Digital Filters

- A better approximation to the ideal lowpass filter is given by a higher-order moving-average filter
- Signals with rapid fluctuations in sample values are generally associated with high-frequency components
- These high-frequency components are essentially removed by an moving-average filter resulting in a smoother output waveform

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Simple FIR Digital Filters

Highpass FIR Digital Filters

- The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing z with $-z$
- This results in

$$H_1(z) = \tfrac{1}{2}(1 - z^{-1})$$

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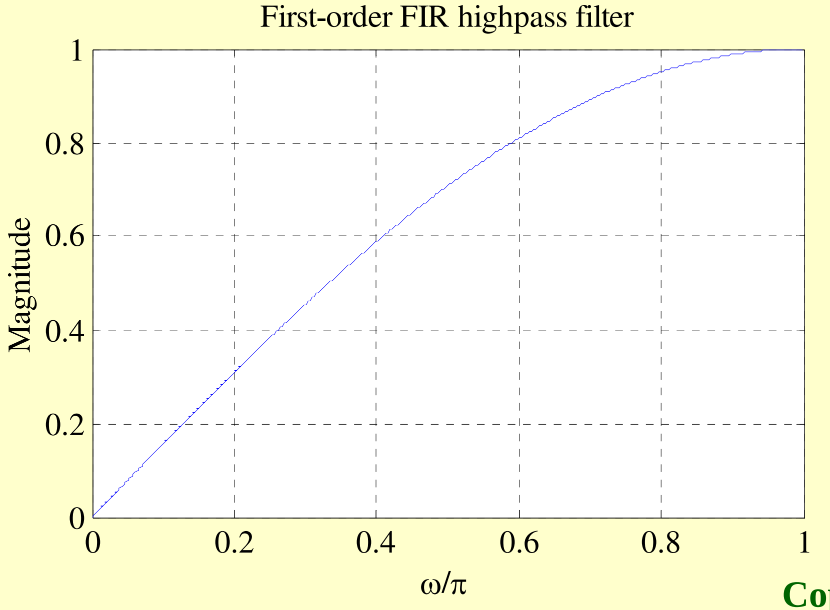
Simple FIR Digital Filters

- Corresponding frequency response is given by

$$H_1(e^{j\omega}) = j\,e^{-j\omega/2}\sin(\omega/2)$$

whose magnitude response is plotted below



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Simple FIR Digital Filters

- The monotonically increasing behavior of the magnitude function can again be demonstrated by examining the pole-zero pattern of the transfer function $H_1(z)$
- The highpass transfer function $H_1(z)$ has a zero at $z = 1$ or $\omega = 0$ which is in the stopband of the filter

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Simple FIR Digital Filters

- Improved highpass magnitude response can again be obtained by cascading several sections of the first-order highpass filter
- Alternately, a higher-order highpass filter of the form

$$H_1(z) = \frac{1}{M}\sum_{n=0}^{M-1}(-1)^n z^{-n}$$

is obtained by replacing z with $-z$ in the transfer function of a moving average filter

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Simple FIR Digital Filters

- An application of the FIR highpass filters is in moving-target-indicator (MTI) radars
- In these radars, interfering signals, called **clutters**, are generated from fixed objects in the path of the radar beam
- The clutter, generated mainly from ground echoes and weather returns, has frequency components near zero frequency (dc)

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Simple FIR Digital Filters

- The clutter can be removed by filtering the radar return signal through a **two-pulse canceler**, which is the first-order FIR highpass filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- For a more effective removal it may be necessary to use a **three-pulse canceler** obtained by cascading two two-pulse cancelers

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Simple IIR Digital Filters

Lowpass IIR Digital Filters

- We have shown earlier that the first-order causal IIR transfer function

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \quad 0 < \alpha < 1$$

has a lowpass magnitude response for $\alpha > 0$

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Simple IIR Digital Filters

- An improved lowpass magnitude response is obtained by adding a factor $(1+z^{-1})$ to the numerator of transfer function

$$H(z)=\frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad 0<\alpha<1$$

- This forces the magnitude response to have a zero at $\omega=\pi$ in the stopband of the filter

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Simple IIR Digital Filters

- On the other hand, the first-order causal IIR transfer function

$$H(z) = \frac{K}{1-\alpha z^{-1}}, \quad -1 < \alpha < 0$$

has a highpass magnitude response for $\alpha < 0$

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Simple IIR Digital Filters

- However, the modified transfer function obtained with the addition of a factor $(1+z^{-1})$ to the numerator

$$H(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad -1 < \alpha < 0$$

exhibits a lowpass magnitude response

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Simple IIR Digital Filters

- The modified first-order lowpass transfer function for both positive and negative values of α is then given by

$$H_{LP}(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0

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Simple IIR Digital Filters

- The maximum values of the magnitude function is $2K/(1-\alpha)$ at $\omega = 0$ and the minimum value is 0 at $\omega = \pi$, i.e.,

$$\left|H_{LP}(e^{j0})\right| = \frac{2K}{1-\alpha}, \qquad \left|H_{LP}(e^{j\pi})\right| = 0$$

- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$

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Simple IIR Digital Filters

- For most applications, it is usual to have a dc gain of 0 dB, that is to have $|H(e^{j0})| = 1$
- To this end, we choose $K = (1-\alpha)/2$ resulting in the first-order IIR lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right), \quad 0 < |\alpha| < 1$$

- The above transfer function has a zero at i.e., at $\omega = \pi$ which is in the stopband

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Simple IIR Digital Filters

Lowpass IIR Digital Filters

- A first-order causal lowpass IIR digital filter has a transfer function given by

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

where $|\alpha| < 1$ for stability

- The above transfer function has a zero at $z = -1$ i.e., at $\omega = \pi$ which is in the stopband

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- $H_{LP}(z)$ has a real pole at $z = \alpha$
- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of α, the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

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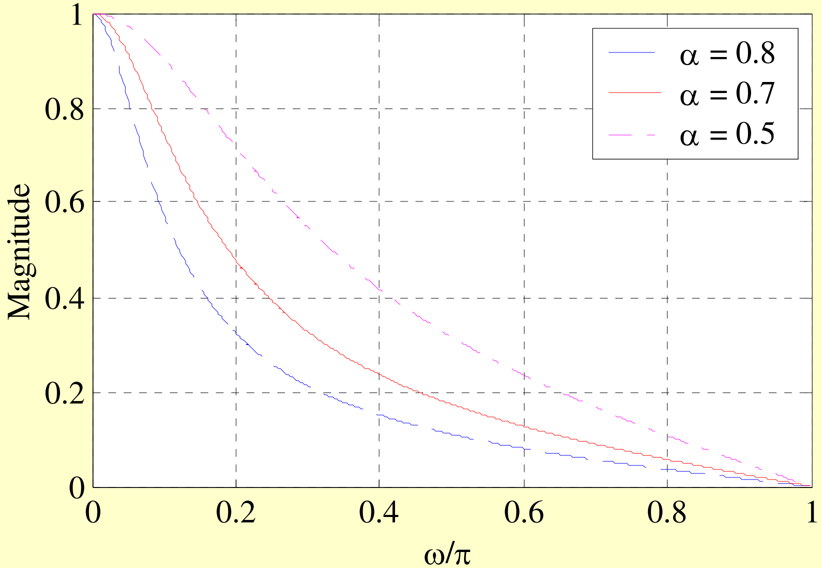
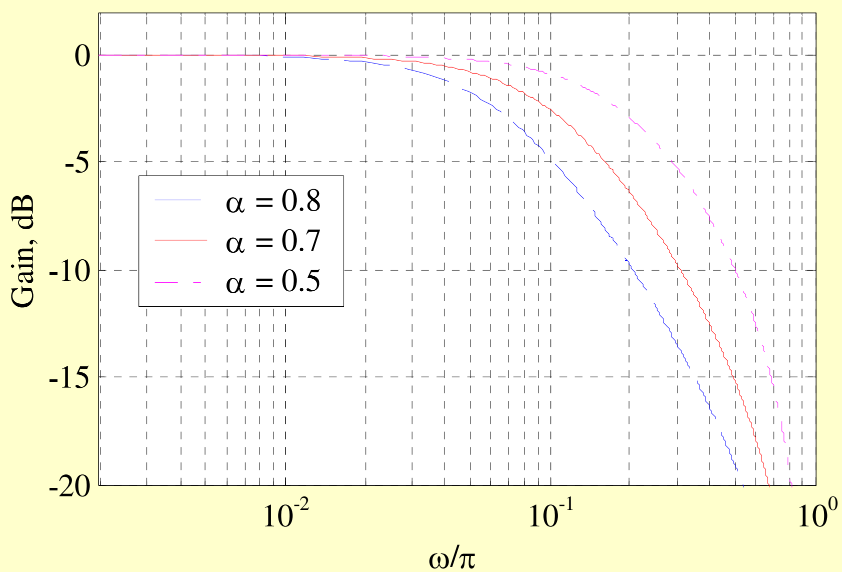
- i.e., $|H_{LP}(e^{j0})| = 1, \qquad |H_{LP}(e^{j\pi})| = 0$
- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$ as indicated below





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- The squared magnitude function is given by

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

- The derivative of $|H_{LP}(e^{j\omega})|^2$ with respect to ω is given by

$$\frac{d\,|H_{LP}(e^{j\omega})|^2}{d\omega} = \frac{-(1-\alpha)^2(1+2\alpha+\alpha^2)\sin\omega}{2(1-2\alpha\cos\omega+\alpha^2)^2}$$

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$d\,|H_{LP}(e^{j\omega})|^2 / d\omega \le 0$ in the range $0 \le \omega \le \pi$ verifying again the monotonically decreasing behavior of the magnitude function

- To determine the 3-dB cutoff frequency we set

$$|H_{LP}(e^{j\omega_c})|^2 = \frac{1}{2}$$

in the expression for the square magnitude function resulting in

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$$\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}=\frac{1}{2}$$

or

$$(1-\alpha)^2(1+\cos\omega_c)=1+\alpha^2-2\alpha\cos\omega_c$$

which when solved yields

$$\cos\omega_c=\frac{2\alpha}{1+\alpha^2}$$

- The above quadratic equation can be solved for α yielding two solutions

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- The solution resulting in a stable transfer function $H_{LP}(z)$ is given by

$$\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$$

- It follows from

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2 (1+\cos\omega)}{2(1+\alpha^2 - 2\alpha\cos\omega)}$$

that $H_{LP}(z)$ is a BR function for $|\alpha| < 1$

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Highpass IIR Digital Filters

- A first-order causal highpass IIR digital filter has a transfer function given by

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

where $|\alpha| < 1$ for stability

- The above transfer function has a zero at $z = 1$ i.e., at $\omega = 0$ which is in the stopband

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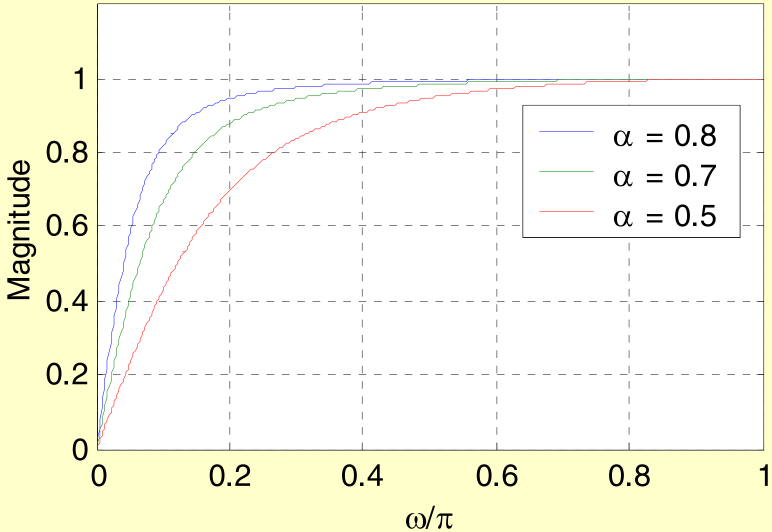
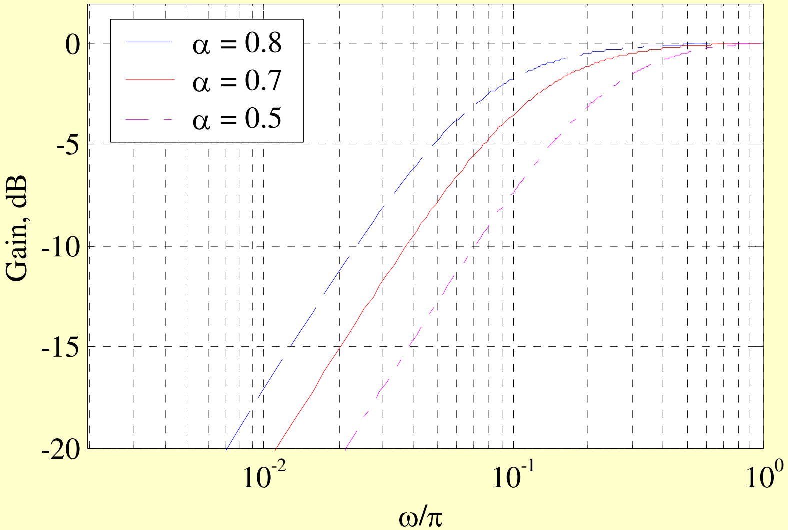
- Its 3-dB cutoff frequency ω_c is given by

$$\alpha = (1 - \sin\omega_c)/\cos\omega_c$$

which is the same as that of $H_{LP}(z)$

- Magnitude and gain responses of $H_{HP}(z)$ are shown below





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- $H_{HP}(z)$ is a BR function for $|\alpha| < 1$
- Example - Design a first-order highpass digital filter with a 3-dB cutoff frequency of 0.8π
- Now, $\sin(\omega_c) = \sin(0.8\pi) = 0.587785$ and $\cos(0.8\pi) = -0.80902$
- Therefore

$$\alpha = (1 - \sin\omega_c)/\cos\omega_c = -0.5095245$$

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- Therefore,

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

$$= 0.245238\left(\frac{1-z^{-1}}{1+0.5095245\,z^{-1}}\right)$$

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Bandpass IIR Digital Filters

- A 2nd-order bandpass digital transfer function is given by

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- Its squared magnitude function is

$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1+\beta^2(1+\alpha)^2+\alpha^2-2\beta(1+\alpha)^2\cos\omega+2\alpha\cos 2\omega]}$$

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Simple IIR Digital Filters

- $|H_{BP}(e^{j\omega})|^2$ goes to zero at $\omega = 0$ and $\omega = \pi$
- It assumes a maximum value of 1 at $\omega = \omega_o$, called the **center frequency** of the bandpass filter, where

$$\omega_o = \cos^{-1}(\beta)$$

- The frequencies ω_{c1} and ω_{c2} where $|H_{BP}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**

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- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

- The transfer function $H_{BP}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

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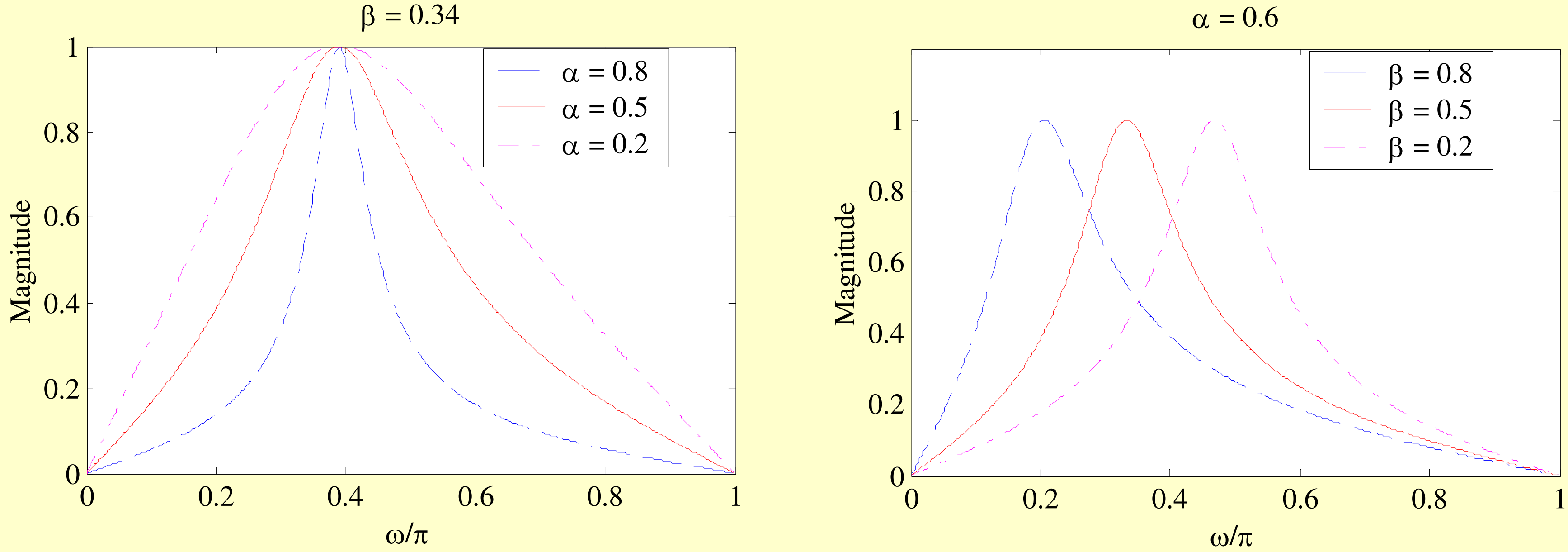
- Plots of $|H_{BP}(e^{j\omega})|$ are shown below



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Simple IIR Digital Filters

- Example - Design a 2nd order bandpass digital filter with center frequency at 0.4π and a 3-dB bandwidth of 0.1π
- Here $\beta = \cos(\omega_o) = \cos(0.4\pi) = 0.309017$

  and

$$\frac{2\alpha}{1+\alpha^2} = \cos(B_w) = \cos(0.1\pi) = 0.9510565$$

- The solution of the above equation yields: $\alpha = 1.376382$ and $\alpha = 0.72654253$

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- The corresponding transfer functions are

$$H'_{BP}(z) = -0.18819\frac{1-z^{-2}}{1-0.7343424z^{-1}+1.37638z^{-2}}$$

and

$$H''_{BP}(z) = 0.13673\frac{1-z^{-2}}{1-0.533531z^{-1}+0.72654253z^{-2}}$$

- The poles of $H'_{BP}(z)$ are at $z = 0.3671712 \pm j1.11425636$ and have a magnitude > 1

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- Thus, the poles of $H'_{BP}(z)$ are outside the unit circle making the transfer function unstable
- On the other hand, the poles of $H''_{BP}(z)$ are at $z = 0.2667655 \pm j0.8095546$ and have a magnitude of 0.8523746
- Hence $H''_{BP}(z)$ is BIBO stable
- Later we outline a simpler stability test

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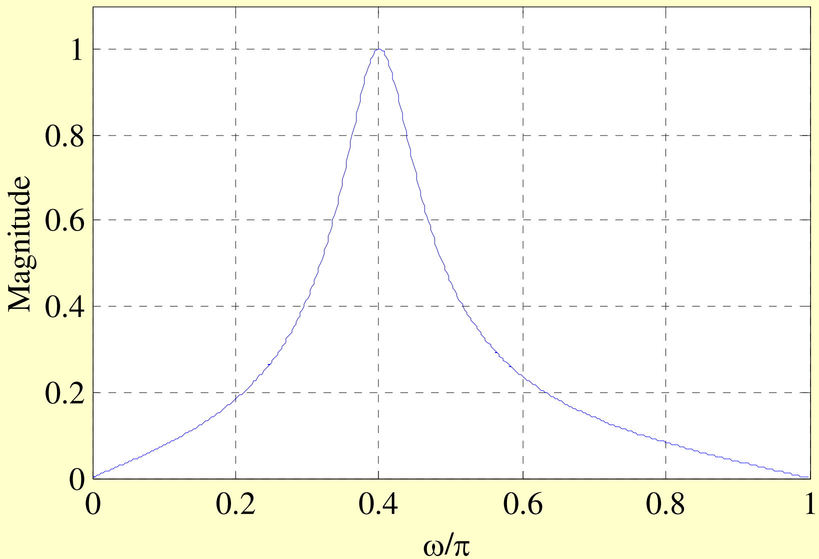
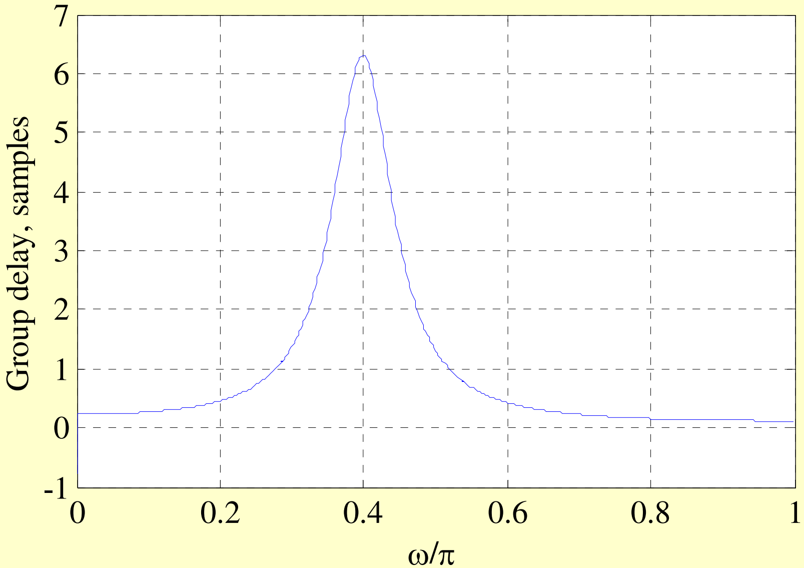
- Figures below show the plots of the magnitude function and the group delay of $H''_{BP}(z)$





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Simple IIR Digital Filters

Bandstop IIR Digital Filters

- A 2nd-order bandstop digital filter has a transfer function given by

$$H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- The transfer function $H_{BS}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

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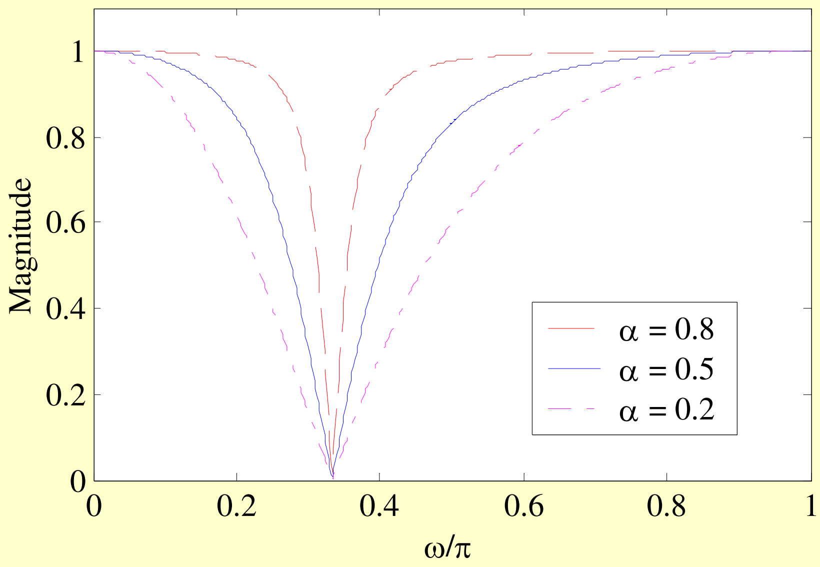
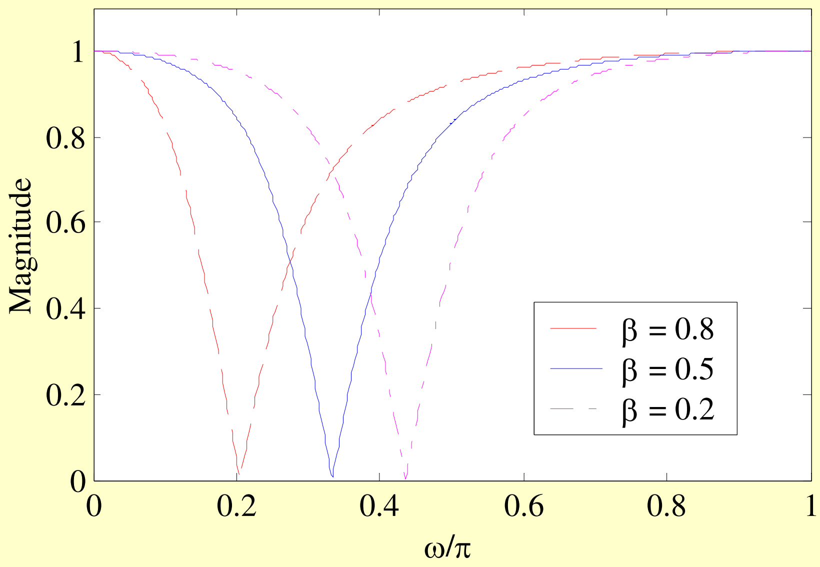
- Its magnitude response is plotted below





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- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_o$, where ω_o , called the **notch frequency**, is given by
$$\omega_o = \cos^{-1}(\beta)$$
- The digital transfer function $H_{BS}(z)$ is more commonly called a **notch filter**

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Simple IIR Digital Filters

- The frequencies ω_{c1} and ω_{c2} where $|H_{BS}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**
- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB notch bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

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Simple IIR Digital Filters

Higher-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of K first-order lowpass sections characterized by the transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

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- The overall structure has a transfer function given by

$$G_{LP}(z) = \left(\frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)^{K}$$

- The corresponding squared-magnitude function is given by

$$|G_{LP}(e^{j\omega})|^2 = \left[\frac{(1-\alpha)^2 (1+\cos\omega)}{2(1+\alpha^2 - 2\alpha\cos\omega)} \right]^{K}$$

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Simple IIR Digital Filters

- To determine the relation between its 3-dB cutoff frequency ω_c and the parameter α, we set

$$\left[\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}\right]^K=\frac{1}{2}$$

which when solved for α, yields for a stable $G_{LP}(z)$:

$$\alpha=\frac{1+(1-C)\cos\omega_c-\sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

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where

$$C = 2^{(K-1)/K}$$

- It should be noted that the expression for α given earlier reduces to

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

for $K = 1$

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Simple IIR Digital Filters

- Example - Design a lowpass filter with a 3-dB cutoff frequency at $\omega_c = 0.4\pi$ using a single first-order section and a cascade of 4 first-order sections, and compare their gain responses
- For the single first-order lowpass filter we have

$$\alpha = \frac{1+\sin\omega_c}{\cos\omega_c} = \frac{1+\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$$

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- For the cascade of 4 first-order sections, we substitute $K = 4$ and get

$$C = 2^{(K-1)/K} = 2^{(4-1)/4} = 1.6818$$

- Next we compute

$$\alpha = \frac{1+(1-C)\cos\omega_c - \sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

$$= \frac{1+(1-1.6818)\cos(0.4\pi)-\sin(0.4\pi)\sqrt{2(1.6818)-(1.6818)^2}}{1-1.6818+\cos(0.4\pi)}$$

$$= -0.251$$

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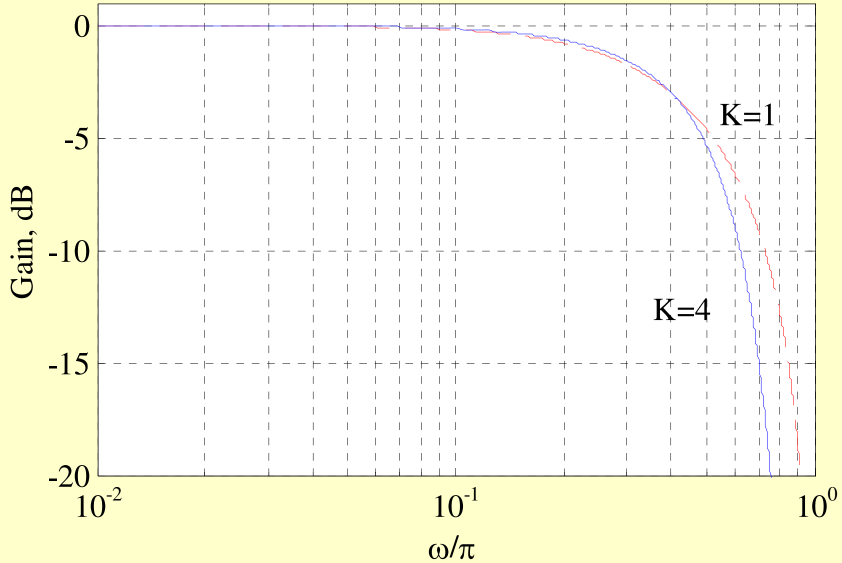
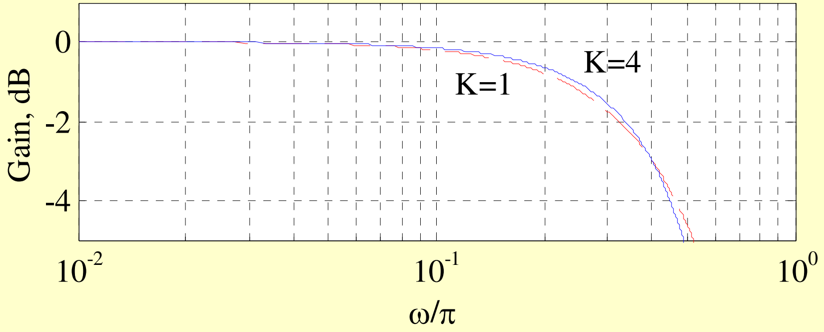
Simple IIR Digital Filters

- The gain responses of the two filters are shown below
- As can be seen, cascading has resulted in a sharper roll-off in the gain response



Passband details



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