1.

(1)

Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g[n]$ where

$$g[n] = g_a(nT), \qquad -\infty < n < \infty, \tag{3.61}$$

with T being the *sampling period*. The reciprocal of T is called the *sampling frequency* F_T; that is, $F_T = 1/T$. Now, the frequency-domain representation of $g_a(t)$ is given by its continuous-time Fourier transform (CTFT):

$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t)e^{-j\Omega t}\, dt, \tag{3.62}$$

whereas the frequency-domain representation of $g[n]$ is given by its discrete-time Fourier transform:

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}. \tag{3.63}$$

To establish the relations between these two different types of Fourier spectra, $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a multiplication of the continuous-time signal $g_a(t)$ by a periodic impulse train $p(t)$:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{3.64}$$

consisting of a train of ideal impulse functions[7] with a period T, as indicated in Figure 3.14. The multiplication operation yields a continuous function of time for all values of t including $t = nT$:

$$g_p(t) = g_a(t)p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT). \tag{3.65}$$

It should be noted that the signal $g_p(t)$ consists of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant and is thus a distribution solely defined by its integration properties.

There are two different forms of the continuous-time Fourier transform $G_p(j\Omega)$ of $g_p(t)$. One form is obtained by taking the CTFT of Eq. (3.64), which results in a weighted sum of the continuous-time Fourier transforms of the shifted impulse functions $\delta(t-nT)$. The CTFT of $\delta(t-nT)$ is given by $e^{-j\Omega nT}$. Hence, from Eq. (3.65), $G_p(j\Omega)$ is given by

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}. \tag{3.66}$$

To derive the second form, we make use of the *Poisson's sum formula*, given by [Pap62]

$$\sum_{n=-\infty}^{\infty} \phi(t+nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)\, e^{jk\Omega_T t}, \tag{3.67}$$

where $\Omega_T = 2\pi/T$ denotes the angular sampling frequency, and $\Phi(j\Omega)$ is the CTFT of the continuous-time function $\phi(t)$. A proof of the above identity is left as an exercise (Problem 3.56). For $t = 0$, Eq. (3.67) reduces to a simpler form

$$\sum_{n=-\infty}^{\infty} \phi(nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T), \tag{3.68}$$

which is used to derive the second form of $G_p(j\Omega)$. Now from the frequency-shifting property of the CTFT, the CTFT of $g_a(t)\, e^{-j\Psi t}$ is given by $G_a\left(j(\Omega+\Psi)\right)$. Substituting $\phi(t) = g_a(t)\, e^{-j\Psi t}$ in Eq. (3.68), we arrive at

$$\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Psi nT} = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j(k\Omega_T+\Psi)\right). \tag{3.69}$$

By replacing Ψ with Ω in the above equation, we arrive at the alternative form of the continuous-time Fourier transform of $g_p(t)$ given by

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j(\Omega+k\Omega_T)\right). \tag{3.70}$$

As can be seen from the above equation, $G_p(j\Omega)$ is a periodic function of frequency Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by $1/T$. The

We now establish the relation between the discrete-time Fourier transform $G(e^{j\omega})$ of the sequence $g[n]$ and the continuous-time Fourier transform $G_a(j\Omega)$ of the analog signal $g_a(t)$. If we compare Eq. (3.63) with Eq. (3.66) and make use of Eq. (3.61), we observe that

$$G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\omega/T}, \tag{3.74a}$$

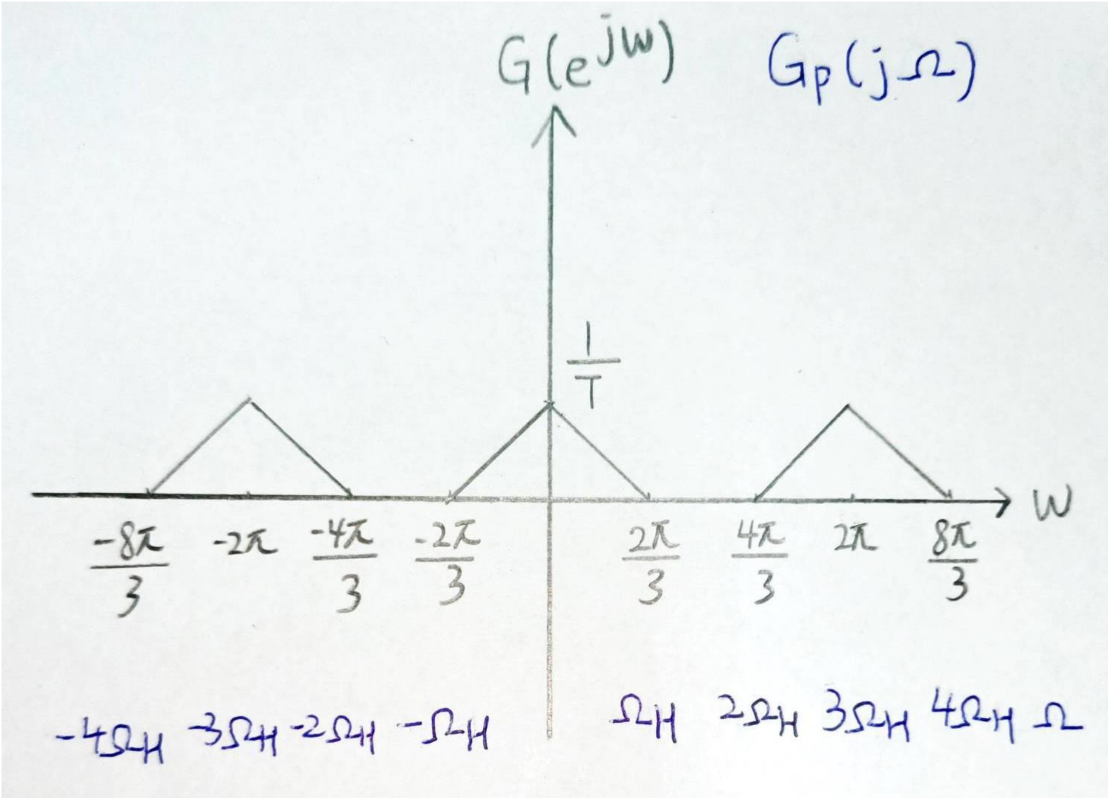
or equivalently,

$$G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}. \tag{3.74b}$$

Therefore, from the above and Eq. (3.69), we arrive at the desired result given by

$$\begin{aligned} G(e^{j\omega}) &= \left.\frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j\Omega + jk\Omega_T)\right|_{\Omega=\omega/T} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} + jk\Omega_T\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} + j\frac{2\pi k}{T}\right), \end{aligned} \tag{3.75a}$$

(2)



2.

(1)

$$g[n] \overset{DTFT}{\longleftrightarrow} G(e^{j\omega})$$

$$g[n-n_d] \overset{DTFT}{\longleftrightarrow} G(e^{j\omega})e^{-jn_d\omega}$$

$$\left|G(e^{j\omega})e^{-jn_d\omega}\right| = \left|G(e^{j\omega})\right|\left|e^{-jn_d\omega}\right| = \left|G(e^{j\omega})\right|$$

(2)

NO

3.

$$G\left(e^{j(\omega+2\pi)}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{(\omega+2\pi)}{T} + j\frac{2\pi k}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} + j\frac{2\pi(k+1)}{T}\right)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} + j\frac{2\pi n}{T}\right)$$

$$= G(e^{j\omega})$$

4.

(1)

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

$$= \sum_{k=0}^{N-1} h[k]\, e^{-j\omega k}$$

$$= \frac{1}{N}\left(1 + e^{-j\omega} + e^{-2j\omega} + \cdots + e^{-j(N-1)\omega}\right)$$

sum of a geometric sequence of N terms

and common ratio $e^{-j\omega}$

$$H(e^{j\omega}) = \frac{1}{N}\,\frac{1-e^{-jN\omega}}{1-e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{N}\,\frac{1-e^{-jN\omega}}{1-e^{-j\omega}}$$

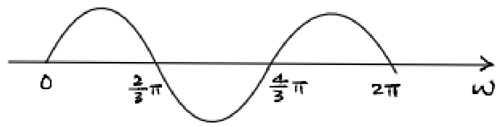
$$= \frac{1}{N}\,\frac{e^{-j\frac{N}{2}\omega}}{e^{-j\frac{\omega}{2}}}\;\frac{e^{j\frac{N}{2}\omega}-e^{-j\frac{N}{2}\omega}}{e^{j\omega/2}-e^{-j\omega/2}}$$

$$= e^{-j\frac{1}{2}(N-1)\omega}\,\frac{1}{N}\,\frac{\sin\frac{N}{2}\omega}{\sin\frac{\omega}{2}}$$

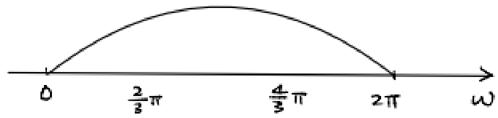
(2)

$$N = 3 \qquad \left|H(e^{j\omega})\right| = \left| \frac{1}{N} \frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega} \right|$$

$\sin \frac{3}{2}\omega$

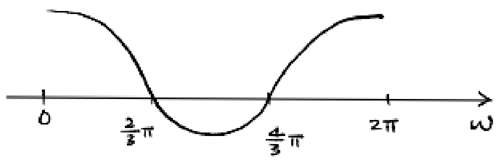


$\sin \frac{1}{2}\omega$

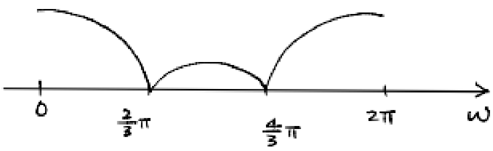


$$\lim_{\omega \to 0} \frac{1}{3} \frac{\sin \frac{3}{2}\omega}{\sin \omega/2} = \frac{1}{3} \lim_{\omega \to 0} \frac{\frac{3}{2}\cos\frac{3}{2}\omega}{\frac{1}{2}\cos\frac{\omega}{2}} = \underline{1}$$

$\frac{1}{3} \frac{\sin \frac{3}{2}\omega}{\sin \omega/2}$

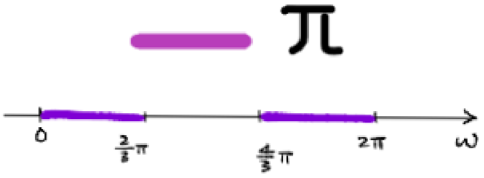


$\left|H(e^{j\omega})\right| = \left|\frac{1}{3} \frac{\sin \frac{3}{2}\omega}{\sin \omega/2}\right|$

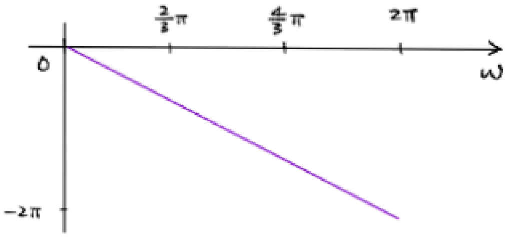


$$\measuredangle H(e^{j\omega}) = \angle e^{-j\frac{1}{2}(N-1)\omega} + \angle \frac{1}{3}\frac{\sin\frac{3}{2}\omega}{\sin\omega/2}$$

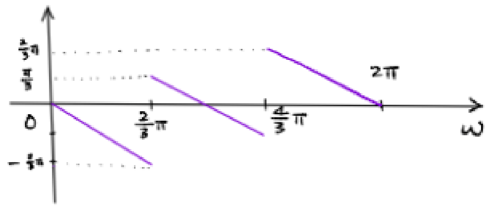
$$\angle \frac{1}{3}\frac{\sin\frac{3}{2}\omega}{\sin\omega/2}$$



$$\angle e^{-j\frac{1}{2}(3-1)\omega} = \angle e^{-j\omega} = -\omega$$



$$\measuredangle H(e^{j\omega}) = \angle e^{-j\frac{1}{2}(N-1)\omega} + \angle \frac{1}{3}\frac{\sin\frac{3}{2}\omega}{\sin\omega/2}$$



(3)

$$\frac{d}{d\omega}\arg\left(H\left(e^{j\omega}\right)\right) = -\frac{1}{2}(N-1)$$

(4)

YES

$$\begin{aligned}
&H\left(e^{j\omega}\right)\\
&= \alpha + \beta e^{-j\omega} + \alpha e^{-j2\omega}\\
&= e^{-j\omega}\left(\alpha e^{j\omega} + \beta + \alpha e^{-j\omega}\right)\\
&= e^{-j\omega}(2\alpha\cos(\omega) + \beta)\\
&\frac{d}{d\omega}\arg\left(H\left(e^{j\omega}\right)\right) = -1
\end{aligned}$$