DTFT Theorems

- Example - Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \ |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], |\alpha| < 1$
- We can therefore write

$$y[n] = n\,x[n] + x[n]$$

- From Table 3.3, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

DTFT Theorems

- Using the differentiation theorem of the DTFT given in Table 3.4, we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next using the linearity theorem of the DTFT given in Table 3.4 we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

DTFT Theorems

- Example - Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.3, the DTFT of $\delta[n]$ is 1
- Using the time-shifting theorem of the DTFT given in Table 3.4 we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Theorems

- Using the linearity theorem of Table 3.4 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- From Parseval's theorem given in Table 3.4 we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$$

  is called the **energy density spectrum**

- The area under this curve in the range $-\pi < \omega \leq \pi$ divided by 2π is the energy of the sequence

Band-limited Discrete-time Signals

- Since the spectrum of a discrete-time signal is a periodic function of ω with a period 2π, a full-band signal has a spectrum occupying the frequency range $-\pi < \omega \leq \pi$
- A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range $-\pi < \omega \leq \pi$

Band-limited Discrete-time Signals

- An ideal band-limited signal has a spectrum that is zero outside a frequency range $0 < \omega_a \le |\omega| \le \omega_b < \pi$, that is

$$X(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| < \omega_a \\ 0, & \omega_b < |\omega| < \pi \end{cases}$$

- An ideal band-limited discrete-time signal cannot be generated in practice

Band-limited Discrete-time Signals

- A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal's energy is concentrated
- A lowpass discrete-time real signal has a spectrum occupying the frequency range $0 < |\omega| \leq \omega_p < \pi$ and has a bandwidth of ω_p

Band-limited Discrete-time Signals

- A highpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_p \le |\omega| < \pi$ and has a bandwidth of $\pi - \omega_p$
- A bandpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_L \le |\omega| \le \omega_H < \pi$ and has a bandwidth of $\omega_H - \omega_L$

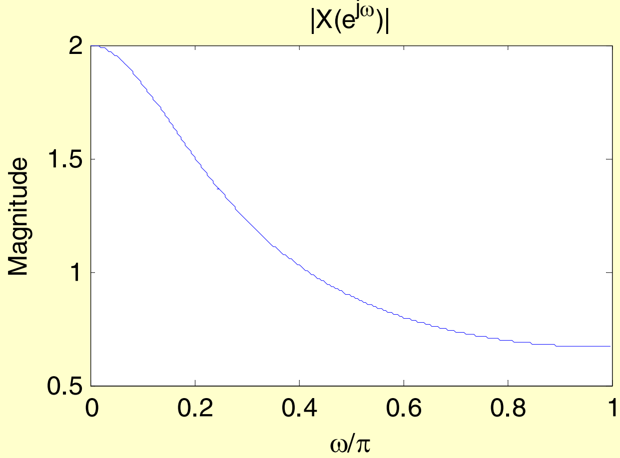
Band-limited Discrete-time Signals

- Example – Consider the sequence

$$x[n] = (0.5)^n \mu[n]$$

- Its DTFT is given below on the left along with its magnitude spectrum shown below on the right

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



Band-limited Discrete-time Signals

- It can be shown that 80% of the energy of this lowpass signal is contained in the frequency range $0 \le |\omega| \le 0.5081\pi$
- Hence, we can define the 80% bandwidth to be 0.5081π radians

Energy Density Spectrum

- Example - Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \ -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) \right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Energy Density Spectrum

- Therefore

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}[n]$ is a finite-energy lowpass sequence

DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \dots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \dots + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_\ell$

DTFT Computation Using MATLAB

- For example, the statement

```
H = freqz(num,den,w)
```

returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector `w`

DTFT Computation Using MATLAB

- There are several other forms of the function `freqz`
- Program 3_1.m in the text can be used to compute the values of the DTFT of a real finite-length sequence
- It computes the real and imaginary parts, and the magnitude and phase of the DTFT

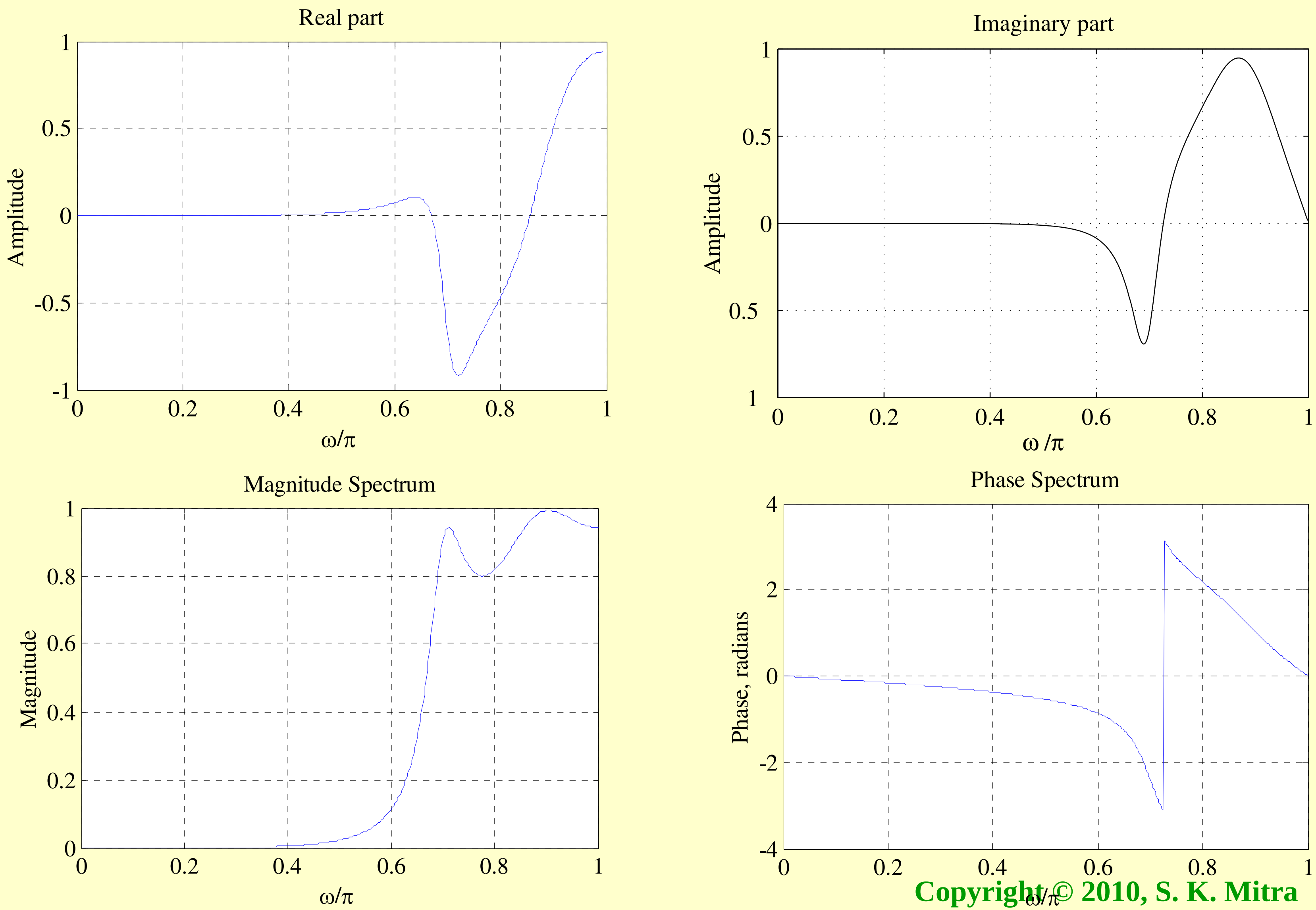
DTFT Computation Using MATLAB

- Example - Plots of the real and imaginary parts, and the magnitude and phase of the DTFT as a function of the normalized angular frequency variable ω/π

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

are shown on the next slide

DTFT Computation Using MATLAB



Linear Convolution Using DTFT

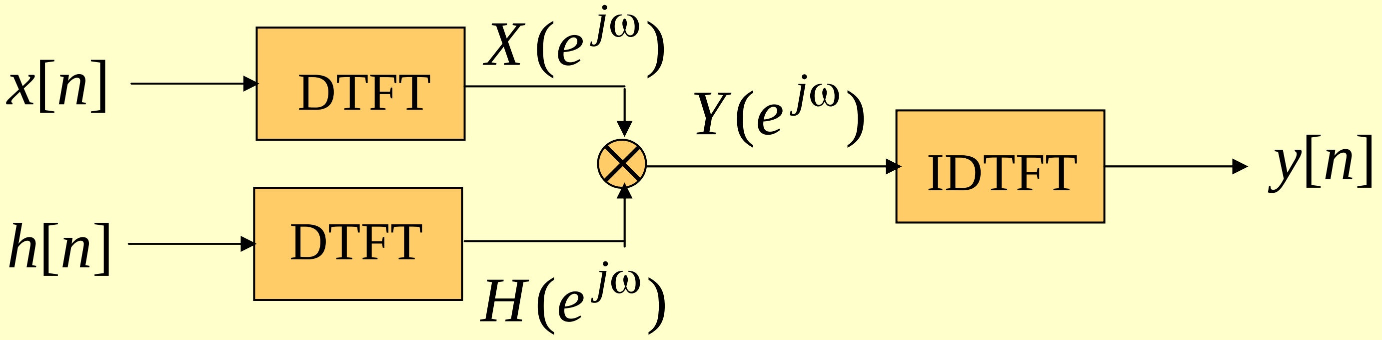
- An important property of the DTFT is given by the convolution theorem in Table 3.4
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Linear Convolution Using DTFT

- 1) Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
- 2) Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDFT $y[n]$ of $Y(e^{j\omega})$



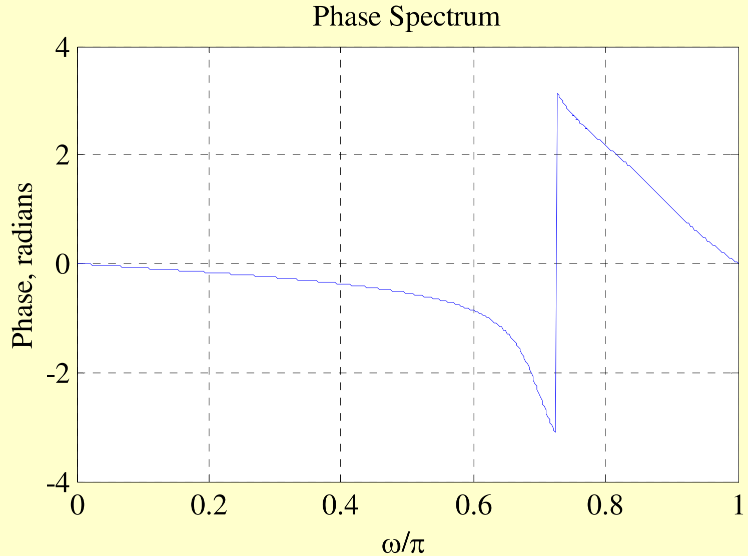
The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range
- Thus. the phase functions of some sequences exhibit discontinuities of 2π radians in the plot

The Unwrapped Phase Function

- For example, there is a discontinuity of 2π at $\omega = 0.72$ in the phase response below

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

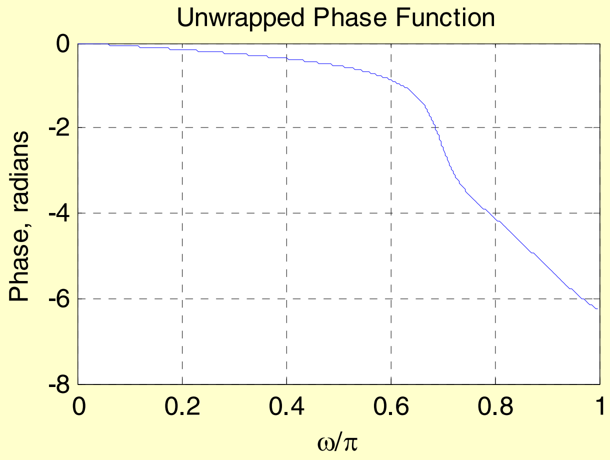


The Unwrapped Phase Function

- In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π
- Process of discontinuity removal is called unwrapping the phase
- The unwrapped phase function will be denoted as $\theta_c(\omega)$

The Unwrapped Phase Function

- In MATLAB, the unwrapping can be implemented using the M-file `unwrap`
- The unwrapped phase function of the DTFT of previous page is shown below



The Unwrapped Phase Function

- The conditions under which the phase function will be a continuous function of ω is next derived
- Now

$$\ln X(e^{j\omega}) = \left|X(e^{j\omega})\right| + j\theta(\omega)$$

where

$$\theta(\omega) = \arg\{H(e^{j\omega})\}$$

The Unwrapped Phase Function

- If $\ln X(e^{j\omega})$ exits, then its derivative with respect to ω also exists and is given by

$$\frac{d \ln X(e^{j\omega})}{d\omega} = \frac{1}{X(e^{j\omega})}\left[\frac{dX(e^{j\omega})}{d\omega}\right]$$

$$= \frac{1}{X(e^{j\omega})}\left[\frac{dX_{\text{re}}(e^{j\omega})}{d\omega} + j\frac{dX_{\text{im}}(e^{j\omega})}{d\omega}\right]$$

The Unwrapped Phase Function

- From $\ln X(e^{j\omega}) = \left| X(e^{j\omega}) \right| + j\theta(\omega)$,

$d \ln X(e^{j\omega}) / d\omega$ is also given by

$$\frac{d \ln X(e^{j\omega})}{d\omega} = \frac{d \left| X(e^{j\omega}) \right|}{d\omega} + j \frac{d\,\theta(\omega)}{d\omega}$$

The Unwrapped Phase Function

- Thus, $d\theta(\omega)/d\omega$ is given by the imaginary part of

$$\frac{1}{X(e^{j\omega})}\left[\frac{dX_{\text{re}}(e^{j\omega})}{d\omega} + j\frac{dX_{\text{im}}(e^{j\omega})}{d\omega}\right]$$

- Hence,

$$\frac{d\theta(\omega)}{d\omega} = \frac{1}{\left|X(e^{j\omega})\right|^2}\Big[X_{\text{re}}(e^{j\omega})\frac{dX_{\text{im}}(e^{j\omega})}{d\omega} - X_{\text{im}}(e^{j\omega})\frac{dX_{\text{re}}(e^{j\omega})}{d\omega}\Big]$$

The Unwrapped Phase Function

- The phase function can thus be defined unequivocally by its derivative $d\theta(\omega)/d\omega$:

$$\theta(\omega) = \int_0^{\omega} \left[\frac{d\theta(\eta)}{d\eta}\right] d\eta,$$

with the constraint

$$\theta(0) = 0$$

The Unwrapped Phase Function

- The phase function defined by

$$\theta(\omega) = \int_0^{\omega} \left[\frac{d\theta(\eta)}{d\eta}\right] d\eta$$

is called the unwrapped phase function of $X(e^{j\omega})$ and it is a continuous function of ω

- $\Rightarrow \ln X(e^{j\omega})$ exists

The Unwrapped Phase Function

- Moreover, the phase function will be an odd function of ω if

$$\frac{1}{\pi}\int_{0}^{2\pi}\left[\frac{d\theta(\eta)}{d\eta}\right]d\eta = 0$$

- If the above constraint is not satisfied, then the computed phase function will exhibit absolute jumps greater than π