DSP Final

Consider a causal system $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

1. For $h[n] = \underset{\uparrow}{1}, a, a^2, a^3, \cdots$ where $a = -0.5$

① What is $H(z)$ and its ROC?

② Where are the one pole and one zero of $H(z)$?

③ Is $h[n]$ a low-pass or high-pass system? Use $H(e^{j\omega})$ and pole-zero locations to explain.

2. ① Show that

$h[n]$ is BIBO stable $\Longleftrightarrow \sum_{n=0}^{\infty} |h[n]| < \infty$

② Show that $H(e^{j\omega})$ exists $\Longleftrightarrow \sum_{n=0}^{\infty} |h[n]| < \infty$

③ Consider $h[n] = 1, a, a^2, a^3, \cdots$

Verify that $H(e^{j\omega})$ exists $\Longleftrightarrow$ ROC of $H(z) \supset |z| = 1$

④ Consider $h[n] = 1, a, a^2, a^3, \cdots$

Show that $H(z)$ is BIBO stable $\Leftrightarrow$ pole of $H(z)$ is inside $|z| = 1$

3. For $h[n] = \underset{\uparrow}{1}, \frac{1}{2}, -\frac{1}{2}$

① What is $H(z)$ and its ROC?

② Where are the two poles and two zeros of $H(z)$?

③ Show that $h[n]$ is BIBO stable.

10% for each subproblem.

1.

① $H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$

$= \sum_{n=0}^{\infty} (-0.5)^n z^{-n}$

$= \frac{1}{1-(-0.5z^{-1})}$

$= \frac{1}{1+0.5z^{-1}}$ #

ROC : $|-0.5z^{-1}| < 1$

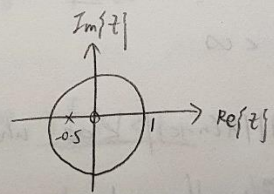
$\therefore |z| > 0.5$ #

② $H(z) = \frac{z}{z+0.5}$

$\begin{cases} \text{zero} : z = 0 \\ \text{pole} : z = -0.5 \end{cases}$ #

③ $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$

$= \frac{e^{j\hat{\omega}}}{e^{j\hat{\omega}}+0.5}$



high-pass system, ∵ pole pulls up and zero pulls down, we can see that low-freq is closer to zero which pulls down, and high-freq is closer to pole which pulls up. #

① BIBO: when input $x(n] < M_x < \infty$

then output $y(n] < M_y < \infty$

$$
\begin{aligned}
y(n] &= h(n] * x(n] \\
&= \sum_{n=0}^{\infty} h[n] x(n-k] \\
&\le \sum_{n=0}^{\infty} | h(n] x(n-k] | \\
&\le \sum_{n=0}^{\infty} | h(n] | | x(n-k] | \\
&< \sum_{n=0}^{\infty} | h(n] | M_x < \infty
\end{aligned}
$$

$\therefore \sum_{n=0}^{\infty} | h(n] | < \infty$

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if $\sum_{n=0}^{\infty} | h(n] | < \infty$

then $\sum_{n=0}^{\infty} | h(n] | | x(n-k] | < \infty$ when $x(n] < M_x < \infty$

$\because y(n] \le \sum_{n=0}^{\infty} | h(n] | | x(n-k] | < \infty$

$\therefore y(n] < M_y < \infty$

$\therefore$ $h(n]$ is BIBO stable $\Leftrightarrow \sum_{n=0}^{\infty} | h(n] | < \infty$ verified.

②

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{\infty} h[n] e^{-j\hat{\omega} n}$$

$$\le \sum_{n=0}^{\infty} |h[n]| \underbrace{|e^{-j\hat{\omega} n}|}_{\to 1}$$

$$= \sum_{n=0}^{\infty} |h[n]|$$

$H(e^{j\hat{\omega}})$ exists means it has a finite value

$\therefore \sum_{n=0}^{\infty} |h[n]| < \infty$

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if $\sum_{n=0}^{\infty} |h[n]| < \infty$

then $\sum_{n=0}^{\infty} |h[n]| |e^{-j\hat{\omega} n}| < \infty$

$\because H(e^{j\hat{\omega}}) \le \sum_{n=0}^{\infty} |h[n]| |e^{-j\hat{\omega} n}| < \infty$

$\therefore H(e^{j\hat{\omega}}) < \infty$

which means $H(e^{j\hat{\omega}})$ exists

$\therefore H(e^{j\hat{\omega}})$ exists $\iff \sum_{n=0}^{\infty} |h[n]| < \infty$ verified.

③

$H(e^{j\hat{\omega}})$ exists means $\sum_{n=0}^{\infty} |h[n]| < \infty$

now $h[n] = 1, a, a^2 \dots$

$\therefore \sum_{n=0}^{\infty} |h[n]| = \frac{1}{1-|a|} < \infty$ $\underline{\text{if } |a| < 1}$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \frac{1}{1-az^{-1}}$$

ROC: $|az^{-1}| < 1$

$\underline{\underline{|z| > |a|}}$

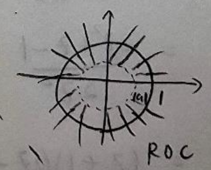
$\because$ If we want $H(e^{j\hat{\omega}})$ exists

then $\sum_{n=0}^{\infty} |h[n]| < \infty$

and $|a| < 1$

ROC of $H(z)$: $|z| > |a|$

which will contain $|z| = 1$ as following figure.



On contrary, ROC of $H(z) \supset |z| = 1$

means $|a| < 1$, which satisfies $\sum_{n=0}^{\infty} |h[n]| < \infty$

then $H(e^{j\hat{\omega}})$ exists

$\therefore H(e^{j\hat{\omega}})$ exists $\iff$ ROC of $H(z) \supset |z| = 1$ verified.

④

$H(z)$ is BIBO stable means $\sum_{n=0}^{\infty} |h[n]| < \infty$

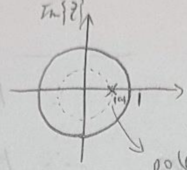
$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$ if $\sum_{n=0}^{\infty} |h[n]| < \infty$ then $|a| < 1$

$= \frac{1}{1 - az^{-1}}, |az^{-1}| < 1$ and $|az^{-1}| < 1$, $|z| > |a|$

$= \frac{z}{z-a}$

pole: $z = a$

(suppose $a > 0$)



pole of $H(z)$ is inside $|z|=1$

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if pole of $H(z)$ is inside $|z|=1$ means RoC contains $|z|=1$

so $H(e^{j\Omega})$ exists $\Leftrightarrow \sum_{n=0}^{\infty} |h[n]| < \infty$ then $H(z)$ is BIBO stable.

$\therefore$ $H(z)$ is BIBO stable $\Leftrightarrow$ pole of $H(z)$ is inside $|z|=1$

verified.

3.

30 ① $H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$

$= 1 + \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}$

$= 1 + \frac{1}{2z} - \frac{1}{2z^2}$

$= \frac{2z^2 + z - 1}{2z^2}$

$= \frac{(z+1)(2z-1)}{2z^2}$

RoC: all z plane but $z \neq 0$

②

pole: $z = 0$, $z = 0$

zero: $z = -1$, $z = \frac{1}{2}$ #

③ $h[n]$ is BIBO stable

if $\sum_{n=0}^{\infty} |h[n]| < \infty$

$\sum_{n=0}^{\infty} |h[n]| = 1 + \frac{1}{2} + \frac{1}{2} = 2 < \infty$

$\therefore$ $h[n]$ is BIBO stable.

verified.