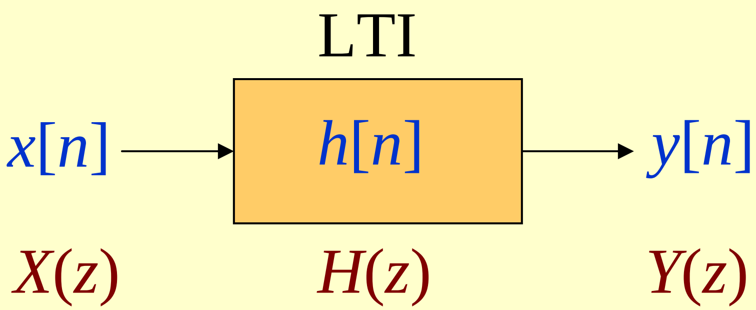
# Types of Transfer Functions




$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$Y(z) = H(z)X(z)$$

# Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function is based on the length of its impulse response $h[n]$:

  \- Finite impulse response (FIR) transfer function

  \- Infinite impulse response (IIR) transfer function

# Types of Transfer Functions

- In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications
- (1) Classification based on the shape of the magnitude function $|H(e^{j\omega})|$
- (2) Classification based on the the form of the phase function $\theta(\omega)$

# Classification Based on Magnitude Characteristics

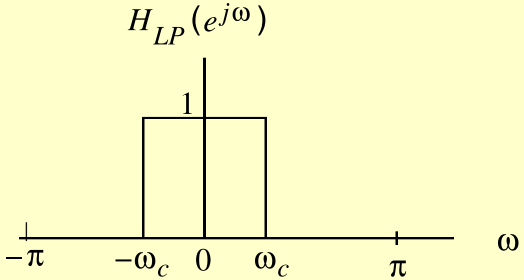
- One common classification is based on an ideal magnitude response
- A digital filter designed to pass signal components of certain frequencies without distortion should have a magnitude response equal to one at these frequencies, and should have a magnitude response equal to zero at all other frequencies

# Ideal Filters

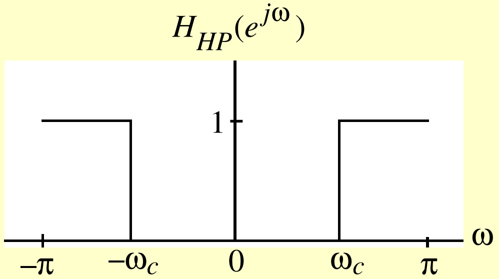
- The range of frequencies where the magnitude response takes the value of one is called the **passband**
- The range of frequencies where the magnitude response takes the value of zero is called the **stopband**

# Ideal Filters

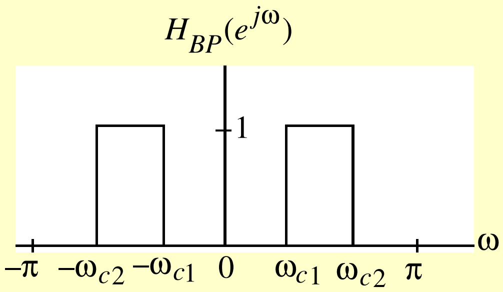
- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:



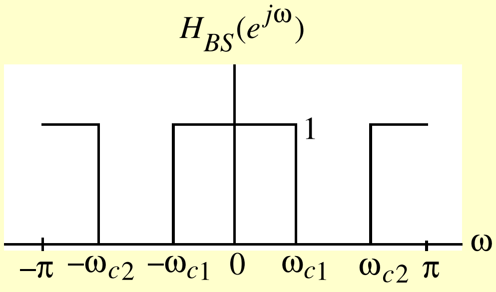

Lowpass




Highpass




Bandpass




Bandstop

# Ideal Filters

- Lowpass filter: Passband - $0 \le \omega \le \omega_c$

  Stopband - $\omega_c < \omega \le \pi$
- Highpass filter: Passband - $\omega_c \le \omega \le \pi$

  Stopband - $0 \le \omega < \omega_c$
- Bandpass filter: Passband - $\omega_{c1} \le \omega \le \omega_{c2}$

  Stopband - $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi$
- Bandstop filter: Stopband - $\omega_{c1} < \omega < \omega_{c2}$

  Passband - $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$

# Ideal Filters

- The frequencies $\omega_c$ , $\omega_{c1}$ , and $\omega_{c2}$ are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere

# Ideal Filters

- Earlier in the course we derived the inverse DTFT of the frequency response $H_{LP}(e^{j\omega})$ of the ideal lowpass filter:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable

# Ideal Filters

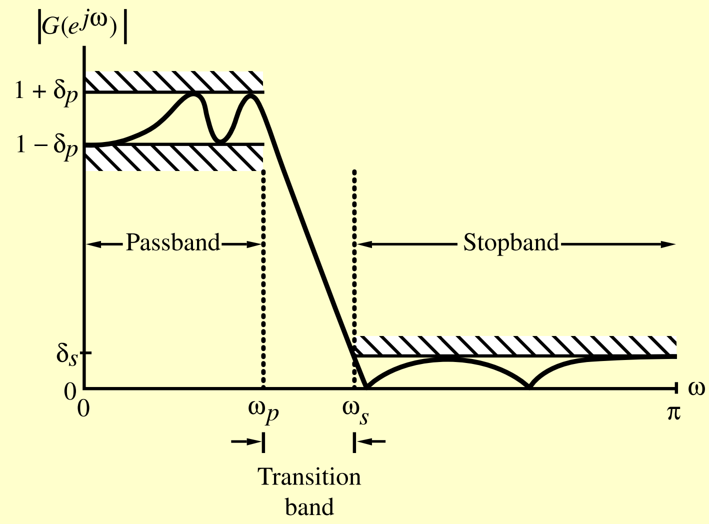
- Also, $h_{LP}[n]$ is not causal and is of doubly infinite length
- The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable
- Thus, the ideal filters with the ideal "brick wall" frequency responses cannot be realized with finite dimensional LTI filter

# Ideal Filters

- To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a **transition band** between the passband and the stopband
- This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband

# Ideal Filters

- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband
- Typical magnitude response specifications of a lowpass filter are shown below



# Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a **bounded real** (BR) **transfer function** if

$$|H(e^{j\omega})| \leq 1 \quad \text{for all values of } \omega$$

- Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs

# Bounded Real Transfer Functions

- Then the condition $|H(e^{j\omega})| \le 1$ implies that

$$\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2$$

- Integrating the above from $-\pi$ to $\pi$, and applying Parseval's theorem we get

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \le \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

# Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a **passive structure**
- If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a **lossless system**

# Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})|=1$ is thus called a **lossless bounded real (LBR) transfer function**
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity

# Bounded Real Transfer Functions

- Example – Consider the causal stable IIR transfer function

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

where $K$ and $\alpha$ are real constants

- Its square-magnitude function is given by

$$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

# Bounded Real Transfer Functions

- The maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha\cos\omega$ in the denominator is a maximum and the minimum value is obtained when $2\alpha\cos\omega$ is a minimum
- For $\alpha > 0$, maximum value of $2\alpha\cos\omega$ is equal to $2\alpha$ at $\omega = 0$, and minimum value is $-2\alpha$ at $\omega = \pi$

# Bounded Real Transfer Functions

- Thus, for $\alpha > 0$, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$
- On the other hand, for $\alpha < 0$, the maximum value of $2\alpha\cos\omega$ is equal to $-2\alpha$ at $\omega = \pi$ and the minimum value is equal to $2\alpha$ at $\omega = 0$

# Bounded Real Transfer Functions

- Here, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = \pi$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = 0$
- Hence, the maximum value can be made equal to 1 by choosing $K = \pm(1-\alpha)$, in which case the minimum value becomes $(1-\alpha)^2/(1+\alpha)^2$

# Bounded Real Transfer Functions

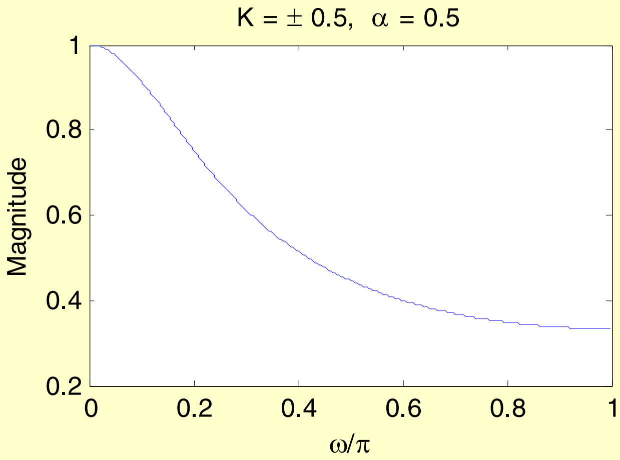
- Hence,

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

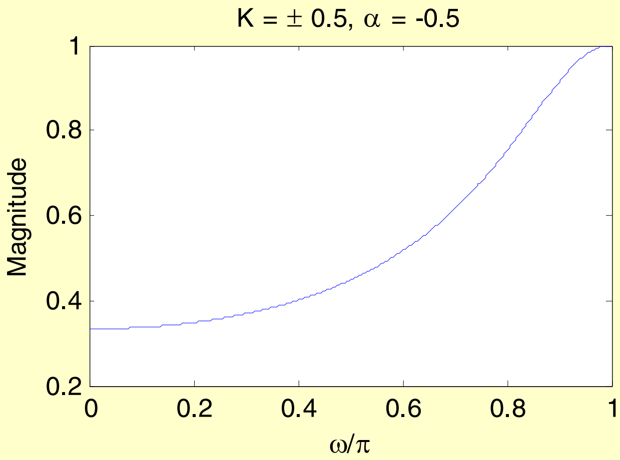
is a BR function for $K = \pm(1 - \alpha)$

- Plots of the magnitude function for $\alpha = \pm 0.5$ with values of $K$ chosen to make $H(z)$ a BR function are shown on the next slide

# Bounded Real Transfer Functions




Lowpass Filter




Highpass Filter

# Allpass Transfer Function

**Definition**

- An IIR transfer function $\mathcal{A}(z)$ with unity magnitude response for all frequencies, i.e.,

$$|\mathcal{A}(e^{j\omega})|^2 = 1, \quad \text{for all } \omega$$

  is called an **allpass transfer function**

- An $M$-th order causal real-coefficient allpass transfer function is of the form

$$\mathcal{A}_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

# Allpass Transfer Function

- If we denote the denominator polynomial of $\mathcal{A}_M(z)$ as $D_M(z)$:

$$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

then it follows that $\mathcal{A}_M(z)$ can be written as:

$$\mathcal{A}_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r} e^{-j\phi}$

# Allpass Transfer Function

- The numerator of a real-coefficient allpass transfer function is said to be the **mirror-image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-$M$ polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z^{-1})$$

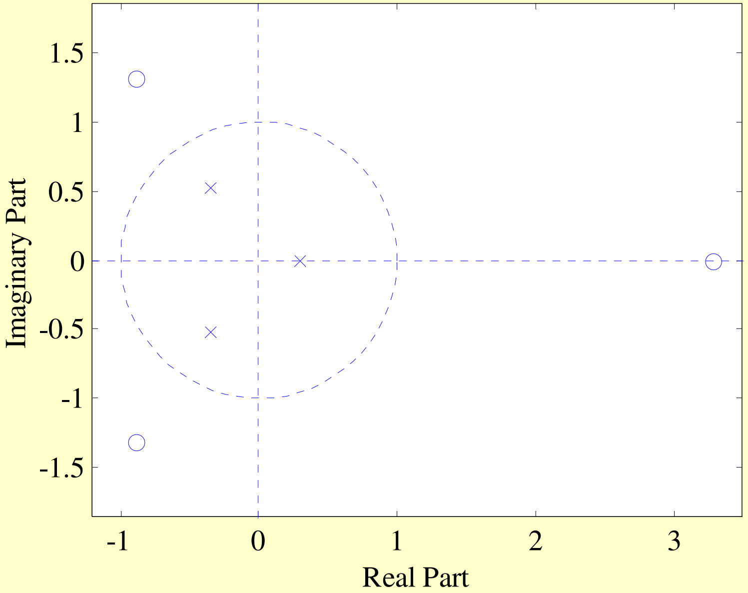
# Allpass Transfer Function

- The expression

$$\mathcal{A}_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** in the $z$-plane

$$\mathcal{A}_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



# Allpass Transfer Function

- To show that $|\mathcal{A}_M(e^{j\omega})| = 1$ we observe that

$$\mathcal{A}_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$\mathcal{A}_M(z)\mathcal{A}_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Hence

$$|\mathcal{A}_M(e^{j\omega})|^2 = \mathcal{A}_M(z)\mathcal{A}_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

# Allpass Transfer Function

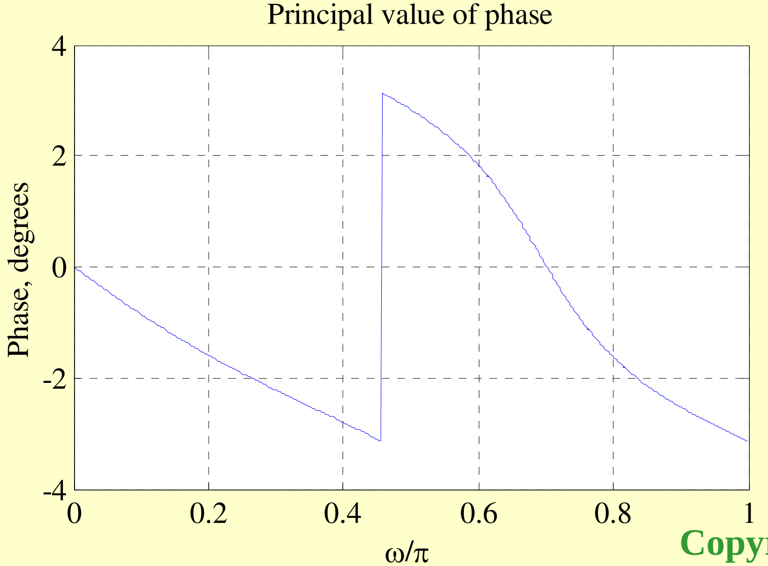
- Now, the poles of a causal stable transfer function must lie inside the unit circle in the $z$-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle

# Allpass Transfer Function

- Figure below shows the principal value of the phase of the 3rd-order allpass function

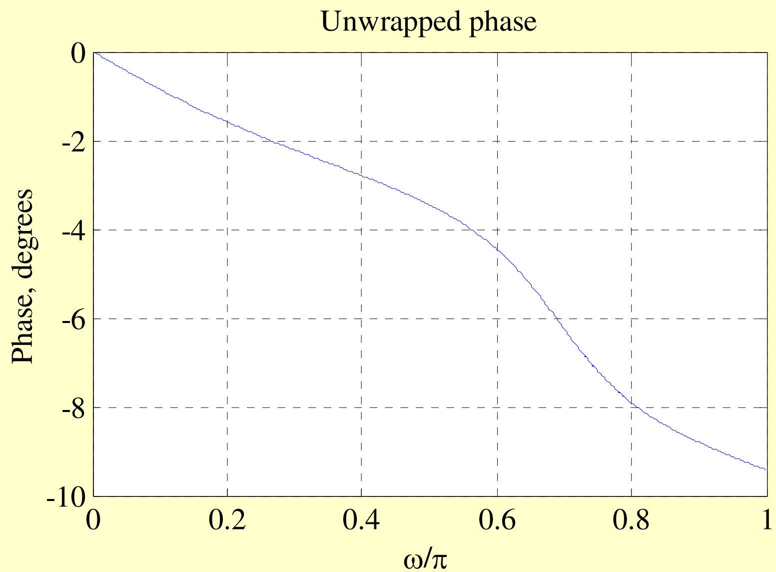
$$\mathcal{A}_3(z)=\frac{-0.2+0.18z^{-1}+0.4z^{-2}+z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

- Note the discontinuity by the amount of $2\pi$ in the phase $\theta(\omega)$



# Allpass Transfer Function

- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below
- Note: The unwrapped phase function is a continuous function of $\omega$



# Allpass Transfer Function

- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of $\omega$

**Properties**

- (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure

# Allpass Transfer Function

- (2) The magnitude function of a stable allpass function $\mathcal{A}(z)$ satisfies:

$$\left|\mathcal{A}(z)\right| \begin{cases} <1, & \text{for } \left|z\right| > 1 \\ =1, & \text{for } \left|z\right| = 1 \\ >1, & \text{for } \left|z\right| < 1 \end{cases}$$

- (3) Let $\tau(\omega)$ denote the group delay function of an allpass filter $\mathcal{A}(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

# Allpass Transfer Function

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of $\omega$ so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an $M$-th order stable real-coefficient allpass transfer function satisfies:

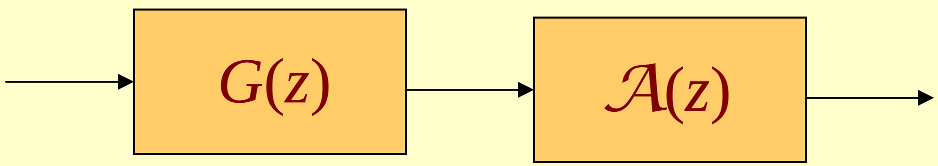
$$\int_0^\pi \tau(\omega)d\omega = M\pi$$

# Allpass Transfer Function

## A Simple Application

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

# Allpass Transfer Function



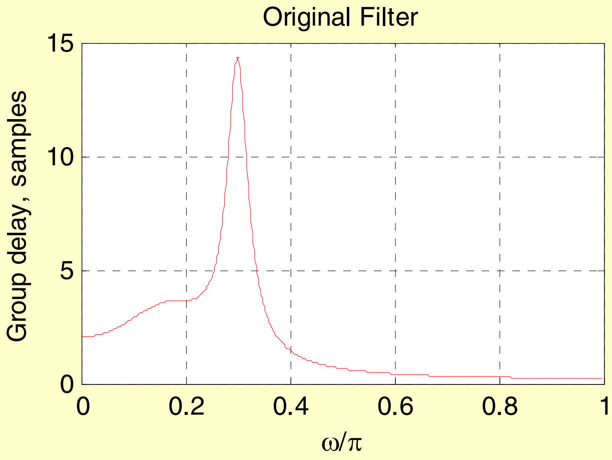

- Since $|\mathcal{A}(e^{j\omega})| = 1$, we have

$$|G(e^{j\omega})\mathcal{A}(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $\mathcal{A}(z)$

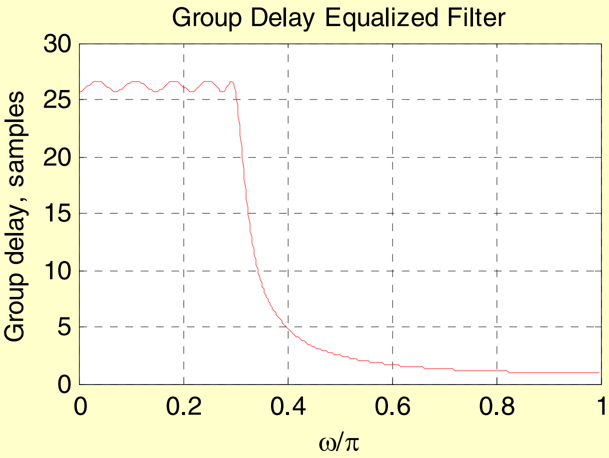
# Allpass Transfer Function

- Example – Figure below shows the group delay of a $4^{th}$ order elliptic filter with the following specifications: $\omega_p = 0.3\pi$, $\delta_p = 1$ dB, $\delta_s = 35$ dB



# Allpass Transfer Function

- Figure below shows the group delay of the original elliptic filter cascaded with an 8th order allpass section designed to equalize the group delay in the passband



# Classification Based on Phase Characteristics

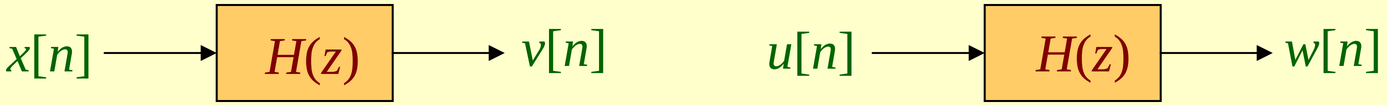
- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

# Zero-Phase Transfer Function

- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is not possible to design a causal digital filter with a zero phase

# Zero-Phase Transfer Function

- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement
- One zero-phase filtering scheme is sketched below




$$u[n] = v[-n], \qquad y[n] = w[-n]$$

# Zero-Phase Transfer Function

- It is easy to verify the above scheme in the frequency domain
- Let $X(e^{j\omega})$, $V(e^{j\omega})$ , $U(e^{j\omega})$ , $W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively
- From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide

# Zero-Phase Transfer Function

$x[n] \longrightarrow \boxed{H(z)} \longrightarrow v[n]$ $\qquad$ $u[n] \longrightarrow \boxed{H(z)} \longrightarrow w[n]$

$$u[n] = v[-n], \qquad y[n] = w[-n]$$

$$V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(e^{j\omega}), \qquad Y(e^{j\omega}) = W^*(e^{j\omega})$$

- Combining the above equations we get

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega})$$

$$= H^*(e^{j\omega})V(e^{j\omega}) = H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega})$$

$$= \left|H(e^{j\omega})\right|^2 X(e^{j\omega})$$

# Zero-Phase Transfer Function

- The function `filtfilt` implements the above zero-phase filtering scheme
- In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a **linear-phase** characteristic in the frequency band of interest

# Linear-Phase Transfer Function

- The most general type of a filter with a linear phase has a frequency response given by

$$H(e^{j\omega}) = e^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$

- Note also $$|H(e^{j\omega})| = 1$$

$$\tau(\omega) = D$$

# Linear-Phase Transfer Function

- The output $y[n]$ of this filter to an input $x[n] = Ae^{j\omega n}$ is then given by

$$y[n] = Ae^{-j\omega D}e^{j\omega n} = Ae^{j\omega(n-D)}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount $D$

# Linear-Phase Transfer Function

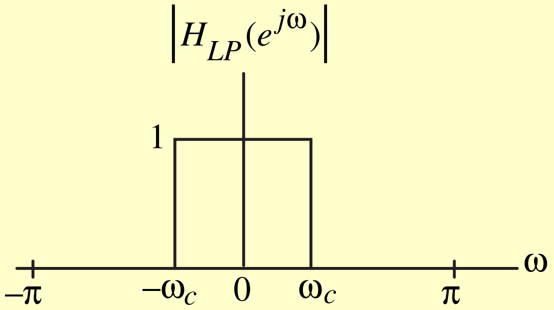
- If $D$ is an integer, then $y[n]$ is identical to $x[n]$, but delayed by $D$ samples
- If $D$ is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed $D$ units of time

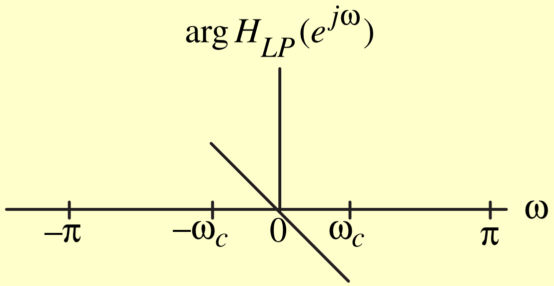
# Linear-Phase Transfer Function

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest

# Linear-Phase Transfer Function

- Figure below shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband



# Linear-Phase Transfer Function

- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape
- Example - Determine the impulse response of an ideal lowpass filter with a linear phase response:

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_o}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

# Linear-Phase Transfer Function

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_o)}{\pi (n - n_o)}, \quad -\infty < n < \infty$$

- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable

# Linear-Phase Transfer Function

- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed
- The truncated approximation may or may not exhibit linear phase, depending on the value of $n_o$ chosen

# Linear-Phase Transfer Function

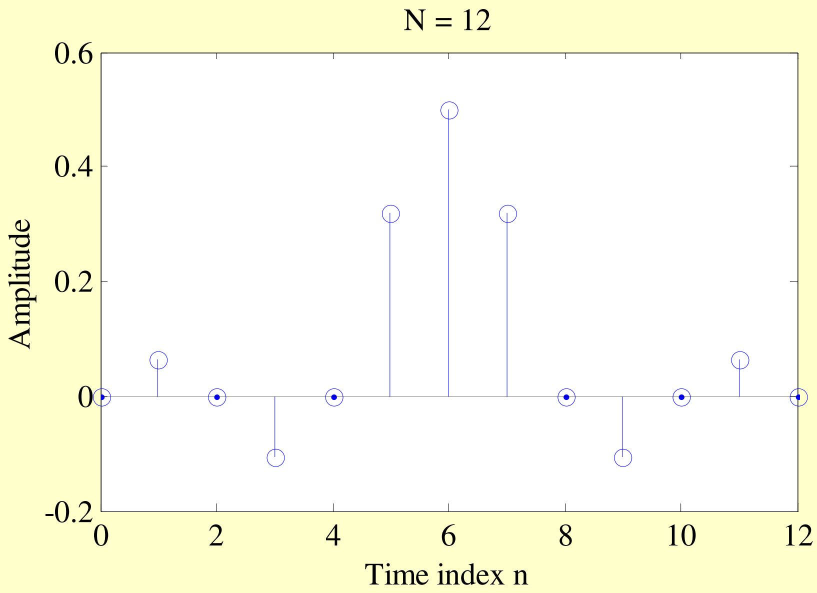
- If we choose $n_o = N/2$ with $N$ a positive integer, the truncated and shifted approximation

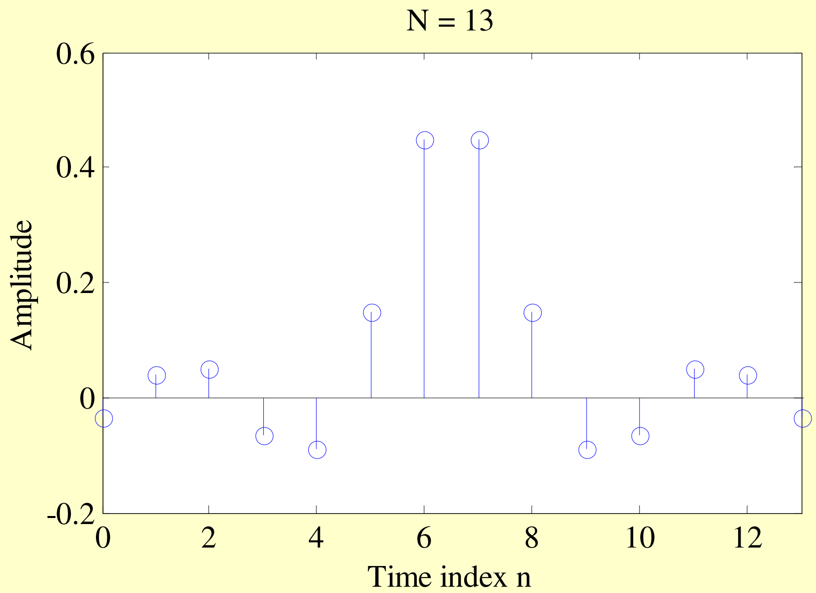
$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi (n - N/2)}, \quad 0 \le n \le N$$

will be a length $N+1$ causal linear-phase FIR filter

# Linear-Phase Transfer Function

- Figure below shows the filter coefficients obtained using the function `sinc` for two different values of $N$



# Zero-Phase Response

- Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} \hat{h}_{LP}[n] e^{-j\omega n} = e^{-j\omega N/2} \tilde{H}_{LP}(\omega)$$

where $\tilde{H}_{LP}(\omega)$, called the **zero-phase response** or **amplitude response**, is a real function of $\omega$

# Minimum-Phase and Maximum-Phase Transfer Functions

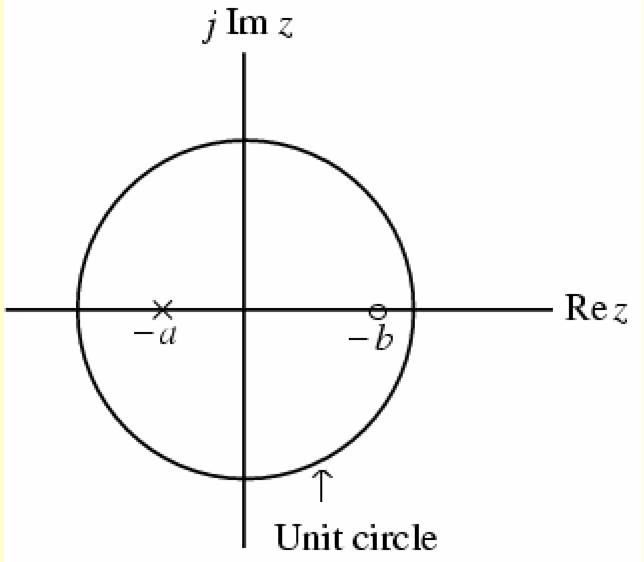
- Consider the two 1st-order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

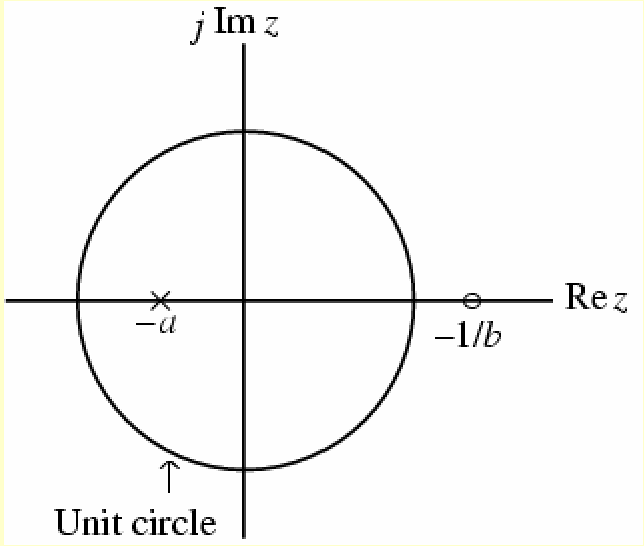
- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$ , whereas, the zero of $H_2(z)$ is at $z = -\frac{1}{b}$ situated in a mirror-image symmetry

# Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions




$H_1(z)$




$H_2(z)$

# Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude function as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

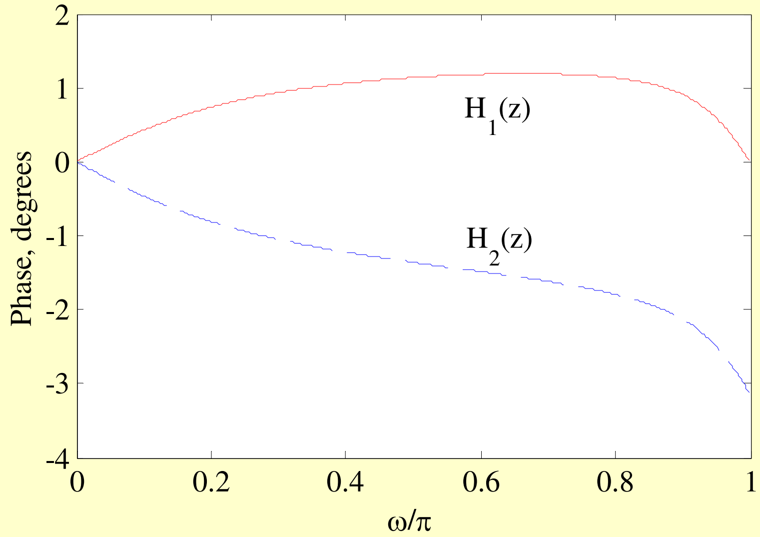
- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

# Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for $a = 0.8$ and $b = -0.5$



# Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- The excess phase lag property of $H_2(z)$ with respect to $H_1(z)$ can also be explained by observing that we can write

$$H_2(z) = \frac{bz+1}{z+a} = \underbrace{\left(\frac{z+b}{z+a}\right)}_{H_1(z)}\underbrace{\left(\frac{bz+1}{z+b}\right)}_{A(z)}$$

# Minimum-Phase and Maximum-Phase Transfer Functions

where $\mathcal{A}(z) = (bz+1)/(z+b)$ is a stable allpass function

- The phase functions of $H_1(z)$ and $H_2(z)$ are thus related through

$$arg[H_2(e^{j\omega})] = arg[H_1(e^{j\omega})] + arg[\mathcal{A}(e^{j\omega})]$$

- As the unwrapped phase function of a stable first-order allpass function is a negative function of ω, it follows from the above that $H_2(z)$ has indeed an excess phase lag with respect to $H_1(z)$

# Minimum-Phase and Maximum-Phase Transfer Functions

- Generalizing the above result, let $H_m(z)$ be a causal stable transfer function with all zeros inside the unit circle and let $H(z)$ be another causal stable transfer function satisfying $\left|H(e^{j\omega})\right| = \left|H_m(e^{j\omega})\right|$
- These two transfer functions are then related through $H(z) = H_m(z)\mathcal{A}(z)$ where $\mathcal{A}(z)$ is a causal stable allpass function

# Minimum-Phase and Maximum-Phase Transfer Functions

- The unwrapped phase functions of $H_m(z)$ and $H(z)$ are thus related through

$$arg[H(e^{j\omega})] = arg[H_m(e^{j\omega})] + arg[\mathcal{A}(e^{j\omega})]$$

- $H(z)$ has an excess phase lag with respect to $H_m(z)$
- A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function**

# Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros outside the unit circle is called a **maximum-phase transfer function**
- A causal stable transfer function with zeros inside and outside the unit circle is called a **mixed-phase transfer function**

# Minimum-Phase and Maximum-Phase Transfer Functions

- Example – Consider the mixed-phase transfer function

$$H(z) = \frac{2(1+0.3z^{-1})(0.4-z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}$$

- We can rewrite $H(z)$ as

$$H(z) = \underbrace{\left[\frac{2(1+0.3z^{-1})(1-0.4z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}\right]}_{\text{Minimum-phase function}}\underbrace{\left(\frac{0.4-z^{-1}}{1-0.4z^{-1}}\right)}_{\text{Allpass function}}$$