Basic IIR Digital Filter Structures

- The causal IIR digital filters we are concerned with in this course are characterized by a real rational transfer function of z^{-1} or, equivalently by a constant real coefficient difference equation
- From the difference equation representation, it can be seen that the realization of the causal IIR digital filters requires some form of feedback

Basic IIR Digital Filter Structures

- An N-th order IIR digital transfer function is characterized by $2N+1$ unique coefficients, and in general, requires $2N+1$ multipliers and $2N$ two-input adders for implementation
- **Direct form IIR filters**: Filter structures in which the multiplier coefficients are precisely the coefficients of the transfer function

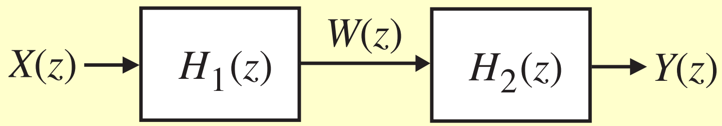
Direct Form IIR Digital Filter Structures

- Consider for simplicity a 3rd-order IIR filter with a transfer function

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

- We can implement $H(z)$ as a cascade of two filter sections as shown on the next slide

Direct Form IIR Digital Filter Structures



where

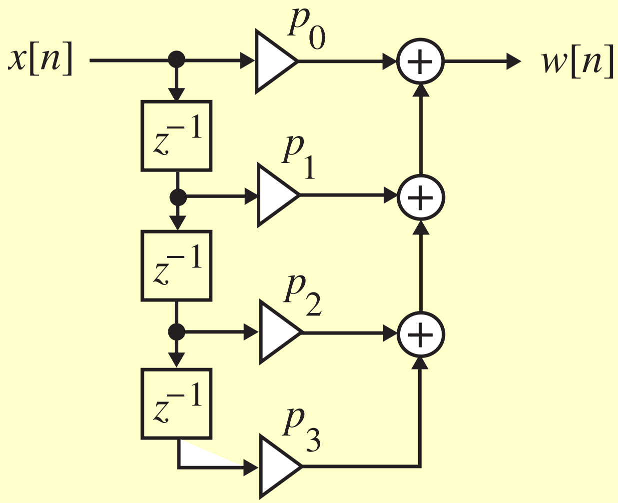
$$H_1(z) = \frac{W(z)}{X(z)} = P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}$$

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{D(z)} = \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

Direct Form IIR Digital Filter Structures

- The filter section $H_1(z)$ can be seen to be an FIR filter and can be realized as shown below

$$w[n] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2] + p_3 x[n-3]$$

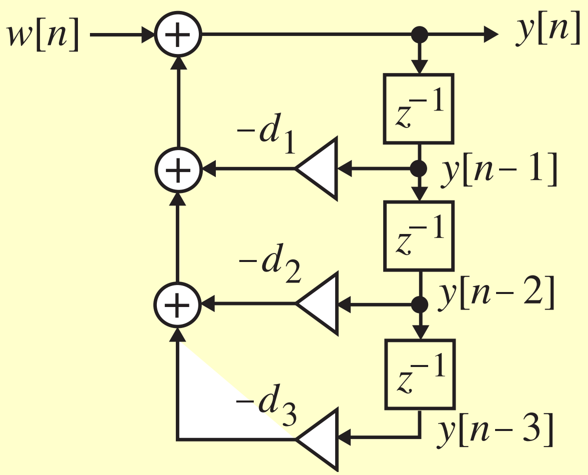


Direct Form IIR Digital Filter Structures

- The time-domain representation of $H_2(z)$ is given by

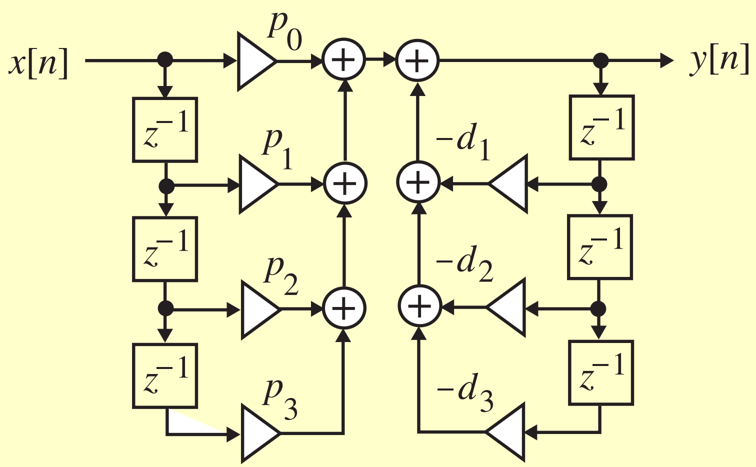
$$y[n] = w[n] - d_1 y[n-1] - d_2 y[n-2] - d_3 y[n-3]$$

Realization of $H_2(z)$ follows from the above equation and is shown on the right



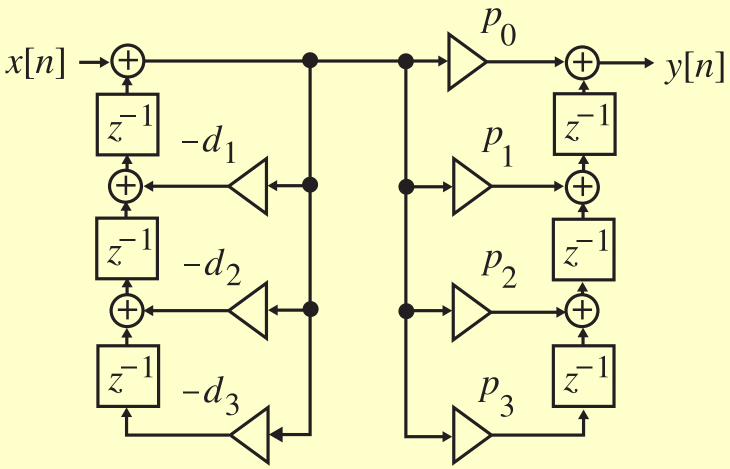
Direct Form IIR Digital Filter Structures

- A cascade of the two structures realizing $H_1(z)$ and $H_2(z)$ leads to the realization of $H(z)$ shown below and is known as the **direct form I** structure



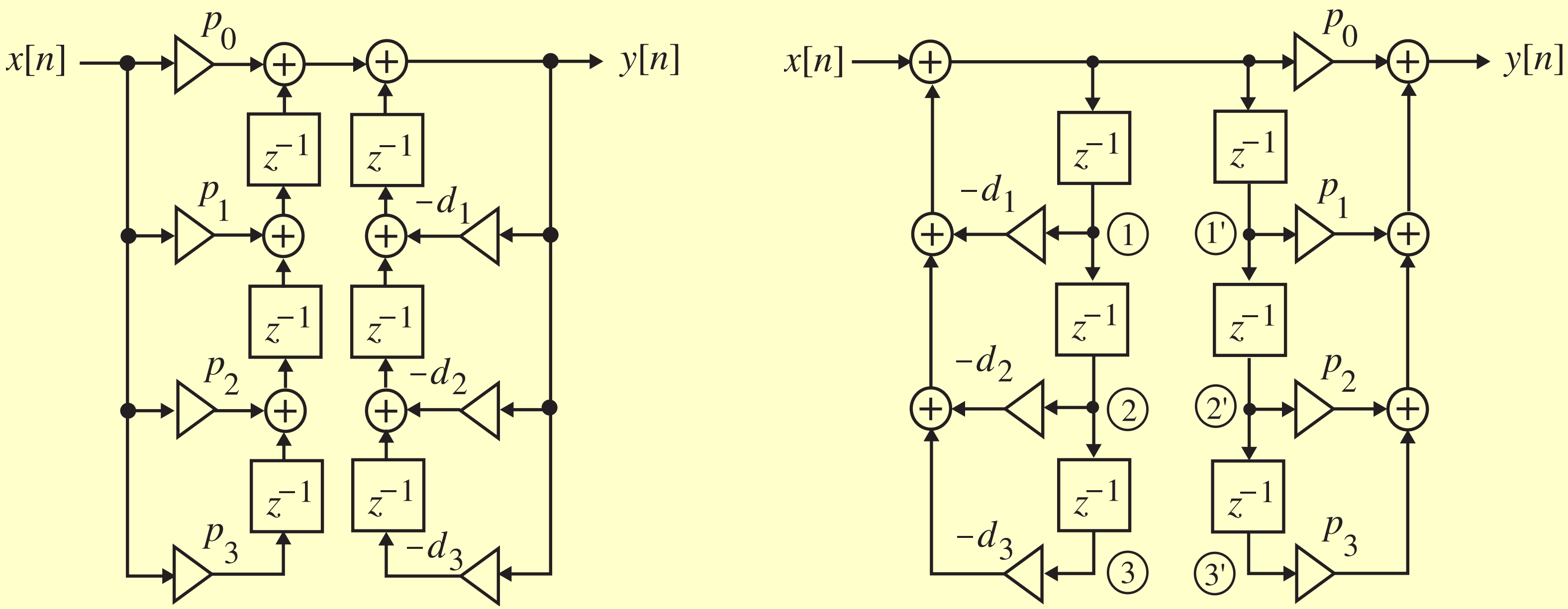
Direct Form IIR Digital Filter Structures

- Note: The direct form I structure is noncanonic as it employs 6 delays to realize a 3rd-order transfer function
- A transpose of the direct form I structure is shown on the right and is called the **direct form I$_t$ structure**



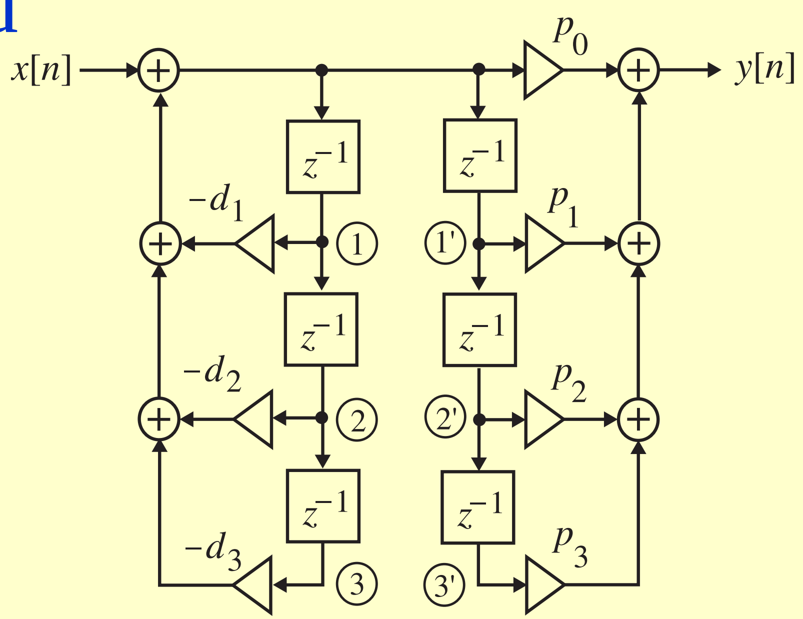
Direct Form IIR Digital Filter Structures

- Various other noncanonic direct form structures can be derived by simple block diagram manipulations as shown below



Direct Form IIR Digital Filter Structures

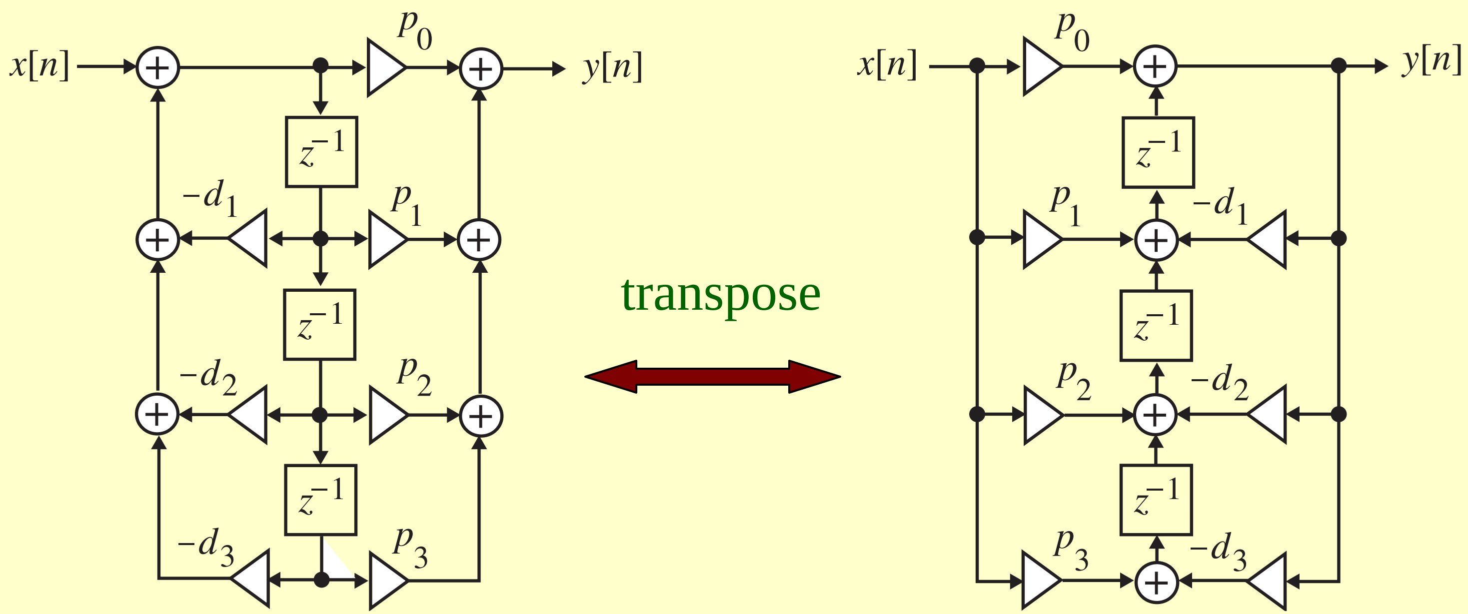
- Observe in the direct form structure shown below, the signal variable at nodes 1 and 1' are the same, and hence the two top delays can be shared



Direct Form IIR Digital Filter Structures

- Likewise, the signal variables at nodes 2 and 2' are the same, permitting the sharing of the middle two delays
- Following the same argument, the bottom two delays can be shared
- Sharing of all delays reduces the total number of delays to 3 resulting in a canonic realization shown on the next slide along with its transpose structure

Direct Form IIR Digital Filter Structures



Direct Form II

Direct Form II$_t$

- Direct form realizations of an *N*-th order IIR transfer function should be evident

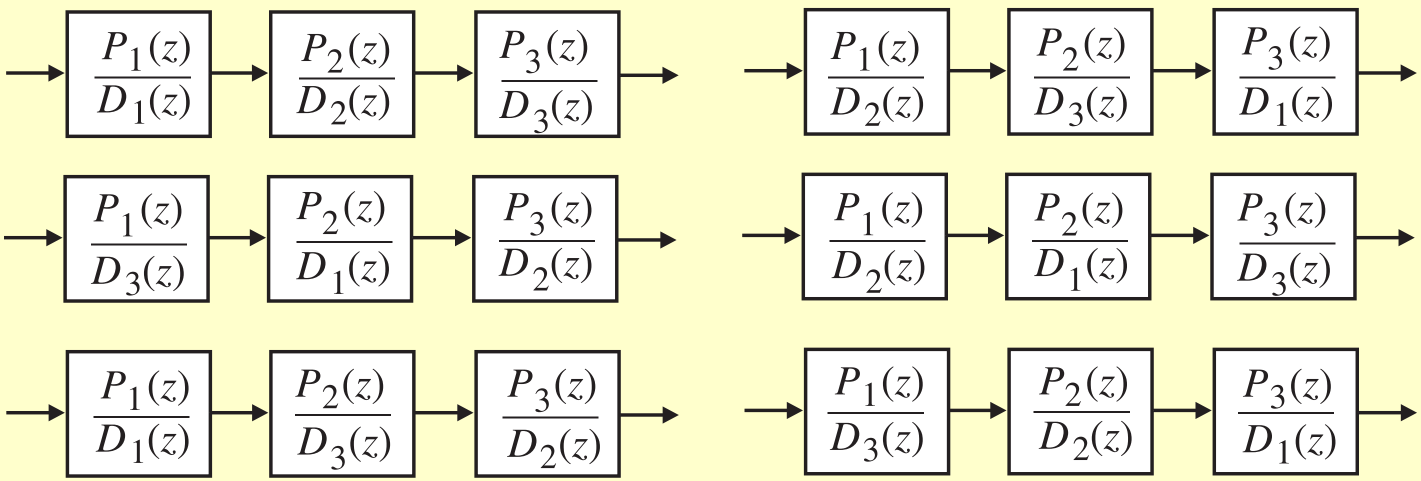
Cascade Form IIR Digital Filter Structures

- By expressing the numerator and the denominator polynomials of the transfer function as a product of polynomials of lower degree, a digital filter can be realized as a cascade of low-order filter sections
- Consider, for example, $H(z) = P(z)/D(z)$ expressed as

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$$

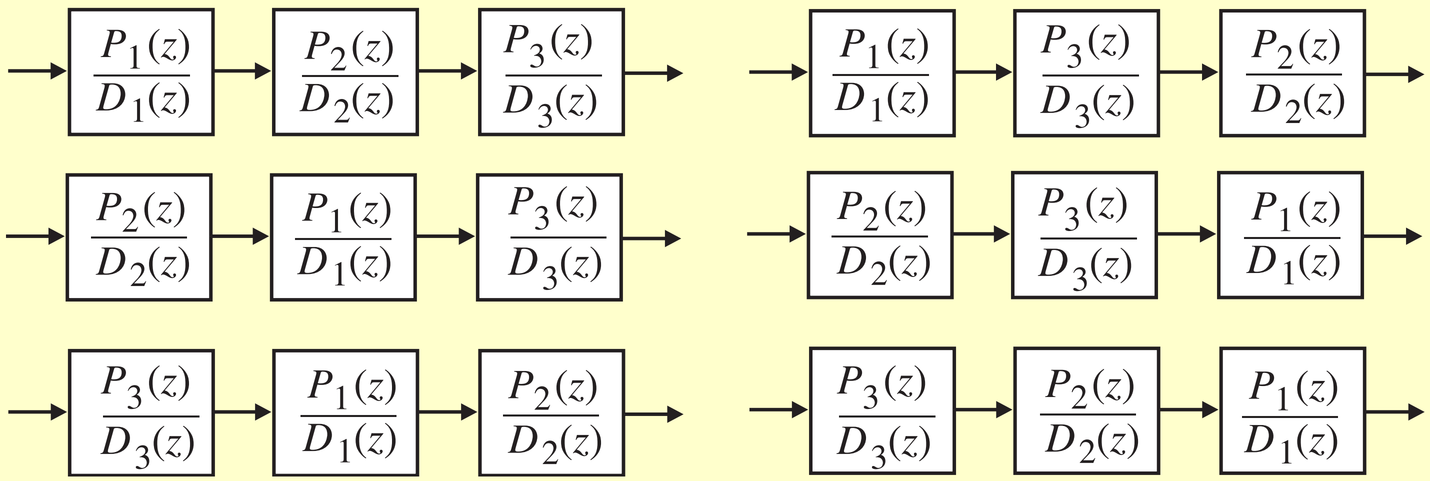
Cascade Form IIR Digital Filter Structures

- Examples of cascade realizations obtained by different pole-zero pairings are shown below



Cascade Form IIR Digital Filter Structures

- Examples of cascade realizations obtained by different ordering of sections are shown below



Cascade Form IIR Digital Filter Structures

- There are altogether a total of 36 different cascade realizations of
$$H(z) = \frac{P_1(z)P_2(z)P_2(z)}{D_1(z)D_2(z)D_3(z)}$$
based on pole-zero-pairings and ordering
- Due to finite wordlength effects, each such cascade realization behaves differently from others

Cascade Form IIR Digital Filter Structures

- Usually, the polynomials are factored into a product of 1st-order and 2nd-order polynomials:

$$H(z) = p_0 \prod_k \left(\frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

- In the above, for a first-order factor

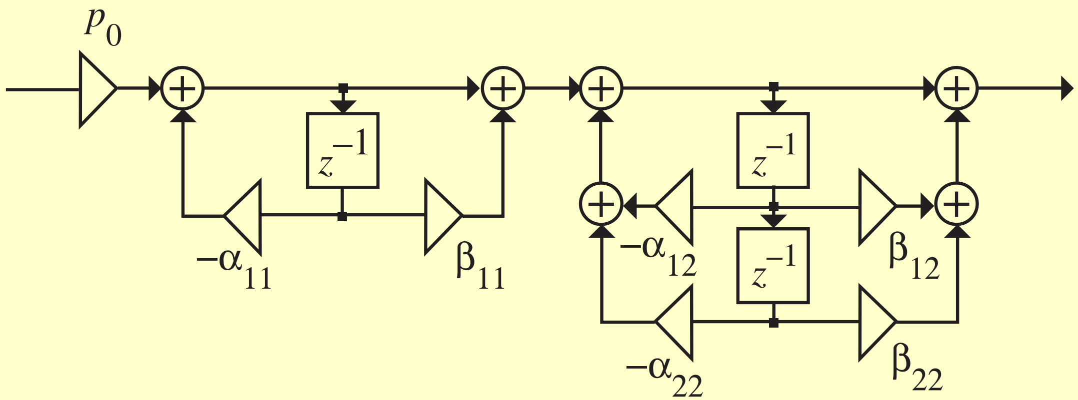
$$\alpha_{2k} = \beta_{2k} = 0$$

Cascade Form IIR Digital Filter Structures

- Consider the 3rd-order transfer function

$$H(z) = p_0 \left(\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}} \right) \left(\frac{1+\beta_{12} z^{-1} + \beta_{22} z^{-2}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}} \right)$$

- One possible realization is shown below



Cascade Form IIR Digital Filter Structures

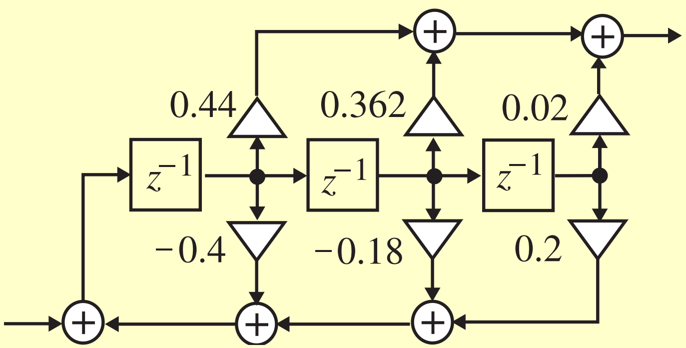
- Example - Direct form II and cascade form realizations of

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

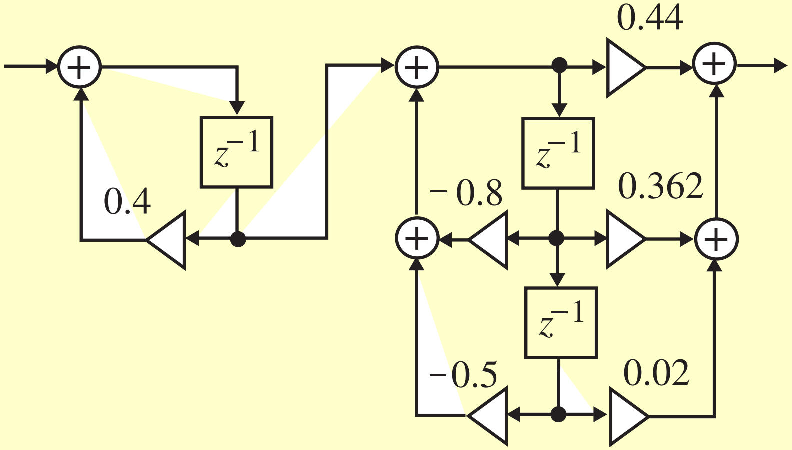
$$= \left(\frac{0.44 + 0.362z^{-1} + 0.02z^{-2}}{1 + 0.8z^{-1} + 0.5z^{-2}} \right) \left(\frac{z^{-1}}{1 - 0.4z^{-1}} \right)$$

are shown on the next slide

Cascade Form IIR Digital Filter Structures



Direct form II



Cascade form

Parallel Form IIR Digital Filter Structures

- A partial-fraction expansion of the transfer function in z^{-1} leads to the **parallel form I** structure
- Assuming simple poles, the transfer function $H(z)$ can be expressed as

$$H(z) = \gamma_0 + \sum_k \left(\frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

- In the above for a real pole $\alpha_{2k} = \gamma_{1k} = 0$

Parallel Form IIR Digital Filter Structures

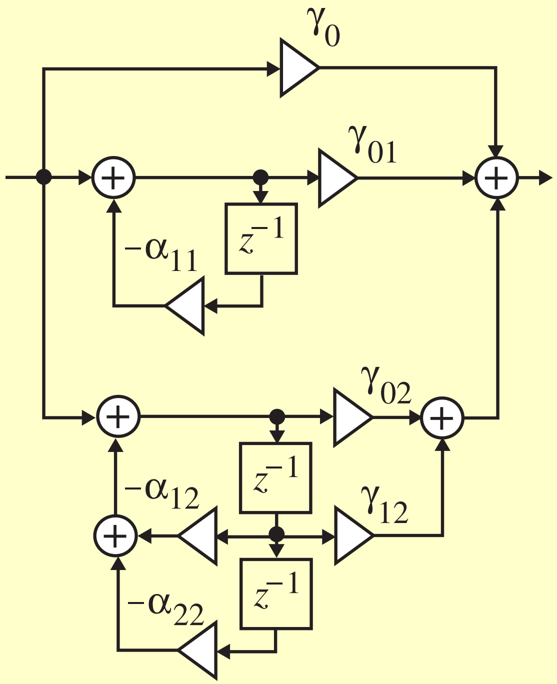
- A direct partial-fraction expansion of the transfer function in z leads to the **parallel form II** structure
- Assuming simple poles, the transfer function $H(z)$ can be expressed as

$$H(z) = \delta_0 + \sum_k \left(\frac{\delta_{1k} z^{-1} + \delta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

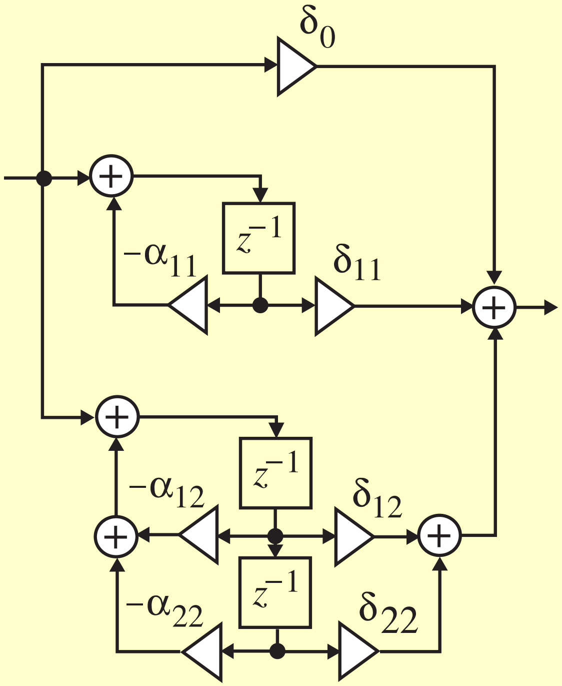
- In the above for a real pole $\alpha_{2k} = \delta_{2k} = 0$

Parallel Form IIR Digital Filter Structures

- The two basic parallel realizations of a 3rd-order IIR transfer function are shown below



Parallel form I



Parallel form II

Parallel Form IIR Digital Filter Structures

- Example - A partial-fraction expansion of

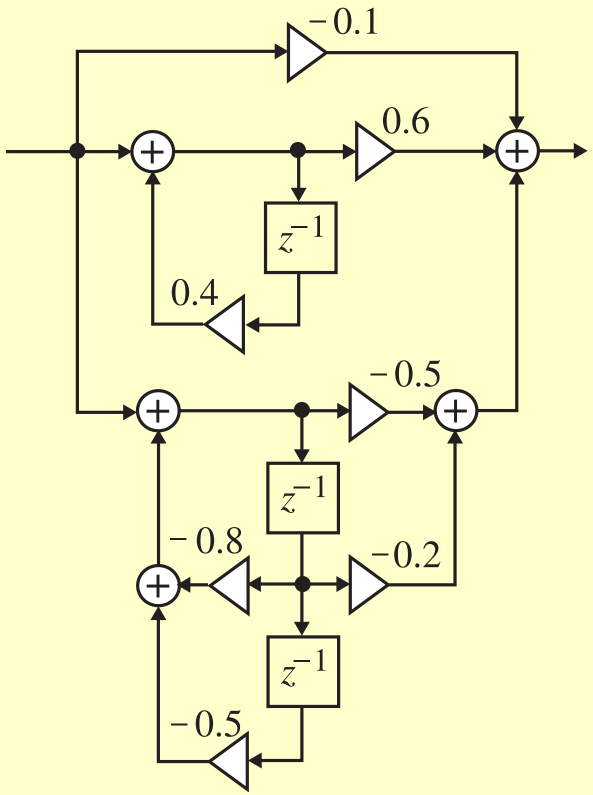
$$H(z)=\frac{0.44z^{-1}+0.362z^{-2}+0.02z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

in z^{-1} yields

$$H(z)=-0.1+\frac{0.6}{1-0.4z^{-1}}+\frac{-0.5-0.2z^{-1}}{1+0.8z^{-1}+0.5z^{-2}}$$

Parallel Form IIR Digital Filter Structures

- The corresponding parallel form I realization is shown below

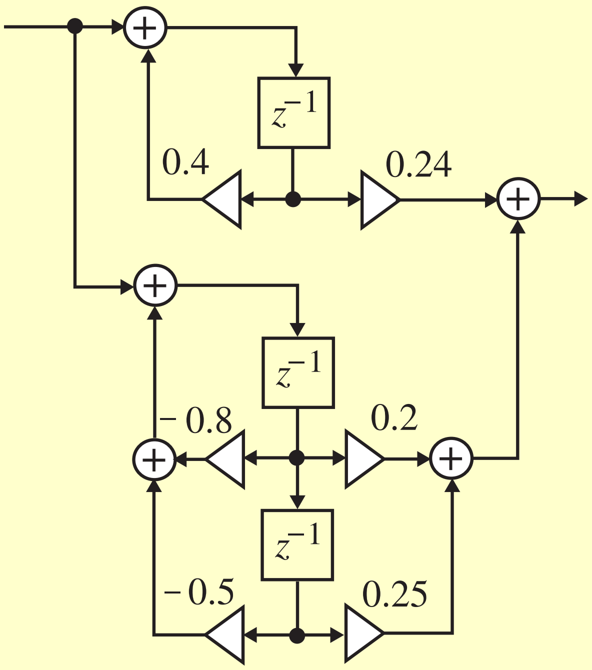


Parallel Form IIR Digital Filter Structures

- Likewise, a partial-fraction expansion of $H(z)$ in z yields

$$H(z) = \frac{0.24z^{-1}}{1-0.4z^{-1}} + \frac{0.2z^{-1}+0.25z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}$$

- The corresponding parallel form II realization is shown on the right



Realization Using MATLAB

- The cascade form requires the factorization of the transfer function which can be developed using the M-file `zp2sos`
- The statement `sos = zp2sos(z,p,k)` generates a matrix `sos` containing the coefficients of each 2nd-order section of the equivalent transfer function $H(z)$ determined from its pole-zero form

Realization Using MATLAB

- `sos` is an $L \times 6$ matrix of the form

$$\texttt{sos} = \begin{bmatrix} \texttt{p}_{\texttt{01}} & \texttt{p}_{\texttt{11}} & \texttt{p}_{\texttt{21}} & \texttt{d}_{\texttt{01}} & \texttt{d}_{\texttt{11}} & \texttt{d}_{\texttt{21}} \\ \texttt{p}_{\texttt{02}} & \texttt{p}_{\texttt{12}} & \texttt{p}_{\texttt{22}} & \texttt{d}_{\texttt{02}} & \texttt{d}_{\texttt{12}} & \texttt{d}_{\texttt{22}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \texttt{p}_{\texttt{0L}} & \texttt{p}_{\texttt{1L}} & \texttt{p}_{\texttt{2L}} & \texttt{d}_{\texttt{0L}} & \texttt{d}_{\texttt{1L}} & \texttt{d}_{\texttt{2L}} \end{bmatrix}$$

whose *i*-th row contains the coefficients $\{p_{i\ell}\}$ and $\{d_{i\ell}\}$, of the the numerator and denominator polynomials of the *i*-th 2nd-order section

Realization Using MATLAB

- L denotes the number of sections
- The form of the overall transfer function is given by

$$H(z) = \prod_{i=1}^{L} H_i(z) = \prod_{i=1}^{L} \frac{p_{0i} + p_{1i} z^{-1} + p_{2i} z^{-2}}{d_{0i} + d_{1i} z^{-1} + d_{2i} z^{-2}}$$

- Program 6_1 can be used to factorize an FIR and an IIR transfer function

Realization Using MATLAB

- Note: An FIR transfer function can be treated as an IIR transfer function with a constant numerator of unity and a denominator which is the polynomial describing the FIR transfer function

Realization Using MATLAB

- Parallel forms I and II can be developed using the functions `residuez` and `residue`, respectively
- Program 6_2 uses these two functions