Realization of Allpass Filters

- An M-th order real-coefficient allpass transfer function $\mathcal{A}_M(z)$ is characterized by M unique coefficients as here the numerator is the mirror-image polynomial of the denominator
- A direct form realization of $\mathcal{A}_M(z)$ requires $2M$ multipliers
- Objective - Develop realizations of $\mathcal{A}_M(z)$ requiring only M multipliers

Realization Using Multiplier Extraction Approach

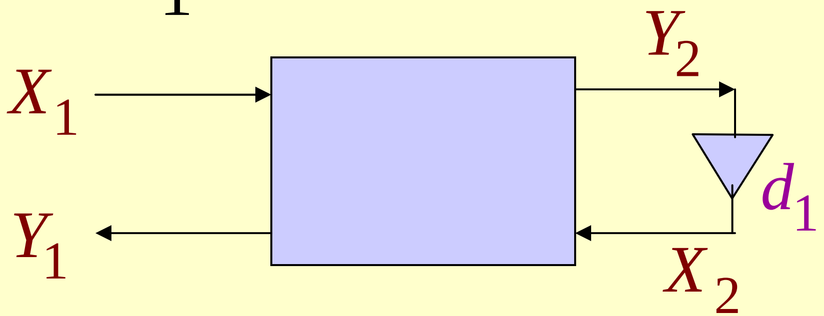
- Now, an arbitrary allpass transfer function can be expressed as a product of 2nd-order and/or 1st-order allpass transfer functions
- We consider first the minimum multiplier realization of a 1st-order and a 2nd-order allpass transfer functions

First-Order Allpass Structures

- Consider first the 1st-order allpass transfer function given by

$$\mathcal{A}_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$$

- We shall realize the above transfer function in the form a structure containing a single multiplier d_1 as shown below



First-Order Allpass Structures

- We express the transfer function $\mathcal{A}_1(z) = Y_1 / X_1$ in terms of the transfer parameters of the two-pair as

$$\mathcal{A}_1(z) = t_{11} + \frac{t_{12}t_{21}d_1}{1-d_1t_{22}} = \frac{t_{11} - d_1(t_{11}t_{22} - t_{12}t_{21})}{1-d_1t_{22}}$$

- A comparison of the above with

$$\mathcal{A}_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$$

yields

$$t_{11} = z^{-1}, \; t_{22} = -z^{-1}, \; t_{11}t_{22} - t_{12}t_{21} = -1$$

First-Order Allpass Structures

- Substituting $t_{11} = z^{-1}$ and $t_{22} = -z^{-1}$ in $t_{11}t_{22} - t_{12}t_{21} = -1$ we get

$$t_{12}t_{21} = 1 - z^{-2}$$

- There are 4 possible solutions to the above equation:

Type 1A: $t_{11} = z^{-1},\ t_{22} = -z^{-1},\ t_{12} = 1 - z^{-2},\ t_{21} = 1$

Type 1B:

$$t_{11} = z^{-1},\ t_{22} = -z^{-1},\ t_{12} = 1 + z^{-1},\ t_{21} = 1 - z^{-1}$$

First-Order Allpass Structures

- Type 1A$_t$: $t_{11} = z^{-1},\ t_{22} = -z^{-1},\ t_{12} = 1,\ t_{21} = 1 - z^{-2}$
- Type 1B$_t$:

$$t_{11} = z^{-1},\ t_{22} = -z^{-1},\ t_{12} = 1 - z^{-1},\ t_{21} = 1 + z^{-1}$$

- We now develop the two-pair structure for the Type 1A allpass transfer function

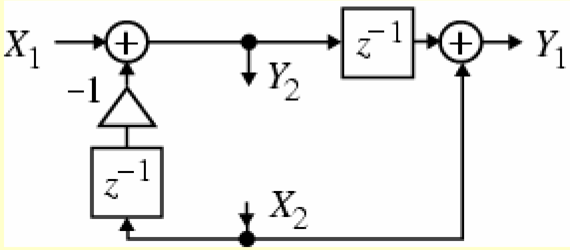
First-Order Allpass Structures

- From the transfer parameters of this allpass we arrive at the input-output relations:

$$Y_2 = X_1 - z^{-1}X_2$$

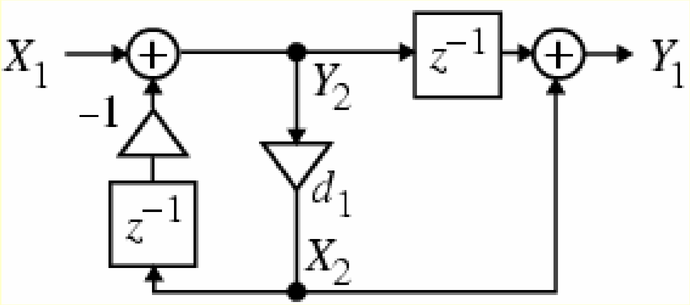
$$Y_1 = z^{-1}X_1 + (1 - z^{-2})X_2 = z^{-1}Y_2 + X_2$$

- A realization of the above two-pair is sketched below



First-Order Allpass Structures

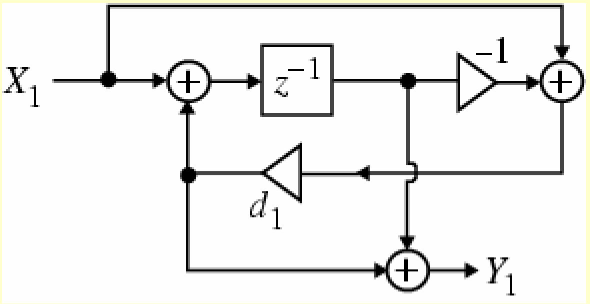
- By constraining the X_2, Y_2 terminal-pair with the multiplier d_1, we arrive at the Type 1A allpass filter structure shown below



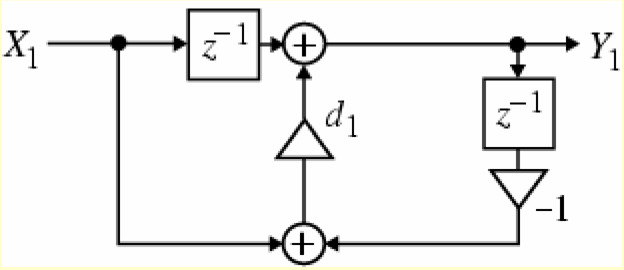
Type 1A

First-Order Allpass Structures

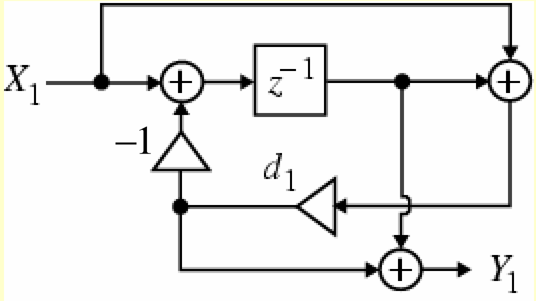
- In a similar fashion, the other three single-multiplier first-order allpass filter structures can be developed as shown below



Type 1B



Type $1A_t$



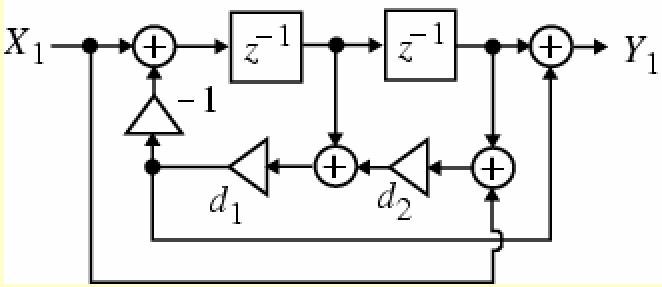
Type $1B_t$

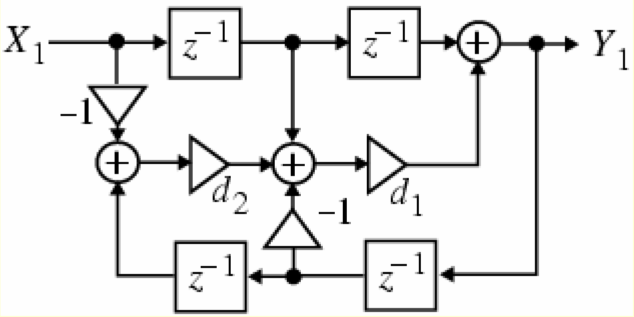
Second-Order Allpass Structures

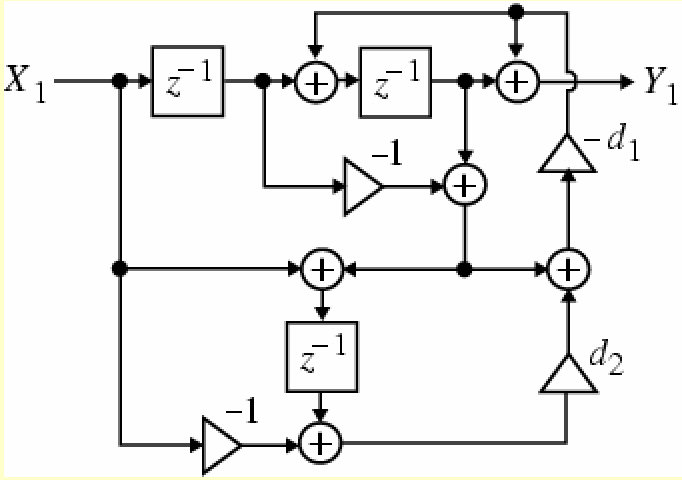
- A 2nd-order allpass transfer function is characterized by 2 unique coefficients
- Hence, it can be realized using only 2 multipliers
- Type 2 allpass transfer function:

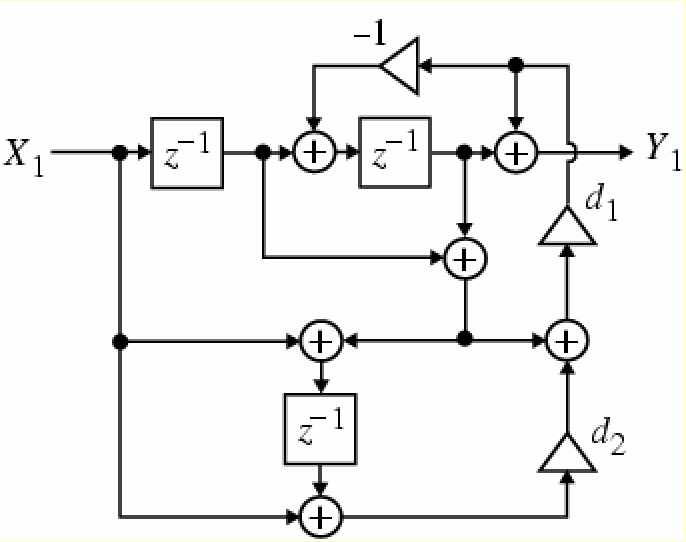
$$\mathcal{A}_2(z) = \frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}$$

Type 2 Allpass Structures







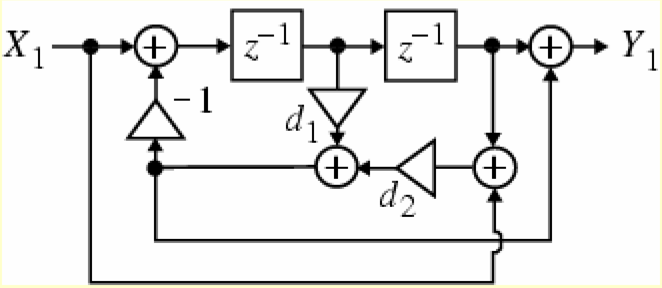


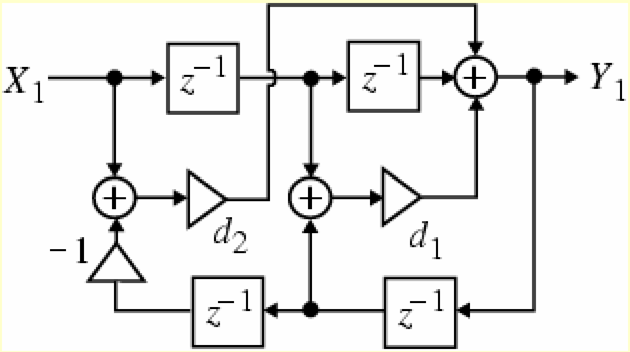
Type 3 Allpass Structures

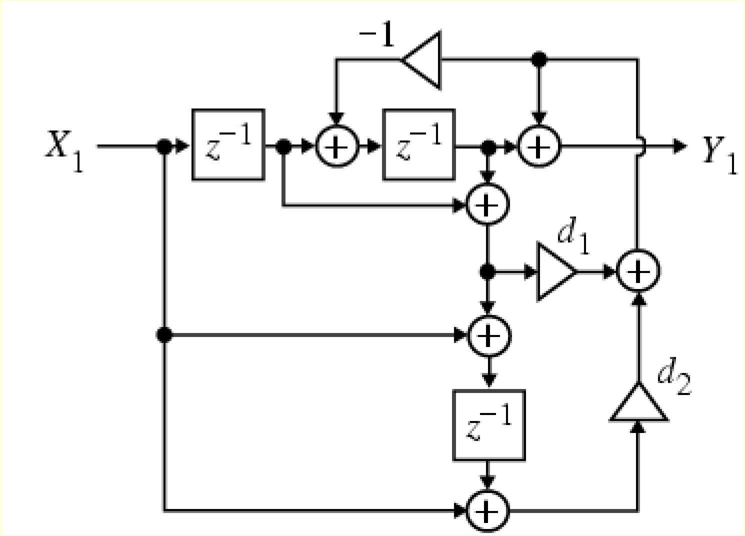
- Type 3 allpass transfer function:

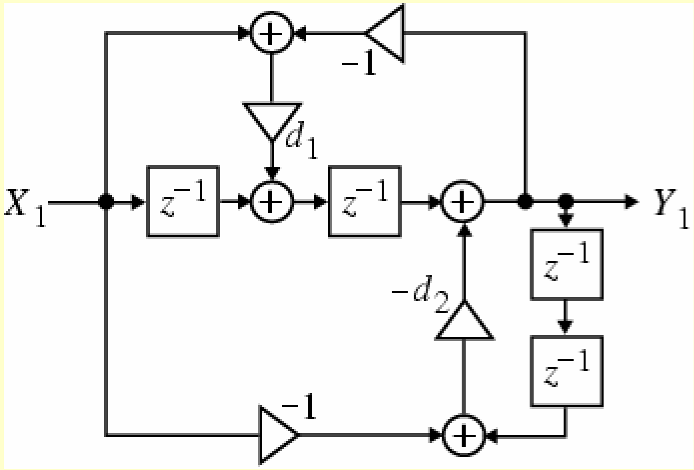
$$\mathcal{A}_3(z) = \frac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$$

Type 3 Allpass Structures









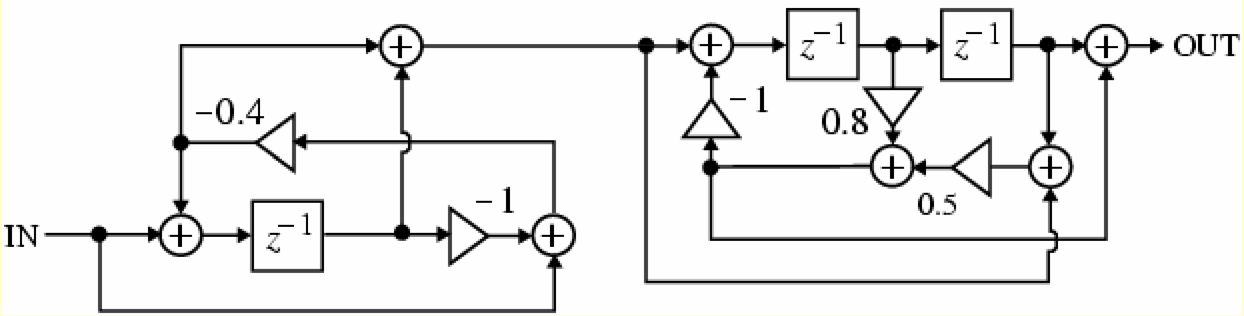
Realization Using Multiplier Extraction Approach

- Example - Realize

$$\mathcal{A}_3(z) = \frac{-0.2+0.18z^{-1}+0.4z^{-2}+z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

$$= \frac{(-0.4+z^{-1})(0.5+0.8z^{-1}+z^{-2})}{(1-0.4z^{-1})(1+0.8z^{-1}+0.5z^{-2})}$$

- A 3-multiplier cascade realization of the above allpass transfer function is shown below



Realization Using Two-Pair Extraction Approach

- The stability test algorithm described earlier in the course also leads to an elegant realization of an Mth-order allpass transfer function
- The algorithm is based on the development of a series of $(m-1)$th-order allpass transfer functions $\mathcal{A}_{m-1}(z)$ from an mth-order allpass transfer function $\mathcal{A}_m(z)$ for $m = M, M-1, \ldots, 1$

Realization Using Two-Pair Extraction Approach

- Let

$$\mathcal{A}_m(z) = \frac{d_m + d_{m-1}z^{-1} + d_{m-2}z^{-2} + \cdots + d_1 z^{-(m-1)} + z^{-m}}{1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_{m-1} z^{-(m-1)} + d_m z^{-m}}$$

- We use the recursion

$$\mathcal{A}_{m-1}(z) = z\left[\frac{\mathcal{A}_m(z) - k_m}{1 - k_m \mathcal{A}_m(z)}\right],\ m = M, M-1, \ldots, 1$$

where $k_m = \mathcal{A}_m(\infty) = d_m$

- It has been shown earlier that $A_M(z)$ is stable if and only if

$$k_m^2 < 1 \quad \text{for } m = M, M-1, \ldots, 1$$

Realization Using Two-Pair Extraction Approach

- If the allpass transfer function $\mathcal{A}_{m-1}(z)$ is expressed in the form

$$\mathcal{A}_{m-1}(z) = \frac{d'_{m-1} + d'_{m-2} z^{-1} + \cdots + d'_1 z^{-(m-2)} + z^{-(m-1)}}{1 + d'_1 z^{-1} + \cdots + d'_{m-2} z^{-(m-2)} + d'_{m-1} z^{-(m-1)}}$$

then the coefficients of $\mathcal{A}_{m-1}(z)$ are simply related to the coefficients of $\mathcal{A}_m(z)$ through

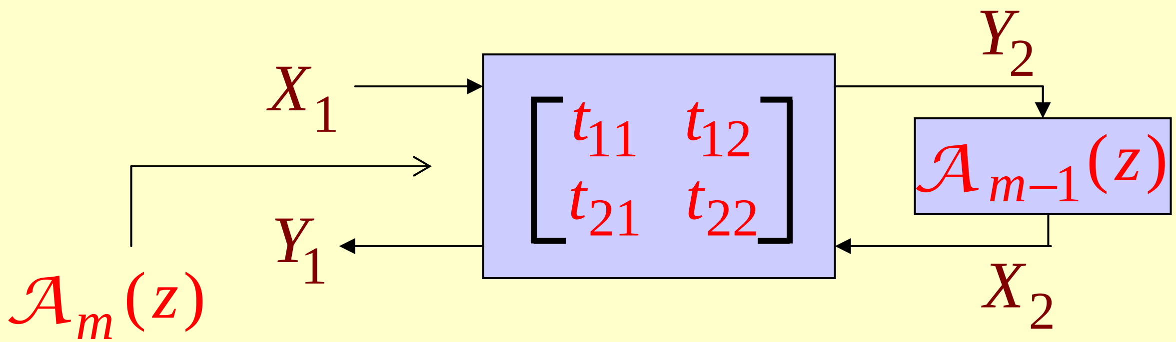
$$d'_i = \frac{d_i - d_m d_{m-i}}{1 - d_m^2}, \quad 1 \le i \le m-1$$

Realization Using Two-Pair Extraction Approach

- To develop the realization method we express $\mathcal{A}_m(z)$ in terms of $\mathcal{A}_{m-1}(z)$:

$$\mathcal{A}_m(z) = \frac{k_m + z^{-1}\mathcal{A}_{m-1}(z)}{1 + k_m z^{-1}\mathcal{A}_{m-1}(z)}$$

- We realize $\mathcal{A}_m(z)$ in the form shown below



Realization Using Two-Pair Extraction Approach

- The transfer function $\mathcal{A}_m(z) = Y_1/X_1$ of the constrained two-pair can be expressed as

$$\mathcal{A}_m(z) = \frac{t_{11} - (t_{11}t_{22} - t_{12}t_{21})\,\mathcal{A}_{m-1}(z)}{1 - t_{22}\,\mathcal{A}_{m-1}(z)}$$

- Comparing the above with

$$\mathcal{A}_m(z) = \frac{k_m + z^{-1}\mathcal{A}_{m-1}(z)}{1 + k_m z^{-1}\mathcal{A}_{m-1}(z)}$$

we arrive at the two-pair transfer parameters

Realization Using Two-Pair Extraction Approach

$$t_{11} = k_m, \quad t_{22} = -k_m z^{-1}$$

$$t_{11}t_{22} - t_{12}t_{21} = -z^{-1}$$

- Substituting $t_{11} = k_m$ and $t_{22} = -k_m z^{-1}$ in the equation above we get

$$t_{12}t_{21} = (1 - k_m^2) z^{-1}$$

- There are a number of solutions for t_{12} and t_{21}

Realization Using Two-Pair Extraction Approach

- Some possible solutions are given below:

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = z^{-1},\ t_{21} = 1 - k_m^2$$

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = (1 - k_m) z^{-1},\ t_{21} = 1 + k_m$$

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = \sqrt{1 - k_m^2}\, z^{-1},\ t_{21} = \sqrt{1 - k_m^2}$$

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = (1 - k_m^2) z^{-1},\ t_{21} = 1$$

Realization Using Two-Pair Extraction Approach

- Consider the solution

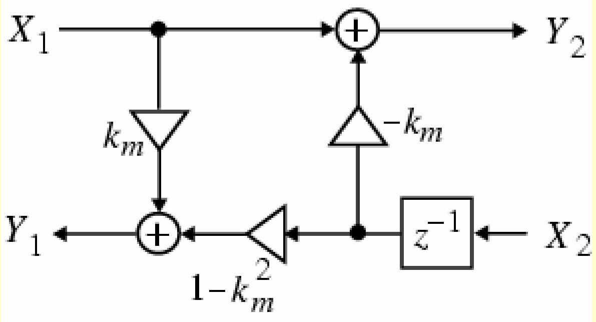
$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = (1-k_m^2)z^{-1},\ t_{21} = 1$$

- Corresponding input-output relations are

$$Y_1 = k_m X_1 + (1-k_m^2)z^{-1}X_2$$
$$Y_2 = X_1 - k_m z^{-1}X_2$$

- A direct realization of the above equations leads to the 3-multiplier two-pair shown on the next slide

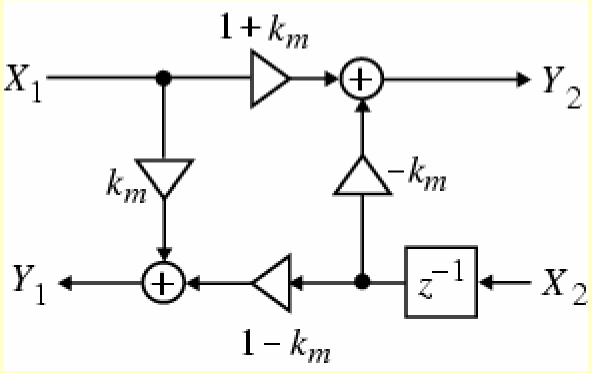
Realization Using Two-Pair Extraction Approach



- The transfer parameters

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = (1-k_m)z^{-1},\ t_{21} = 1+k_m$$

lead to the 4-multiplier two-pair structure shown below

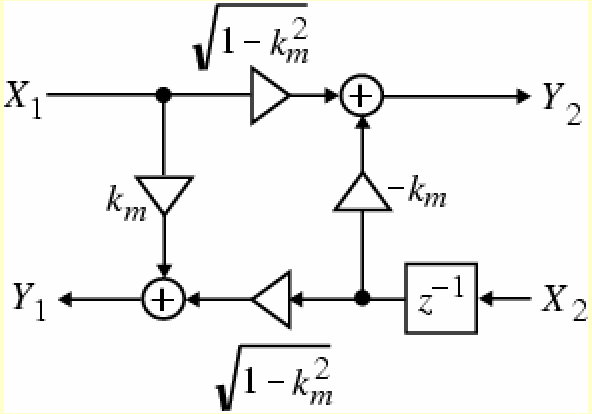


Realization Using Two-Pair Extraction Approach

- Likewise, the transfer parameters

$$t_{11} = k_m,\ t_{22} = -k_m z^{-1},\ t_{12} = \sqrt{1-k_m^2}\,z^{-1},\ t_{21} = \sqrt{1-k_m^2}$$

lead to the 4-multiplier two-pair structure shown below



Realization Using Two-Pair Extraction Approach

- A 2-multiplier realization can be derived by manipulating the input-output relations:

$$Y_1 = k_m X_1 + (1 - k_m^2) z^{-1} X_2$$

$$Y_2 = X_1 - k_m z^{-1} X_2$$

- Making use of the second equation, we can rewrite the first equation as

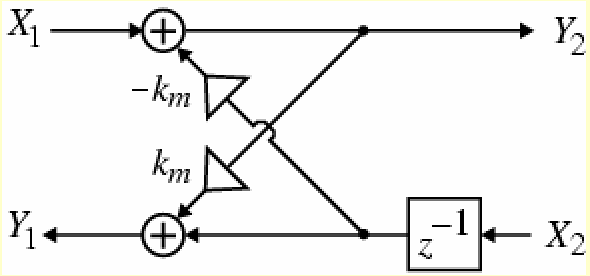
$$Y_1 = k_m Y_2 + z^{-1} X_2$$

Realization Using Two-Pair Extraction Approach

- A direct realization of

$$Y_1 = k_m Y_2 + z^{-1} X_2$$
$$Y_2 = X_1 - k_m z^{-1} X_2$$

lead to the 2-multiplier two-pair structure, known as the **lattice structure**, shown below



Realization Using Two-Pair Extraction Approach

- Consider the two-pair described by

$$t_{11} = k_m,\; t_{22} = -k_m z^{-1},\; t_{12} = (1 - k_m) z^{-1},\; t_{21} = 1 + k_m$$

- Its input-output relations are given by

$$Y_1 = k_m X_1 + (1 - k_m) z^{-1} X_2$$

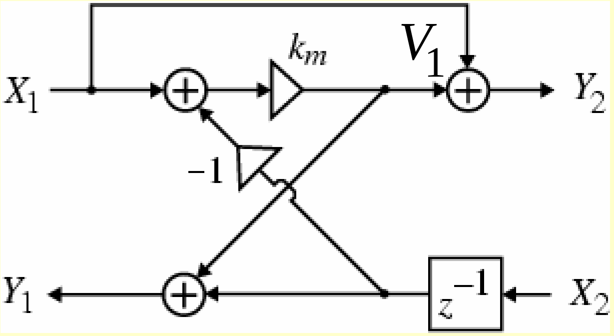
$$Y_2 = (1 + k_m) X_1 - k_m z^{-1} X_2$$

- Define

$$V_1 = k_m (X_1 - z^{-1} X_2)$$

Realization Using Two-Pair Extraction Approach

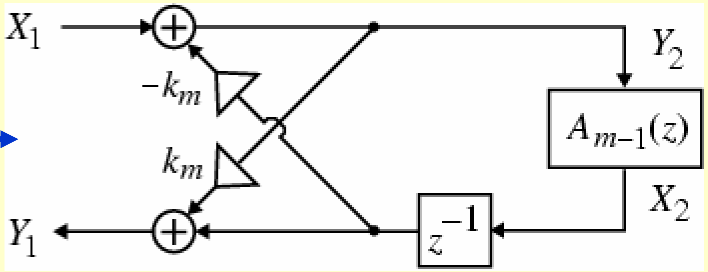
- We can then rewrite the input-output relations as $Y_1 = V_1 + z^{-1} X_2$ and $Y_2 = X_1 + V_1$
- The corresponding 1-multiplier realization is shown below



Realization Using Two-Pair Extraction Approach

- An mth-order allpass transfer function $\mathcal{A}_m(z)$ is then realized by constraining any one of the two-pairs developed earlier by the $(m-1)$th-order allpass transfer function $\mathcal{A}_{m-1}(z)$

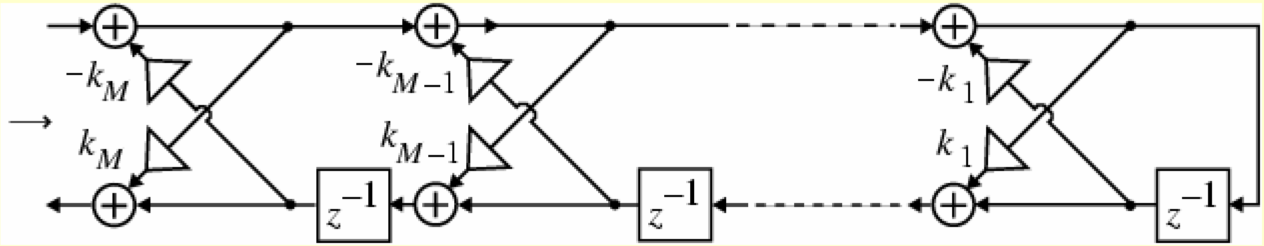
$\mathcal{A}_m(z)$ →



Realization Using Two-Pair Extraction Approach

- The process is repeated until the constraining transfer function is $\mathcal{A}_0(z) = 1$
- The complete realization of $\mathcal{A}_M(z)$ based on the extraction of the two-pair lattice is shown below

$\mathcal{A}_m(z) \rightarrow$



Realization Using Two-Pair Extraction Approach

- It follows from our earlier discussion that $\mathcal{A}_M(z)$ is stable if the magnitudes of all multiplier coefficients in the realization are less than 1, i.e., $|k_m| < 1$ for $m = M, M-1, \ldots, 1$
- The cascaded lattice allpass filter structure requires $2M$ multipliers
- A realization with M multipliers is obtained if instead the single multiplier two-pair is used

Realization Using Two-Pair Extraction Approach

- Example - Realize

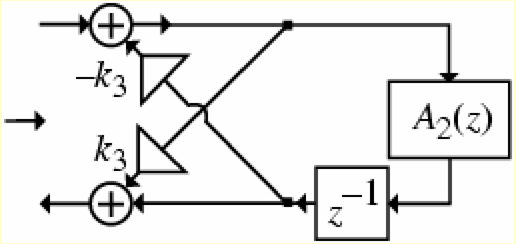
$$\mathcal{A}_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

$$= \frac{d_3 + d_2 z^{-1} + d_1 z^{-2} + z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

Realization Using Two-Pair Extraction Approach

- We first realize $\mathcal{A}_3(z)$ in the form of a lattice two-pair characterized by the multiplier coefficient $k_3 = d_3 = -0.2$ and constrained by a 2nd-order allpass $\mathcal{A}_2(z)$ as indicated below

$\mathcal{A}_3(z) \rightarrow$



$$k_3 = -0.2$$

Realization Using Two-Pair Extraction Approach

- The allpass transfer function $\mathcal{A}_2(z)$ is of the form

$$\mathcal{A}_2(z) = \frac{d_2' + d_1' z^{-1} + z^{-2}}{1 + d_1' z^{-1} + d_2' z^{-2}}$$

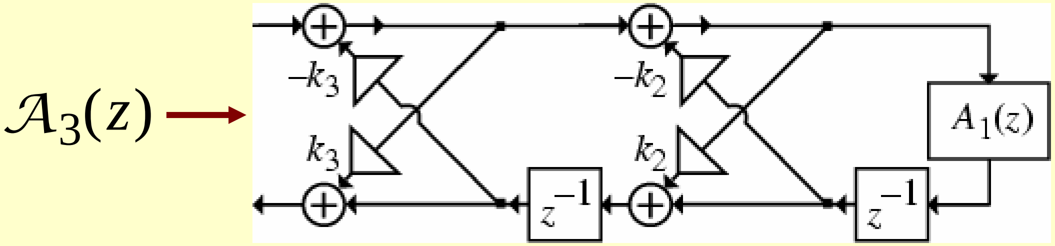
- Its coefficients are given by

$$d_1' = \frac{d_1 - d_3 d_2}{1 - d_3^2} = \frac{0.4 - (-0.2)(0.18)}{1 - (-0.2)^2} = 0.4541667$$

$$d_2' = \frac{d_2 - d_3 d_1}{1 - d_3^2} = \frac{0.18 - (-0.2)(0.4)}{1 - (-0.2)^2} = 0.2708333$$

Realization Using Two-Pair Extraction Approach

- Next, the allpass $\mathcal{A}_2(z)$ is realized as a lattice two-pair characterized by the multiplier coefficient $k_2 = d_2' = 0.2708333$ and constrained by an allpass $\mathcal{A}_1(z)$ as indicated below



$k_3 = -0.2, \quad k_2 = 0.2708333$

Realization Using Two-Pair Extraction Approach

- The allpass transfer function $\mathcal{A}_1(z)$ is of the form

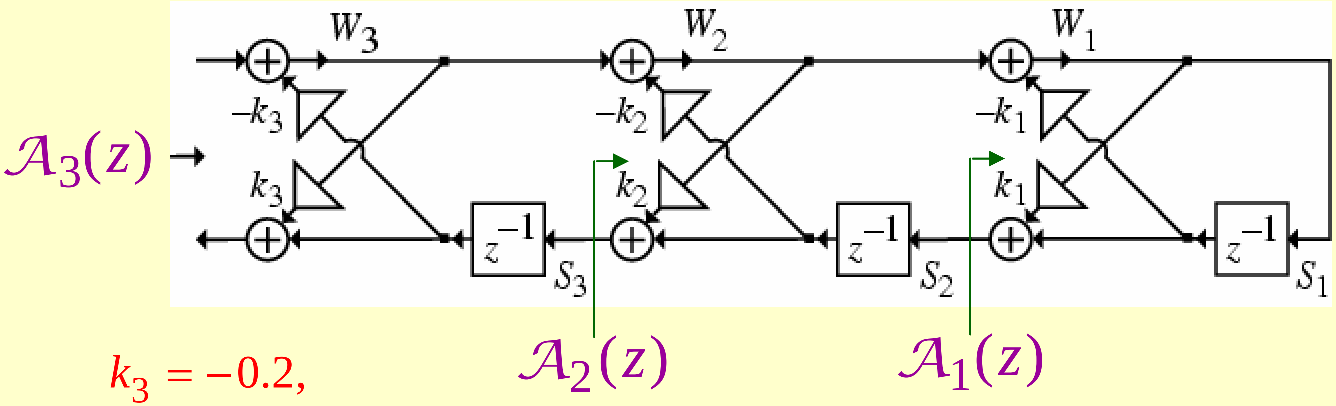
$$\mathcal{A}_1(z) = \frac{d_1'' + z^{-1}}{1 + d_1'' z^{-1}}$$

- It coefficient is given by

$$d_1'' = \frac{d_1' - d_2' d_1'}{1 - (d_2')^2} = \frac{d_1'}{1 + d_2'} = \frac{0.4541667}{1.2708333} = 0.3573771$$

Realization Using Two-Pair Extraction Approach

- Finally, the allpass $\mathcal{A}_1(z)$ is realized as a lattice two-pair characterized by the multiplier coefficient $k_1 = d_1'' = 0.3573771$ and constrained by an allpass $\mathcal{A}_0(z) = 1$ as indicated below



$k_3 = -0.2,$

$k_2 = 0.2708333, \quad k_1 = 0.3573771$

Cascaded Lattice Realization Using MATLAB

- The M-file `poly2rc` can be used to realize an allpass transfer function in the cascaded lattice form
- To this end Program 8_4 can be employed