

z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

leads to the z-transform

- z-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of z-transform techniques permits simple algebraic manipulations

z-Transform

- Consequently, z -transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z -transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where $z = \mathcal{Re}(z) + j\mathcal{Im}(z)$ is a complex variable

z-Transform

- If we let $z = r e^{j\omega}$, then the z -transform reduces to

$$G(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

- Thus $G(r e^{j\omega})$ can be interpreted as the DTFT of the modified sequence $\{g[n] r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z -transform reduces to its DTFT, provided the latter exists

z-Transform

- The contour $|z| = 1$ is a circle in the z -plane of unity radius and is called the **unit circle**
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set \mathcal{R} of values of z for which its z -transform converges is called the **region of convergence (ROC)**

z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty$$

z-Transform

- If $\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty$ for $r = \mathcal{R}_{g-}$ and $r = \mathcal{R}_{g+}$ with $0 \leq \mathcal{R}_{g-} < \mathcal{R}_{g+} < \infty$ then the sequence $g[n]r^{-n}$ is absolutely summable for all values of r ; that is,

$$\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty$$

for all values of r in the range

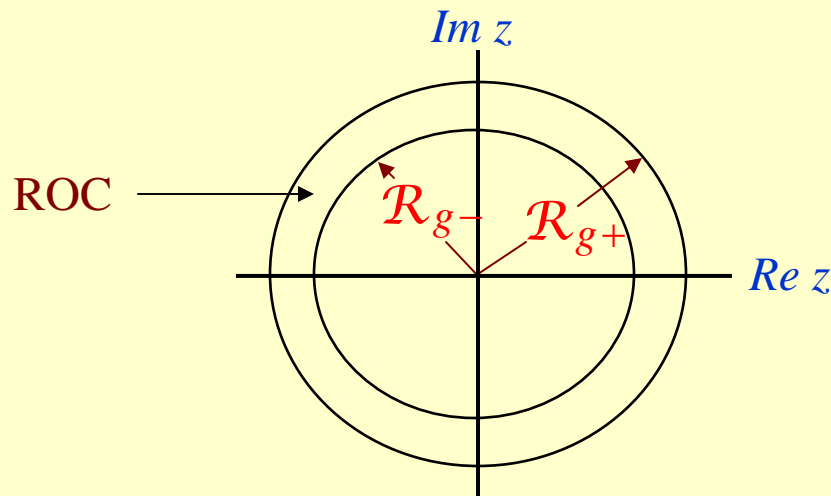
$$0 \leq \mathcal{R}_{g-} \leq r \leq \mathcal{R}_{g+} < \infty$$

z-Transform

- The annular region defined by

$$0 \leq \mathcal{R}_{g-} \leq r \leq \mathcal{R}_{g+} < \infty$$

is called the **region of convergence (ROC)** of $G(re^{j\omega}) = G(z)$



z-Transform

- Example - Determine the z -transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC

- Now
$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

- ROC is the annular region $|z| > |\alpha|$

z-Transform

- Example - The z -transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z^{-1}| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

z-Transform

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly
- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

z-Transform

- Its z -transform is given by

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = - \sum_{m=1}^{\infty} \alpha^{-m} z^m \\ &= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z} \\ &= \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha^{-1} z| < 1 \end{aligned}$$

- ROC is the annular region $|z| < |\alpha|$

z-Transform

- Note: The z -transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z -transform is by specifying its ROC

z-Transform

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z -transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the z -transform

z-Transform

- Example - The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

which converges in the mean-square sense

z-Transform

- However, $h_{LP}[n]$ does not have a z -transform as it is not absolutely summable for any value of r
- Some commonly used z -transform pairs are listed on the next slide

Table 6.1: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
$\alpha^n \mu[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
$(r^n \cos \omega_o n) \mu[n]$	$\frac{1 - (r \cos \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$ z > r$
$(r^n \sin \omega_o n) \mu[n]$	$\frac{(r \sin \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$ z > r$

Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z -transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1} :

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational z-Transforms

- The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z :

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational z-Transforms

- A rational z-transform can be alternately written in factored form as

$$\begin{aligned} G(z) &= \frac{p_0 \prod_{\ell=1}^M (1 - \xi_{\ell} z^{-1})}{d_0 \prod_{\ell=1}^N (1 - \lambda_{\ell} z^{-1})} \\ &= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^M (z - \xi_{\ell})}{d_0 \prod_{\ell=1}^N (z - \lambda_{\ell})} \end{aligned}$$

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the **zeros** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \rightarrow \infty$, and as a result, these values of z are known as the **poles** of $G(z)$

Rational z-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^M (z - \xi_{\ell})}{d_0 \prod_{\ell=1}^N (z - \lambda_{\ell})}$$

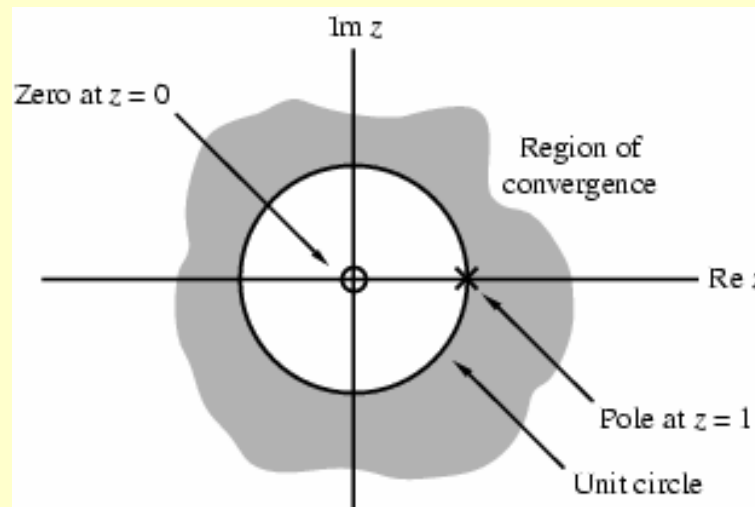
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z -plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational z-Transforms

- Example - The z -transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

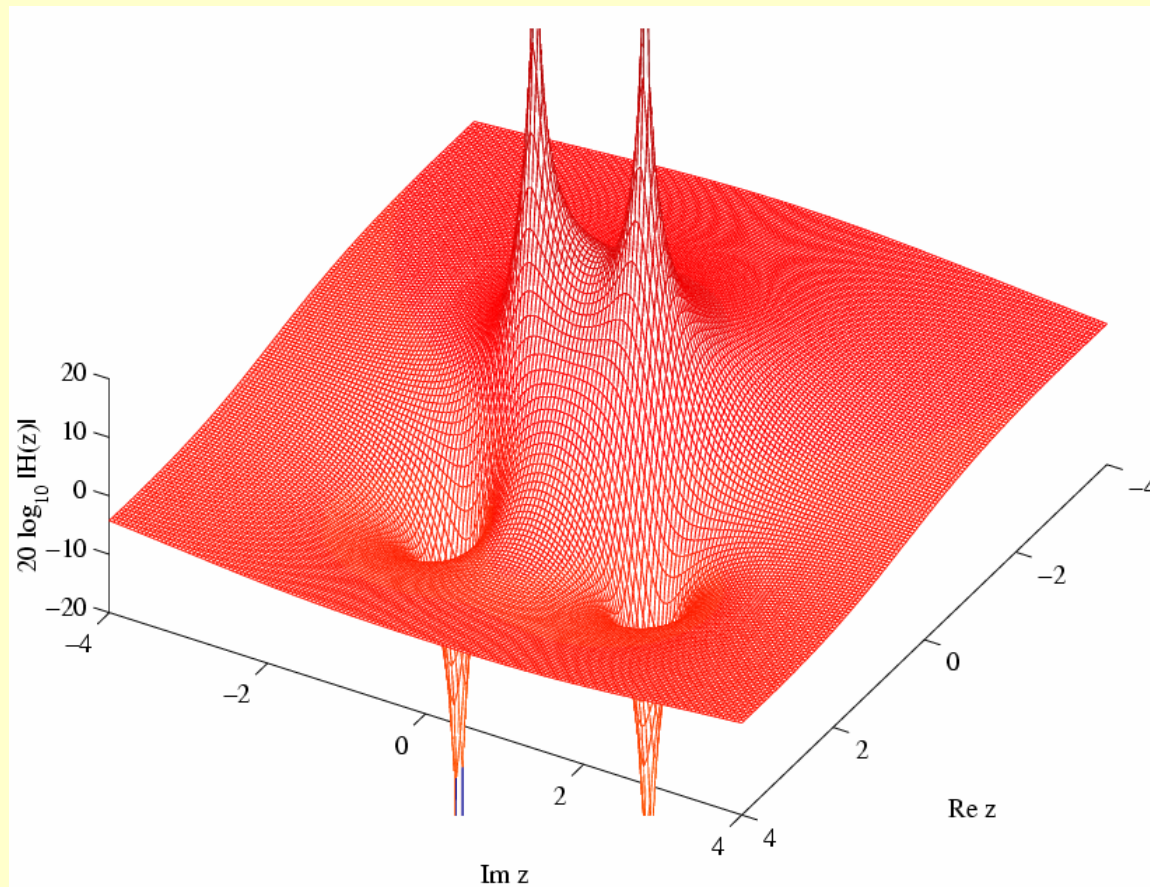


Rational z-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

Rational z-Transforms



Rational z-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

ROC of a Rational z-Transform

- ROC of a z -transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z -transform
- Hence, the z -transform must always be specified with its ROC

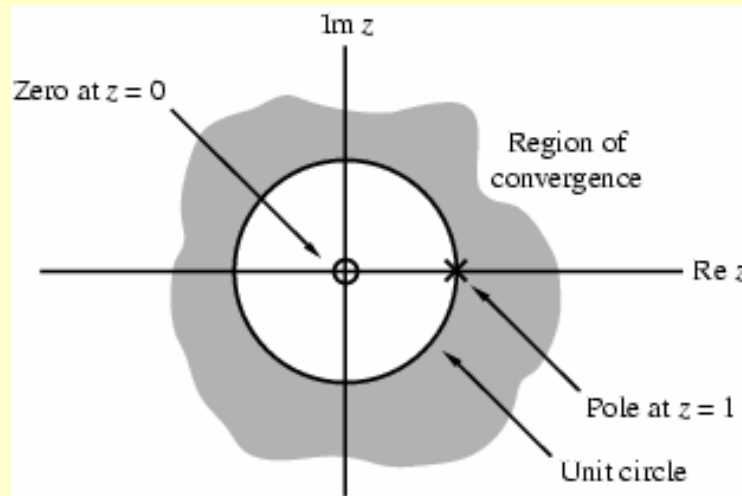
ROC of a Rational z-Transform

- Moreover, if the ROC of a z -transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z -transform on the unit circle
- There is a relationship between the ROC of the z -transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z -Transform

- The ROC of a rational z -transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z -transform
- Consider again the pole-zero plot of the z -transform $\mu(z)$

ROC of a Rational z-Transform

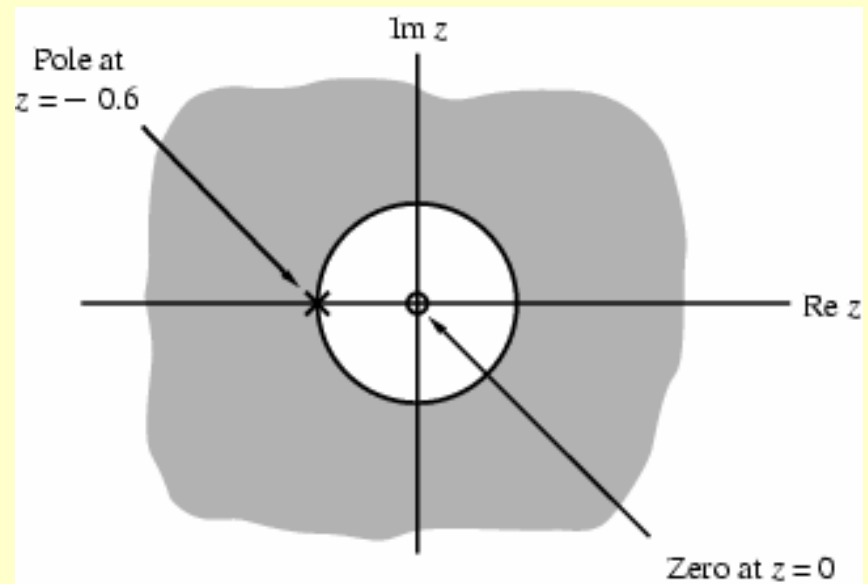


- In this plot, the ROC, shown as the shaded area, is the region of the z -plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z -transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1 + 0.6z^{-1}},$$
$$|z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational z-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

ROC of a Rational z-Transform

- Example - Consider a finite-length sequence $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z -transform is given by

$$G(z) = \sum_{n=-M}^N g[n] z^{-n} = \frac{\sum_0^{N+M} g[n-M] z^{N+M-n}}{z^N}$$

ROC of a Rational z-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$
- As can be seen from the expression for $G(z)$, the z -transform of a finite-length bounded sequence converges everywhere in the z -plane except possibly at $z = 0$ and/or at $z = \infty$

ROC of a Rational z-Transform

- Example - A right-sided sequence with nonzero sample values for $n \geq 0$ is sometimes called a causal sequence
- Consider a causal sequence $u_1[n]$
- Its z -transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z -transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

ROC of a Rational z-Transform

- Example - A left-sided sequence with nonzero sample values for $n \leq 0$ is sometimes called a anticausal sequence
- Consider an anticausal sequence $v_1[n]$
- Its z -transform is given by

$$V_1(z) = \sum_{n=-\infty}^0 v_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N nonnegative has a z -transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

ROC of a Rational z-Transform

- Example - The z -transform of a two-sided sequence $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the z -transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

ROC of a Rational z-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n] z^{-n}$, can be interpreted as the z -transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the z -transform does not exist

ROC of a Rational z-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

ROC of a Rational z-Transform

- There is no overlap between these two regions
- Hence, the z-transform of $u[n] = \alpha^n$ does not exist

ROC of a Rational z -Transform

- The ROC of a rational z -transform cannot contain any poles and is bounded by the poles
- To show that the z -transform is bounded by the poles, assume that the z -transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$
- Assume that the corresponding sequence $x[n]$ is a right-sided sequence

ROC of a Rational z-Transform

- Then $x[n]$ has the form

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[n - N_o], \quad |\alpha| < |\beta|$$

where N_o is a positive or negative integer

- Now, the z -transform of the right-sided sequence $\gamma^n \mu[n - N_o]$ exists if

$$\sum_{n=N_o}^{\infty} |\gamma^n z^{-n}| < \infty$$

for some z

ROC of a Rational z-Transform

- The condition

$$\sum_{n=N_o}^{\infty} |\gamma^n z^{-n}| < \infty$$

holds for $|z| > |\gamma|$ but not for $|z| \leq |\gamma|$

- Therefore, the z-transform of

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[n - N_o], \quad |\alpha| < |\beta|$$

has an ROC defined by $|\beta| < |z| \leq \infty$

ROC of a Rational z-Transform

- Likewise, the z -transform of a left-sided sequence

$$x[n] = (r_1\alpha^n + r_2\beta^n)\mu[-n - N_o], \quad |\alpha| < |\beta|$$

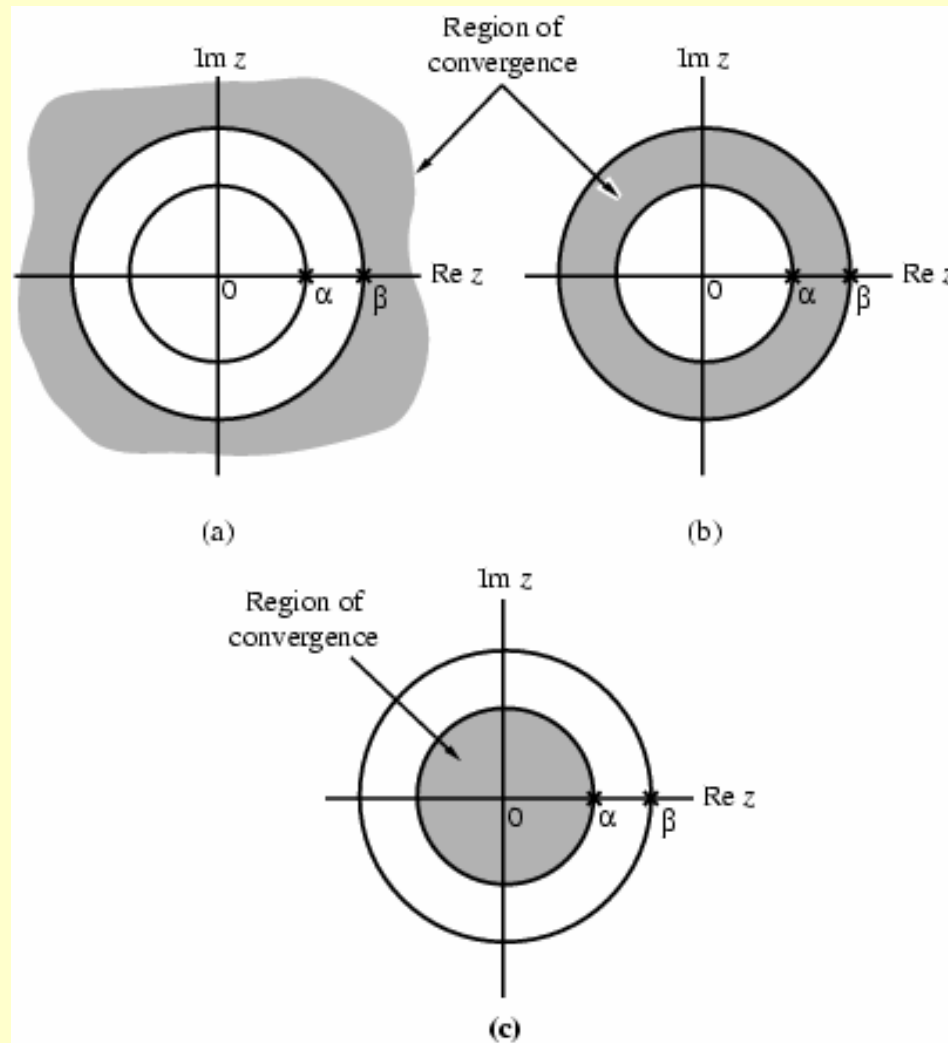
has an ROC defined by $0 \leq |z| < |\alpha|$

- Finally, for a two-sided sequence, some of the poles contribute to terms in the parent sequence for $n < 0$ and the other poles contribute to terms $n \geq 0$

ROC of a Rational z-Transform

- The ROC is thus bounded on the outside by the pole with the smallest magnitude that contributes for $n < 0$ and on the inside by the pole with the largest magnitude that contributes for $n \geq 0$
- There are three possible ROCs of a rational z-transform with poles at $z = \alpha$ and $z = \beta$ ($|\alpha| < |\beta|$)

ROC of a Rational z-Transform



ROC of a Rational z -Transform

- In general, if the rational z -transform has N poles with R distinct magnitudes, then it has $R + 1$ ROCs
- Thus, there are $R + 1$ distinct sequences with the same z -transform
- Hence, a rational z -transform with a specified ROC has a unique sequence as its inverse z -transform

ROC of a Rational z-Transform

- The ROC of a rational z -transform can be easily determined using MATLAB

$[z, p, k] = \text{tf2zp}(\text{num}, \text{den})$

determines the zeros, poles, and the gain constant of a rational z -transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

ROC of a Rational z-Transform

- `[num,den] = zp2tf(z,p,k)`
implements the reverse process
- The factored form of the z -transform can be obtained using `sos = zp2sos(z,p,k)`
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix `sos`

ROC of a Rational z-Transform

$$sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^L \frac{b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}}{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}$$

ROC of a Rational z-Transform

- The pole-zero plot is determined using the function `zplane`
- The z -transform can be either described in terms of its zeros and poles:
`zplane(zeros, poles)`
- or, it can be described in terms of its numerator and denominator coefficients:
`zplane(num, den)`

ROC of a Rational z-Transform

- Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below

