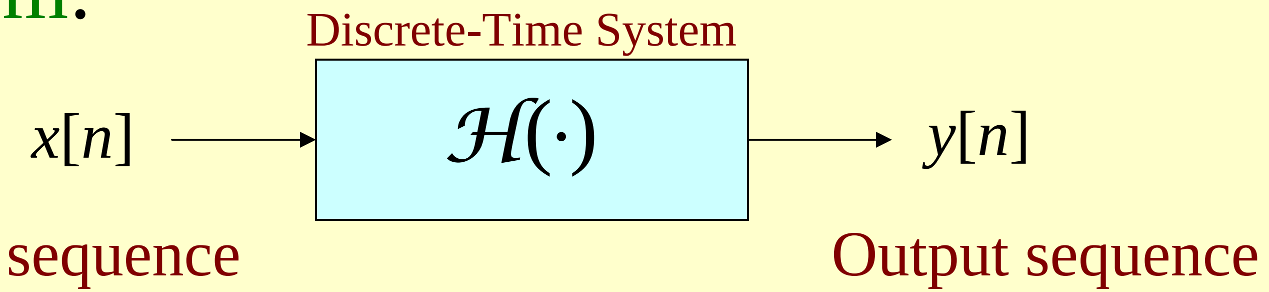
Discrete-Time Systems

- A discrete-time system processes a given **input sequence** $x[n]$ to generates an **output sequence** $y[n]$ with more desirable properties
- In most applications, the discrete-time system is a single-input, single-output system:

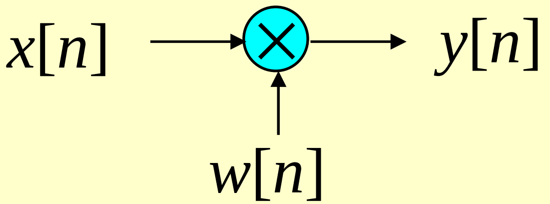


Discrete-Time Systems

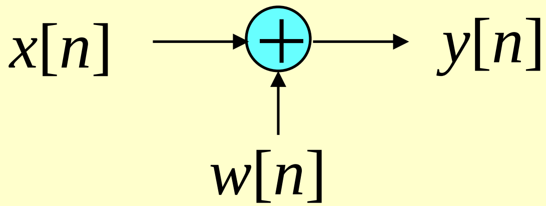
- Mathematically, the discrete-time system is characterized by an operator $\mathcal{H}(\cdot)$ that transforms the input sequence $x[n]$ into another sequence $y[n]$ at the output
- The discrete-time system may also have more than one input and/or more than one output

Discrete-Time Systems: Examples

- 2-input, 1-output discrete-time systems -

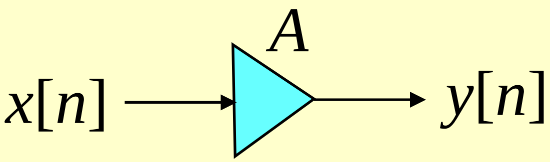


Modulator

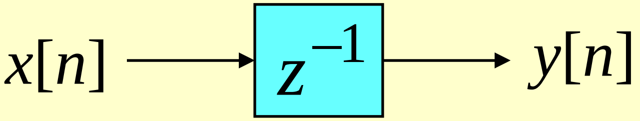


Adder

- 1-input, 1-output discrete-time systems -

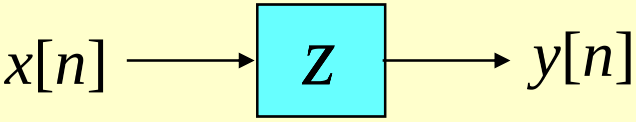


Multiplier



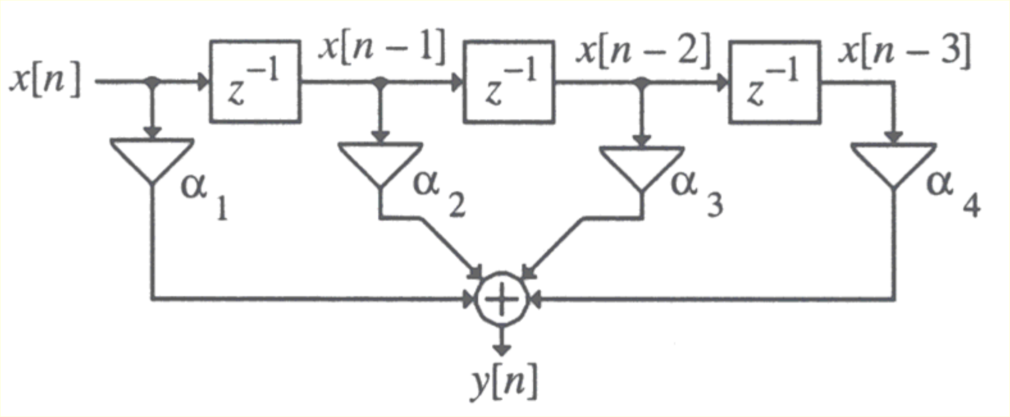
Unit Delay

Unit Advance



Discrete-Time Systems: Examples

- A more complex example of an one-input, one-output discrete-time system is shown below



Discrete-Time Systems: Examples

- **Accumulator -** $y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$

$$= \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n] = y[n-1] + x[n]$$

- The output $y[n]$ at time instant n is the sum of the input sample $x[n]$ at time instant n and the previous output $y[n-1]$ at time instant $n-1$, which is the sum of all previous input sample values from $-\infty$ to $n-1$
- The system cumulatively adds, i.e., it accumulates all input sample values

Discrete-Time Systems:Examples

- **Accumulator** - Input-output relation can also be written in the form

$$y[n] = \sum_{\ell=-\infty}^{-1} x[\ell] + \sum_{\ell=0}^{n} x[\ell]$$

$$= y[-1] + \sum_{\ell=0}^{n} x[\ell], \; n \geq 0$$

- The second form is used for a causal input sequence, in which case $y[-1]$ is called the **initial condition**

Discrete-Time Systems:Examples

- **M-point Moving-Average System** -

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- Used in smoothing random variations in data
- In most applications, the data $x[n]$ is a bounded sequence
- $\Rightarrow$ M-point average $y[n]$ is also a bounded sequence

Discrete-Time Systems:Examples

- If there is no bias in the measurements, an improved estimate of the noisy data is obtained by simply increasing M
- A direct implementation of the M-point moving average system requires $M-1$ additions, 1 division, and storage of $M-1$ past input data samples
- A more efficient implementation is developed next

Discrete-Time Systems:Examples

$$y[n] = \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-\ell] + x[n-M] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{\ell=1}^{M} x[n-\ell] + x[n] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-1-\ell] + x[n] - x[n-M]\right)$$

Hence

$$y[n] = y[n-1] + \frac{1}{M}\left(x[n] - x[n-M]\right)$$

Discrete-Time Systems:Examples

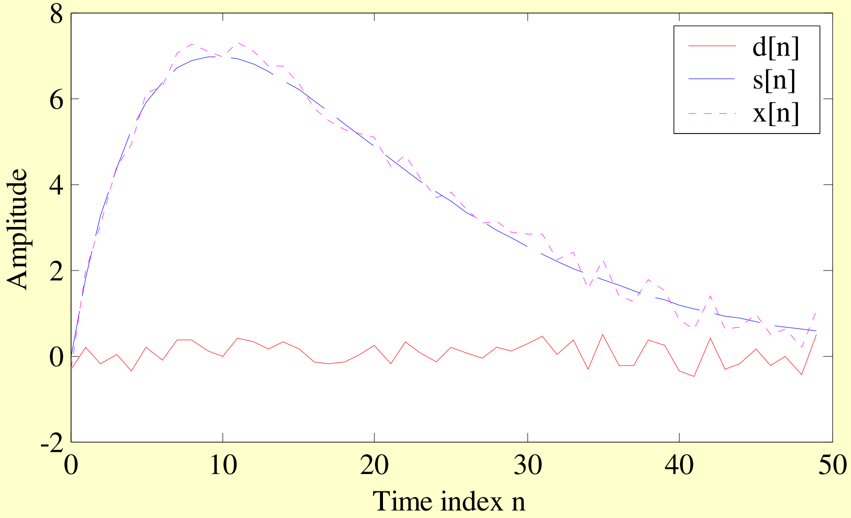
- Computation of the modified M-point moving average system using the recursive equation now requires 2 additions and 1 division
- An application: Consider

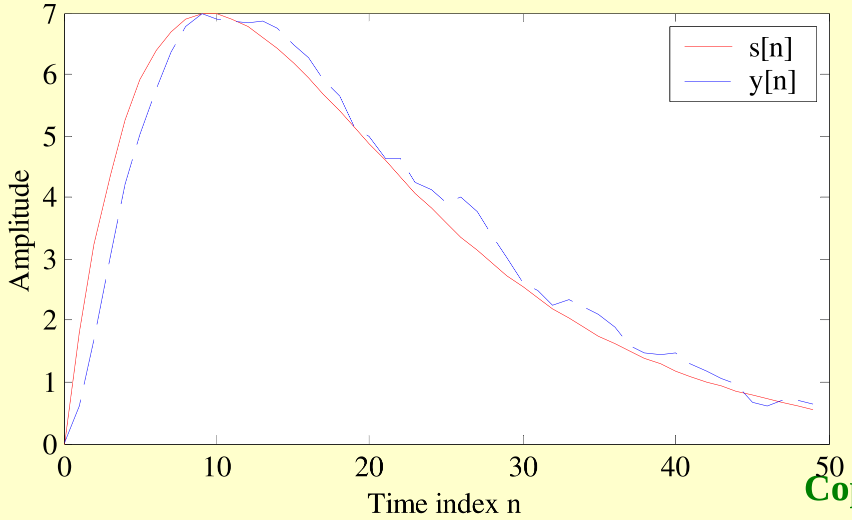
$$x[n] = s[n] + d[n],$$

where $s[n]$ is the signal corrupted by a noise $d[n]$

Discrete-Time Systems:Examples

$s[n] = 2[n(0.9)^n]$, $d[n]$ - random signal





Discrete-Time Systems:Examples

- **Exponentially Weighted Running Average Filter**

$$y[n] = \alpha y[n-1] + x[n], \quad 0 < \alpha < 1$$

- Computation of the running average requires only 2 additions, 1 multiplication and storage of the previous running average
- Does not require storage of past input data samples

Discrete-Time Systems:Examples

- For $0 < \alpha < 1$, the exponentially weighted average filter places more emphasis on current data samples and less emphasis on past data samples as illustrated below

$$y[n] = \alpha\big(\alpha y[n-2] + x[n-1]\big) + x[n]$$

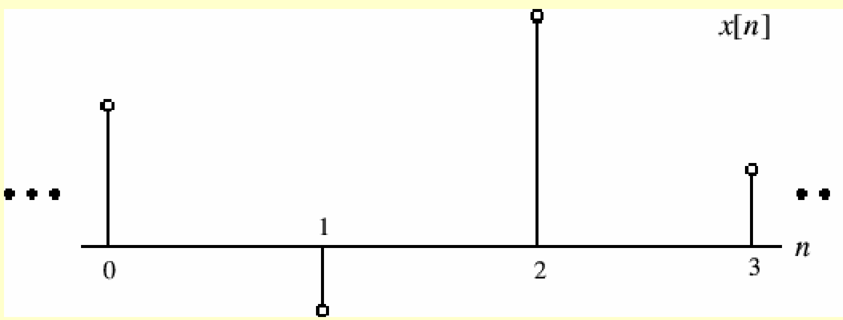
$$= \alpha^2 y[n-2] + \alpha x[n-1] + x[n]$$

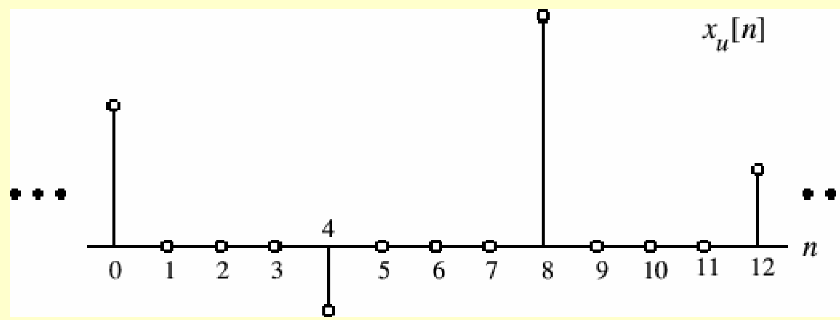
$$= \alpha^2\big(\alpha y[n-3] + x[n-2]\big) + \alpha x[n-1] + x[n]$$

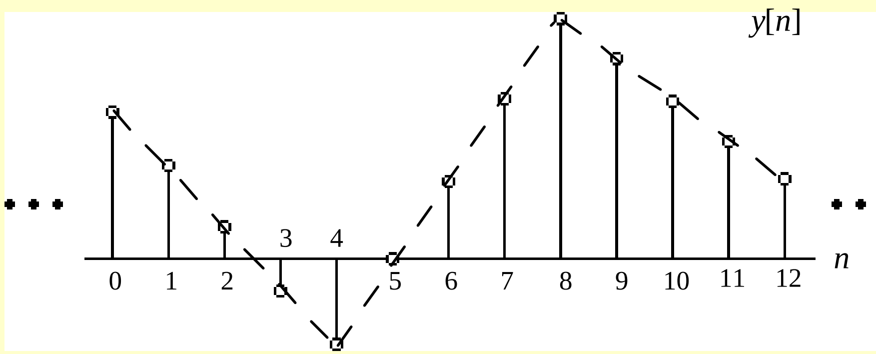
$$= \alpha^3 y[n-3] + \alpha^2 x[n-2] + \alpha x[n-1] + x[n]$$

Discrete-Time Systems:Examples

- **Linear interpolation** - Employed to estimate sample values between pairs of adjacent sample values of a discrete-time sequence
- **Factor-of-4 interpolation**







Discrete-Time Systems: Examples

- **Factor-of-2 interpolator** -

$$y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

- **Factor-of-3 interpolator** -

$$y[n] = x_u[n] + \frac{1}{3}\left(x_u[n-1] + x_u[n+2]\right)$$

$$+ \frac{2}{3}\left(x_u[n-2] + x_u[n+1]\right)$$

Discrete-Time Systems: Examples

- **Factor-of-2 interpolator** –

Original (512×512)

Down−sampled (256×256)

Interpolated (512 × 512)

Discrete-Time Systems: Examples

Median Filter –

- The median of a set of $(2K+1)$ numbers is the number such that K numbers from the set have values greater than this number and the other K numbers have values smaller
- Median can be determined by rank-ordering the numbers in the set by their values and choosing the number at the middle

Discrete-Time Systems: Examples

Median Filter –

- **Example**: Consider the set of numbers

$$\{2, \quad -3, \quad 10, \quad 5, \quad -1\}$$

- Rank-ordered set is given by

$$\{-3, \quad -1, \quad 2, \quad 5, \quad 10\}$$

- Hence,

$$\text{med}\{2, \quad -3, \quad 10, \quad 5, \quad -1\} = 2$$

Discrete-Time Systems: Examples

Median Filter –

- Implemented by sliding a window of odd length over the input sequence $\{x[n]\}$ one sample at a time
- Output $y[n]$ at instant n is the median value of the samples inside the window centered at n

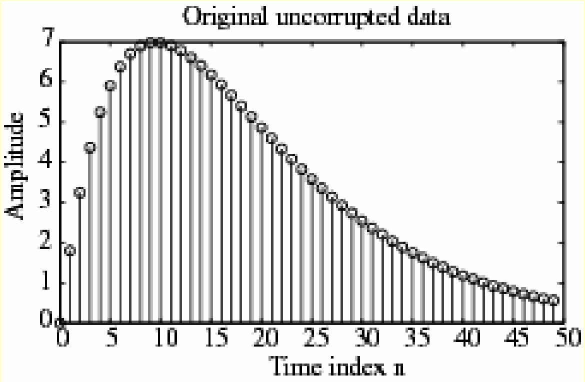
Discrete-Time Systems: Examples

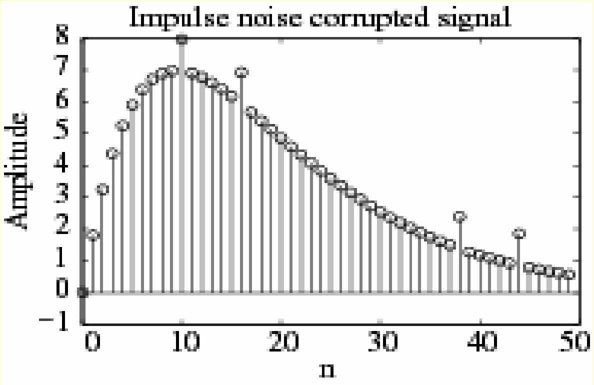
Median Filter –

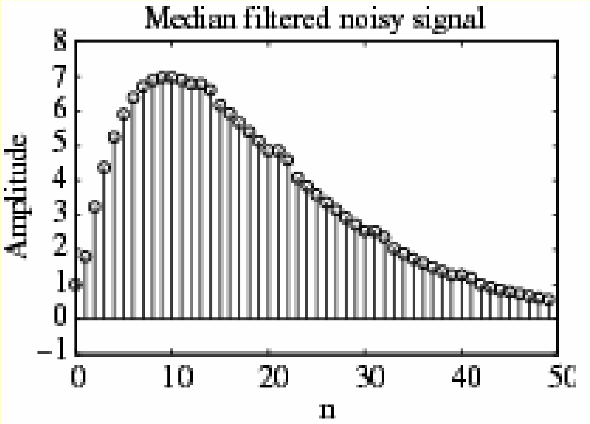
- Finds applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal
- Usually used for the smoothing of signals corrupted by impulse noise

Discrete-Time Systems: Examples

Median Filtering Example –







Discrete-Time Systems: Classification

- Linear System
- Shift-Invariant System
- Causal System
- Stable System
- Passive and Lossless Systems

Linear Discrete-Time Systems

- **Definition** - If $y_1[n]$ is the output due to an input $x_1[n]$ and $y_2[n]$ is the output due to an input $x_2[n]$ then for an input

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

the output is given by

$$y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$$

- Above property must hold for any arbitrary constants α and β, and for all possible inputs $x_1[n]$ and $x_2[n]$

Linear Discrete-Time Systems

- **Accumulator** - $y_1[n] = \sum_{\ell=-\infty}^{n} x_1[\ell], \quad y_2[n] = \sum_{\ell=-\infty}^{n} x_2[\ell]$

  For an input

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

  the output is

$$y[n] = \sum_{\ell=-\infty}^{n} \left(\alpha\, x_1[\ell] + \beta\, x_2[\ell]\right)$$

$$= \alpha \sum_{\ell=-\infty}^{n} x_1[\ell] + \beta \sum_{\ell=-\infty}^{n} x_2[\ell] = \alpha\, y_1[n] + \beta\, y_2[n]$$

- Hence, the above system is **linear**

Linear Discrete-Time Systems

- The outputs $y_1[n]$ and $y_2[n]$ for inputs $x_1[n]$ and $x_2[n]$ are given by

$$y_1[n] = y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell]$$

$$y_2[n] = y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell]$$

- The output $y[n]$ for an input $\alpha\, x_1[n] + \beta\, x_2[n]$ is given by

$$y[n] = y[-1] + \sum_{\ell=0}^{n} (\alpha\, x_1[\ell] + \beta\, x_2[\ell])$$

Linear Discrete-Time Systems

- Now $\alpha\, y_1[n] + \beta\, y_2[n]$

$$= \alpha(y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell]) + \beta(y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell])$$

$$= (\alpha\, y_1[-1] + \beta\, y_2[-1]) + (\alpha \sum_{\ell=0}^{n} x_1[\ell] + \beta \sum_{\ell=0}^{n} x_2[\ell])$$

- Thus $y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$ if

$$y[-1] = \alpha\, y_1[-1] + \beta\, y_2[-1]$$

Linear Discrete-Time System

- For the accumulator with a causal input to be **linear** the condition

$$y[-1] = \alpha\, y_1[-1] + \beta\, y_2[-1]$$

  must hold for all initial conditions $y[-1]$, $y_1[-1]$, $y_2[-1]$, and all constants α and β

- This condition cannot be satisfied unless the accumulator is initially at rest with zero initial condition

Nonlinear Discrete-Time System

- The median filter described earlier is a nonlinear discrete-time system
- To show this, consider a median filter with a window of length 3
- Output of the filter for an input

$$\{x_1[n]\} = \{3, \quad 4, \quad 5\}, 0 \le n \le 2$$

is

$$\{y_1[n]\} = \{3, \quad 4, \quad 4\}, 0 \le n \le 2$$

Nonlinear Discrete-Time System

- Output for an input

$$\{x_2[n]\} = \{2, \quad -1, \quad -1\}, 0 \le n \le 2$$

is

$$\{y_2[n]\} = \{0, \quad -1, \quad -1\}, 0 \le n \le 2$$

- However, the output for an input

$$\{x[n]\} = \{x_1[n] + x_2[n]\}$$

is

$$\{y[n]\} = \{3, \quad 4, \quad 3\}$$

Nonlinear Discrete-Time System

- Note: $\{y_1[n] + y_2[n]\} = \{3, \quad 3, \quad 3\} \neq \{y[n]\}$
- Hence, the median filter is a nonlinear discrete-time system
- The second form of the accumulator with non-zero initial condition is another example

Shift-Invariant System

- For a shift-invariant system, if $y_1[n]$ is the response to an input $x_1[n]$, then the response to an input

$$x[n] = x_1[n - n_o]$$

is simply

$$y[n] = y_1[n - n_o]$$

where n_o is any positive or negative integer

- The above relation must hold for any arbitrary input and its corresponding output

Shift-Invariant System

- In the case of sequences and systems with indices n related to discrete instants of time, the above property is called **time-invariance** property
- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied

Shift-Invariant System

- Example - Consider the up-sampler

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

with an input-output relation given by

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

Shift-Invariant System

- For an input $x_1[n] = x[n - n_o]$ the output $x_{1,u}[n]$ is given by

$$x_{1,u}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x[(n - Ln_o)/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

Shift-Invariant System

- However from the definition of the up-sampler

$$x_u[n-n_o]$$

$$=\begin{cases} x[(n-n_o)/L], & n = n_o, n_o \pm L, n_o \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

$$\neq x_{1,u}[n]$$

- Hence, the up-sampler is a time-varying system

Linear Time-Invariant System

- **Linear Time-Invariant (LTI) System** - A system satisfying both the linearity and the time-invariance property
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design
- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades

Causal System

- In a **causal system**, the n_o-th output sample $y[n_o]$ depends only on input samples $x[n]$ for $n \le n_o$ and does not depend on input samples for $n > n_o$
- Let $y_1[n]$ and $y_2[n]$ be the responses of a causal discrete-time system to the inputs $x_1[n]$ and $x_2[n]$, respectively

Causal System

- Then

$$x_1[n] = x_2[n] \text{ for } n < N$$

implies also that

$$y_1[n] = y_2[n] \text{ for } n < N$$

- For a causal system, changes in output samples do not precede changes in the input samples

Causal System

- Examples of causal systems:

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] \\ + a_1 y[n-1] + a_2 y[n-2]$$

$$y[n] = y[n-1] + x[n]$$

- Examples of noncausal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) \\ + \frac{2}{3}(x_u[n-2] + x_u[n+1])$$

Causal System

- A noncausal system can be implemented as a causal system by delaying the output by an appropriate number of samples
- For example a causal implementation of the factor-of-2 interpolator is given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

Stable System

- There are various definitions of stability
- We consider here the **bounded-input, bounded-output** (**BIBO**) **stability**
- If $y[n]$ is the response to an input $x[n]$ and if

$$|x[n]| \leq B_x \quad \text{for all values of } n$$

then

$$|y[n]| \leq B_y \quad \text{for all values of } n$$

Stable System

- Example - The M-point moving average filter is BIBO stable:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- For a bounded input $|x[n]| \le B_x$ we have

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right| \le \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]|$$

$$\le \frac{1}{M}(MB_x) \le B_x$$

Passive and Lossless Systems

- A discrete-time system is defined to be **passive** if, for every finite-energy input $x[n]$, the output $y[n]$ has, at most, the same energy, i.e.

$$\sum_{n=-\infty}^{\infty} \left| y[n] \right|^2 \leq \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 < \infty$$

- For a **lossless** system, the above inequality is satisfied with an equal sign for every input

Passive and Lossless Systems

- Example - Consider the discrete-time system defined by $y[n] = \alpha\, x[n-N]$ with N a positive integer
- Its output energy is given by

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 = \left|\alpha\right|^2 \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

- Hence, it is a passive system if $\left|\alpha\right| < 1$ and is a lossless system if $\left|\alpha\right| = 1$

Impulse and Step Responses

- The response of a discrete-time system to a unit sample sequence $\{\delta[n]\}$ is called the **unit sample response** or simply, the **impulse response**, and is denoted by $\{h[n]\}$
- The response of a discrete-time system to a unit step sequence $\{\mu[n]\}$ is called the **unit step response** or simply, the **step response**, and is denoted by $\{s[n]\}$

Impulse Response

- Example - The impulse response of the system

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4 \delta[n-3]$$

- The impulse response is thus a finite-length sequence of length 4 given by

$$\{h[n]\} = \{\underset{\uparrow}{\alpha_1}, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4\}$$

Impulse Response

- Example - The impulse response of the discrete-time accumulator

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

Impulse Response

- Example - The impulse response $\{h[\mathrm{n}]\}$ of the factor-of-2 interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

- is obtained by setting $x_u[n] = \delta[n]$ and is given by

$$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

- The impulse response is thus a finite-length sequence of length 3:

$$\{h[n]\} = \{0.5, \quad \underset{\uparrow}{1} \quad 0.5\}$$