DSP HW3 蔡侑霖 311512038

Part I:

- $\mathcal{Z}\left[(r^n \cos \omega_0 n)\mu[n]\right]$

$$= \sum_{n=0}^{\infty} (r^n \cos \omega_0 n) z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} z^{-n}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} (r e^{j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (r e^{-j\omega_0} z^{-1})^n\right)$$

$$= \frac{1}{2}\left(\frac{1}{1 - r e^{j\omega_0} z^{-1}} + \frac{1}{1 - r e^{-j\omega_0} z^{-1}}\right) \quad , \ |r e^{j\omega_0} z^{-1}| = |r e^{-j\omega_0} z^{-1}| < 1 \Rightarrow |z| > r$$

$$= \frac{1}{2} \frac{1 - r e^{-j\omega_0} z^{-1} + 1 - r e^{j\omega_0} z^{-1}}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})}$$

$$= \frac{1}{2} \frac{2 - r e^{-j\omega_0} z^{-1} - r e^{j\omega_0} z^{-1}}{1 - r e^{j\omega_0} z^{-1} - r e^{-j\omega_0} z^{-1} + r^2 z^{-2}}$$

$$= \frac{1}{2} \frac{2 - 2r \frac{e^{-j\omega_0} + e^{j\omega_0}}{2} z^{-1}}{1 - 2r \frac{e^{-j\omega_0} + e^{j\omega_0}}{2} z^{-1} + r^2 z^{-2}}$$

$$= \frac{1 - r \cos \omega_0 \, z^{-1}}{1 - 2r \cos \omega_0 \, z^{-1} + r^2 z^{-2}} \quad , \ |z| > r \quad \#$$

- $\mathcal{Z}\left[(r^n \sin \omega_0 n)\mu[n]\right]$

$$= \sum_{n=0}^{\infty} (r^n \sin \omega_0 n) z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} z^{-n}$$

$$= \frac{1}{2j}\left(\sum_{n=0}^{\infty} (r e^{j\omega_0} z^{-1})^n - \sum_{n=0}^{\infty} (r e^{-j\omega_0} z^{-1})^n\right)$$

$$= \frac{1}{2j}\left(\frac{1}{1 - r e^{j\omega_0} z^{-1}} - \frac{1}{1 - r e^{-j\omega_0} z^{-1}}\right) \quad , \ |r e^{j\omega_0} z^{-1}| = |r e^{-j\omega_0} z^{-1}| < 1 \Rightarrow |z| > r$$

$$= \frac{1}{2j} \frac{1 - r e^{-j\omega_0} z^{-1} - 1 + r e^{j\omega_0} z^{-1}}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})}$$

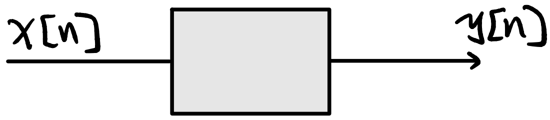
$$= \frac{1}{2j} \frac{- r e^{-j\omega_0} z^{-1} + r e^{j\omega_0} z^{-1}}{1 - r e^{j\omega_0} z^{-1} - r e^{-j\omega_0} z^{-1} + r^2 z^{-2}}$$

$$= \frac{1}{2j} \frac{2j\, r \frac{e^{j\omega_0} - e^{-j\omega_0}}{2j} z^{-1}}{1 - 2r \frac{e^{-j\omega_0} + e^{j\omega_0}}{2} z^{-1} + r^2 z^{-2}}$$

$$= \frac{(r \sin \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}} \quad , \ |z| > r \quad \#$$

Part II :

5 point moving average filter :



ⓐ $y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right)$

$\#$

ⓑ When $x[n] = \delta[n]$

Apparently, we can obtain

$$y[n] = \frac{1}{5}\left(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\right)$$

$$\triangleq g[n]$$

or by the transfer function of the difference equation :

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)$$

We can also obtain the impulse response of the system by the inverse z transform of $G(z)$

$$\mathcal{Z}^{-1}[G(z)] = \frac{1}{5}\left(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\right)$$

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ⓒ let $\mathcal{Z}[x[n]] = X(z)$

$\mathcal{Z}[y[n]] = Y(z)$

the z transform of the difference equation:

$$Y(z) = \frac{1}{5}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4})X(z)$$

$$\Rightarrow G(z) \triangleq \frac{Y(z)}{X(z)} = \frac{1}{5}\frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

\#

because $\frac{Y(z)}{X(z)}$ is the sum of $1, z^{-1}, z^{-2}, z^{-3}, z^{-4}$

and we know $\begin{cases} \mathcal{Z}[\delta[n]] = 1, & \text{ROC: all values of } z \\ \mathcal{Z}[\delta[n-n_0]] = z^{-n_0}, & \text{ROC: } |z| > 0 \end{cases}$

therefore, the ROC of the transfer function is: $|z| > 0$

\#

(d) Since the ROC of the transfer function includes unit circle $\Rightarrow$ DTFT exists.

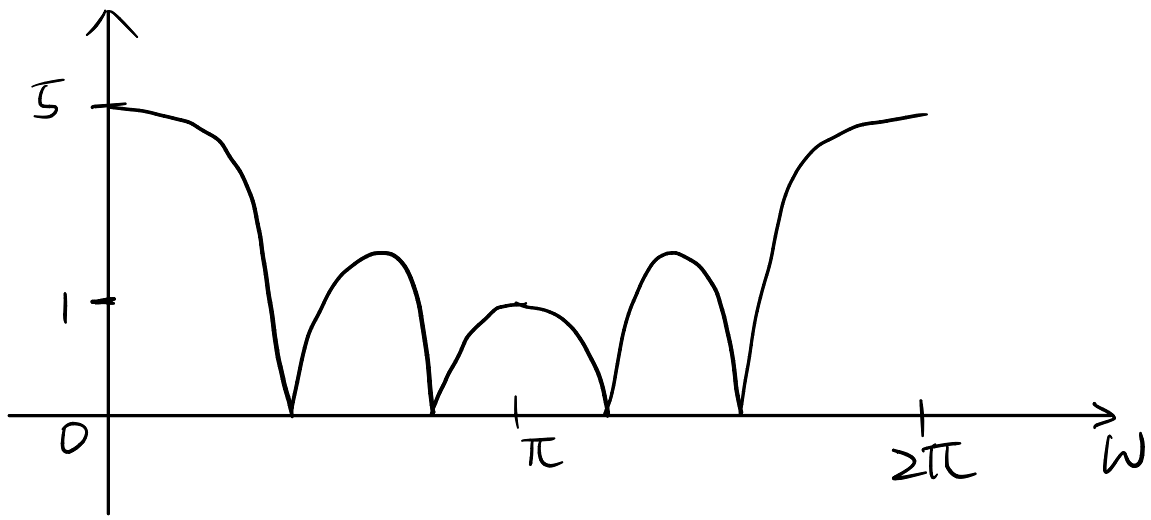
we can obtain the DTFT of transfer function by replacing z with e^{jw}

$$\Rightarrow \frac{1}{5}(1+e^{-jw}+e^{-2jw}+e^{-3jw}+e^{-4jw})$$

$$=\frac{1}{5}e^{-2jw}\left(e^{2jw}+e^{-2jw}+e^{jw}+e^{-jw}+1\right)$$

$$=\frac{1}{5}e^{-2jw}(2\cos(2w)+2\cos(w)+1) \quad \#$$

magnitude spectrum



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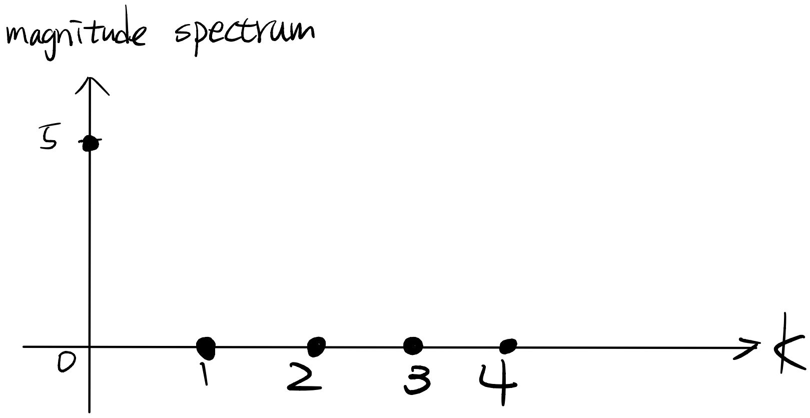
ⓔ the DFT can be obtain

by replacing w with $w_k = \frac{2\pi k}{N}$, $0 \le k \le N-1$

(1) $N=5$, $w_k = \frac{2\pi}{5}k$, the DFT of $g[n]$ is

$$\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{5}}+e^{-j\frac{4\pi k}{5}}+e^{-j\frac{6\pi k}{5}}+e^{-j\frac{8\pi k}{5}}\right),\ 0 \le k \le 4$$

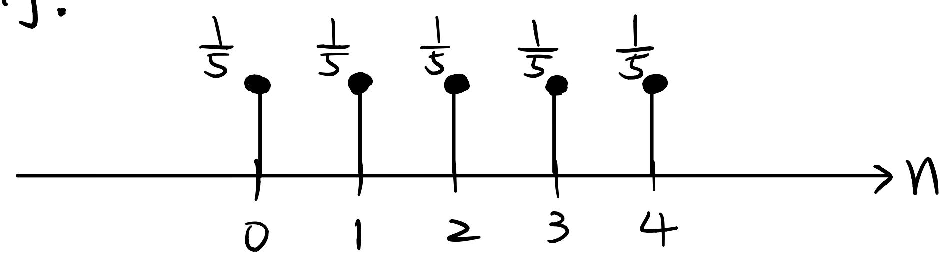


the IDFT of $\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{5}}+e^{-j\frac{4\pi k}{5}}+e^{-j\frac{6\pi k}{5}}+e^{-j\frac{8\pi k}{5}}\right)$ is

$$g_1[n] = \frac{1}{5}\sum_{k=0}^{4} \frac{1}{5}\left(1+e^{-j\frac{2\pi k}{5}}+e^{-j\frac{4\pi k}{5}}+e^{-j\frac{6\pi k}{5}}+e^{-j\frac{8\pi k}{5}}\right)e^{j\frac{2n\pi k}{5}}$$

$$= \underset{\uparrow}{\frac{1}{5}},\ \frac{1}{5},\ \frac{1}{5},\ \frac{1}{5},\ \frac{1}{5}$$

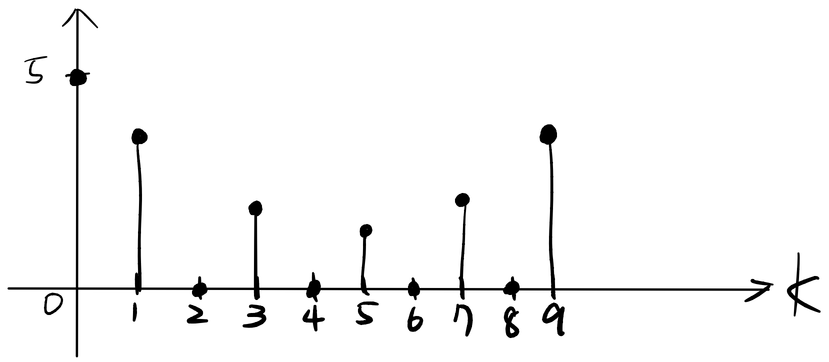
$g_1[n]$:



(2) $N=10$, $w_k=\frac{2\pi}{10}k$, the DFT of $g[n]$ is

$$\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{10}}+e^{-j\frac{4\pi k}{10}}+e^{-j\frac{6\pi k}{10}}+e^{-j\frac{8\pi k}{10}}\right),\ 0\le k\le 9$$

magnitude spectrum

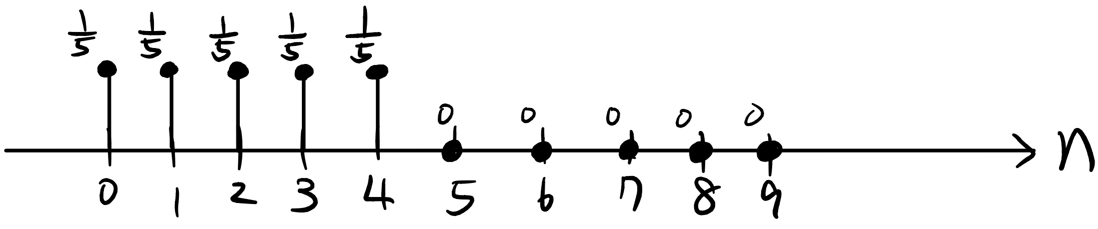


the IDFT of $\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{10}}+e^{-j\frac{4\pi k}{10}}+e^{-j\frac{6\pi k}{10}}+e^{-j\frac{8\pi k}{10}}\right)$ is

$$g_2[n]=\frac{1}{10}\sum_{k=0}^{9}\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{10}}+e^{-j\frac{4\pi k}{10}}+e^{-j\frac{6\pi k}{10}}+e^{-j\frac{8\pi k}{10}}\right)e^{j\frac{2n\pi k}{10}}$$

$$=\underset{\uparrow}{\frac{1}{5}},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},0,0,0,0,0$$

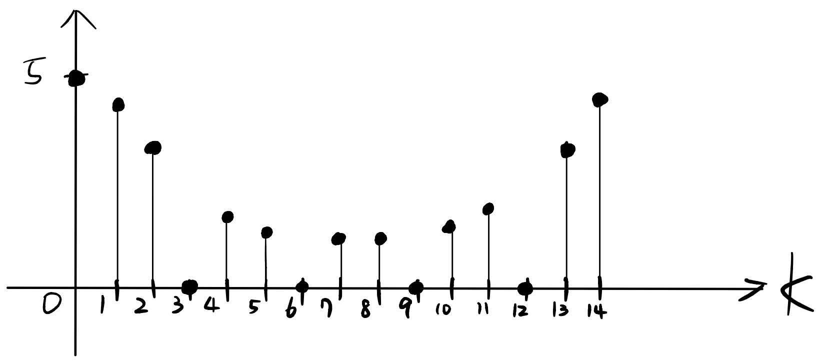
$g_2[n]$:



(3) $N=15$, $w_k = \frac{2\pi}{15}k$, the DFT of $g[n]$ is

$$\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{15}}+e^{-j\frac{4\pi k}{15}}+e^{-j\frac{6\pi k}{15}}+e^{-j\frac{8\pi k}{15}}\right),\ 0 \le k \le 14$$

magnitude spectrum

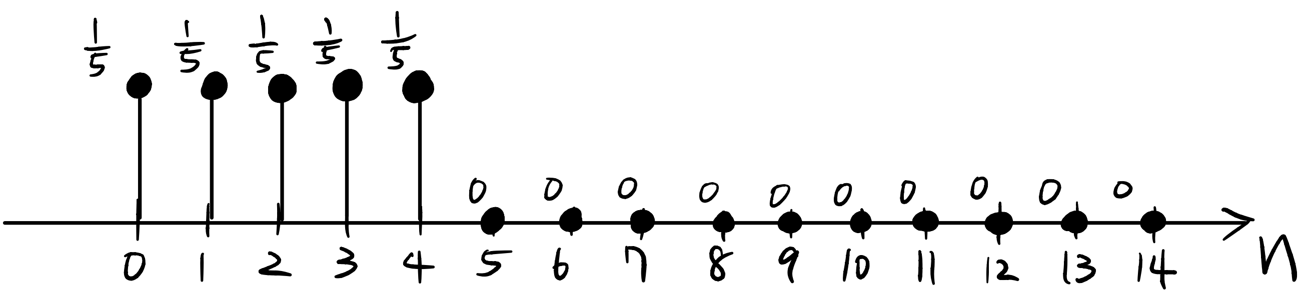


the IDFT of $\frac{1}{5}\left(1+e^{-j\frac{2\pi k}{15}}+e^{-j\frac{4\pi k}{15}}+e^{-j\frac{6\pi k}{15}}+e^{-j\frac{8\pi k}{15}}\right)$ is

$$g_3[n] = \frac{1}{15}\sum_{k=0}^{14} \frac{1}{5}\left(1+e^{-j\frac{2\pi k}{15}}+e^{-j\frac{4\pi k}{15}}+e^{-j\frac{6\pi k}{15}}+e^{-j\frac{8\pi k}{15}}\right)e^{j\frac{2n\pi k}{15}}$$

$$= \underset{\uparrow}{\frac{1}{5}}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$$

$g_3[n]$:



(f)

$$G(z) = \frac{1}{5}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}) = \frac{1}{5}\frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

by $z^4 + z^3 + z^2 + z + 1 = \frac{(z^5 - 1)}{(z - 1)}$

We know the root of $z^4 + z^3 + z^2 + z + 1$ are the roots of $(z^5 - 1)$ except $z = 1$

$\therefore$ the zeros of $G(z)$ are $e^{j\frac{2\pi}{5}}, e^{j\frac{4\pi}{5}}, e^{-j\frac{2\pi}{5}}, e^{-j\frac{4\pi}{5}}$

and the poles of $G(z)$ are $z = 0$