Inverse z-Transform

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z)\, z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

Inverse z-Transform

- But the integral remains unchanged when C' is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum \begin{bmatrix} \text{residues of } G(z)z^{n-1} \\ \text{at the poles inside } C \end{bmatrix}$$

- The above equation needs to be evaluated at all values of n and is not pursued here

Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**
- To develop the proper fraction part $P_1(z)/D(z)$ from $G(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reverse order until the remainder polynomial $P_1(z)$ is of lower degree than that of the denominator $D(z)$

Inverse Transform by Partial-Fraction Expansion

- Example - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division in reverse order we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \underbrace{\frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}}_{\text{Proper fraction}}$$

Inverse Transform by Partial-Fraction Expansion

- **Simple Poles**: In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of $G(z)$ be at $z = \lambda_k, 1 \le k \le N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

Inverse Transform by Partial-Fraction Expansion

- The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) G(z)\Big|_{z=\lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_\ell|$ and, thus has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Note: The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence

Inverse Transform by Partial-Fraction Expansion

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

Inverse Transform by Partial-Fraction Expansion

- Now

$$\rho_1 = (1 - 0.2\, z^{-1}) H(z)\Big|_{z=0.2} = \left.\frac{1 + 2\, z^{-1}}{1 + 0.6\, z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1 + 0.6\, z^{-1}) H(z)\Big|_{z=-0.6} = \left.\frac{1 + 2\, z^{-1}}{1 - 0.2\, z^{-1}}\right|_{z=-0.6} = -1.75$$

Inverse Transform by Partial-Fraction Expansion

- Hence

$$H(z) = \frac{2.75}{1 - 0.2\,z^{-1}} - \frac{1.75}{1 + 0.6\,z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- **Multiple Poles**: If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, $1 \le \ell \le N - L$

Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z) \right]_{z=\nu},$$

$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Partial-Fraction Expansion Using MATLAB

- `[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors `num` and `den`
- Vector `r` contains the residues
- Vector `p` contains the poles
- Vector `k` contains the constants η_ℓ

Partial-Fraction Expansion Using MATLAB

- `[num,den]=residuez(r,p,k)` converts a *z*-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division

Inverse z-Transform via Long Division

- Example - Consider

$$H(z) = \frac{1+2z^{-1}}{1+0.4z^{-1}-0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result

$$\{h[n]\} = \{\underset{\uparrow}{1} \quad 1.6 \quad -0.52 \quad 0.4 \quad -0.2224 \quad \cdots\}, \quad n \geq 0$$

Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational *z*-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

Table 6.2: z-Transform Theorems

Theorems	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

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z-Transform Theorems

- Example - Consider the two-sided sequence

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1]$$

- Let $x[n] = \alpha^n \mu[n]$ and $y[n] = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms
- Now $X(z) = \dfrac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$

and $Y(z) = \dfrac{1}{1-\beta z^{-1}}, \quad |z| < |\beta|$

z-Transform Theorems

- Using the linearity theorem we arrive at

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist

z-Transform Theorems

- Example - Determine the z-transform and its ROC of the causal sequence

$$x[n] = r^n (\cos \omega_o n) \mu[n]$$

- We can express $x[n] = v[n] + v^*[n]$ where

$$v[n] = \tfrac{1}{2} r^n e^{j\omega_o n} \mu[n] = \tfrac{1}{2} \alpha^n \mu[n]$$

- The z-transform of $v[n]$ is given by

$$V(z) = \tfrac{1}{2} \cdot \frac{1}{1 - \alpha z^{-1}} = \tfrac{1}{2} \cdot \frac{1}{1 - r e^{j\omega_o} z^{-1}}, \qquad |z| > |\alpha| = r$$

z-Transform Theorems

- Using the conjugation theorem we obtain the z-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2}\cdot\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-r\,e^{-j\omega_o}\,z^{-1}},$$

$$|z| > |\alpha|$$

- Finally, using the linearity property we get

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1-r\,e^{j\omega_o}z^{-1}} + \frac{1}{1-r\,e^{-j\omega_o}z^{-1}}\right)$$

z-Transform Theorems

- or,

$$X(z) = \frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}, \quad |z| > r$$

- Example - Determine the z-transform $Y(z)$ and the ROC of the sequence

$$y[n] = (n+1)\alpha^n\mu[n]$$

- We can write $y[n] = n\,x[n] + x[n]$ where

$$x[n] = \alpha^n\mu[n]$$

z-Transform Theorems

- Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

- Using the differentiation theorem, we arrive at the z-transform of $n\,x[n]$ as

$$-z \frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \qquad |z| > |\alpha|$$

z-Transform Theorems

- Using the linearity theorem we finally obtain

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$= \frac{1}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$

Linear Convolution Using z-Transform

- Let $\{x[n]\}, 0 \le n \le L,$ denote a finite-length sequence of length $L+1$
- Let $\{h[n]\}, 0 \le n \le M,$ denote a finite-length sequence of length $M+1$
- We shall evaluate $y[n] = x[n] \circledast h[\mathrm{n}]$ using *z*-transform
- Note: $\{y[n]\}$ is a sequence of length $L+M+1$

Linear Convolution Using z-Transform

- Let $X(z)$ denote the z-transform of $\{x[n]\}$ which is a polynomial of degree L in z^{-1}, i.e.,

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[L]z^{-L}$$

- Let $H(z)$ denote the z-transform of $\{h[n]\}$ which is a polynomial of degree M in z^{-1}, i.e.,

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[M]z^{-M}$$

Linear Convolution Using z-Transform

- From the convolution property of the z-transform it follows that the z-transform of $\{y[n]\}$ is simply given by $Y(z) = X(z)H(z)$ which is a polynomial of degree $L+M$ in z^{-1} i.e.,

$$Y(z) = y[0] + y[1]z^{-1} + y[2]z^{-2} + \cdots + y[L+M]z^{-(L+M)}$$

Linear Convolution Using z-Transform

where

$$y[n] = \sum_{k=0}^{L+M} x[k]h[n-k], \quad 0 \le n \le L+M$$

- In the above we have assumed

$$x[n] = 0 \quad \text{for } n > L$$

$$h[n] = 0 \quad \text{for } n > M$$

Linear Convolution Using z-Transform

- **Example** – $X(z) = -2 + z^{-2} - z^{-3} + 3z^{-4}$

  $H(z) = 1 + 2z^{-1} - z^{-3}$

- Therefore

$$Y(z) = (-2 + z^{-2} - z^{-3} + 3z^{-4})(1 + 2z^{-2} - z^{-3})$$

$$= -2 + z^{-2} - z^{-3} + 3z^{-4} - 4z^{-1} + 2z^{-3}$$

$$-2z^{-4} + 6z^{-5} + 2z^{-3} - z^{-5} + z^{-6} - 3z^{-7}$$

Linear Convolution Using z-Transform

$$= -2 + z^{-2} - z^{-3} + 3z^{-4} - 4z^{-1} + 2z^{-3}$$
$$\quad - 2z^{-4} + 6z^{-5} + 2z^{-3} - z^{-5} + z^{-6} - 3z^{-7}$$
$$= -2 - 4z^{-1} + z^{-2} + (2z^{-3} + 2z^{-3} - z^{-3})$$
$$\quad + (3z^{-4} - 2z^{-4}) + (6z^{-5} - z^{-5}) + z^{-6} - 3z^{-7}$$
$$= -2 - 4z^{-1} + z^{-2} + 3z^{-3} + z^{-4}$$
$$\quad + 5z^{-5} + z^{-6} - 3z^{-7}$$

Linear Convolution Using z-Transform

- Hence

$$\{y[n]\} = \{-2, \quad -4, \quad 1, \quad 3, \quad 1, \quad 5, \quad 1, \quad -3\}$$

Circular Convolution Using z-Transform

- Let $\{x[n]\}$ and $\{h[n]\}$ be two length-N sequences defined for $0 \leq n \leq N-1$ with $X(z)$ and $H(z)$ denoting their z-transforms
- Let $y_C[n] = x[n] \, Ⓝ \, h[n]$ denote the N-point circular convolution of $x[n]$ and $h[n]$
- Let $y_L[n] = x[n] \circledast h[n]$ denote the linear convolution of $x[n]$ and $h[n]$

Circular Convolution Using z-Transform

- Let $Y_C(z)$ and $Y_L(z)$ denote the *z*-transforms of $y_C[n]$ and $y_L[n]$
- It can be shown that

$$Y_C(z) = \langle Y_L(z) \rangle_{(z^{-N}-1)}$$

- The modulo operation with respect to $z^{-N} - 1$ is taken by setting $z^{-N} = 1$

Circular Convolution Using z-Transform

- **Example** –

$$G(z) = g[0] + g[1]z^{-1} + g[2]z^{-2} + g[3]z^{-3}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$

- Then

$$Y_L(z) = G(z)H(z)$$

$$= y_L[0] + y_L[1]z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3}$$

$$+ y_L[4]z^{-4} + y_L[5]z^{-5} + y_L[6]z^{-6}$$

Circular Convolution Using z-Transform

where

$$y_L[0] = g[0]h[0]$$

$$y_L[1] = g[0]h[1] + g[1]h[0]$$

$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$

$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$y_L[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$y_L[5] = g[2]h[3] + g[3]h[2]$$

$$y_L[6] = g[3]h[3]$$

Circular Convolution Using z-Transform

- Now $Y_C(z) = \langle Y_L(z) \rangle_{(z^{-4}-1)}$

$$= y_L[0] + y_L[1]z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3} + y_L[4] + y_L[5]z^{-1} + y_L[6]z^{-2}$$

$$= g[0]h[0] + \big(g[0]h[1] + g[1]h[0]\big)z^{-1} + \big(g[0]h[2] + g[1]h[1] + g[2]h[0]\big)z^{-2} + \big(g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]\big)z^{-3} + \big(g[1]h[3] + g[2]h[2] + g[3]h[1]\big) + \big(g[2]h[3] + g[3]h[2]\big)z^{-1} + g[3]h[3]z^{-2}$$

Circular Convolution Using z-Transform

$$= \underbrace{g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]}_{y_C[0]}$$

$$+ \underbrace{\left(g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]\right)}_{y_C[1]} z^{-1}$$

$$+ \underbrace{\left(g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]\right)}_{y_C[2]} z^{-2}$$

$$+ \underbrace{\left(g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]\right)}_{y_C[3]} z^{-3}$$