Orthogonal Transforms

- Let $x[n], 0 \leq n \leq N-1$, denote a length-N time-domain sequence
- Let $\mathcal{X}[k], 0 \leq k \leq N-1$, denote the coefficients of the N-point orthogonal transform of $x[n]$

Orthogonal Transforms

- A general form of the orthogonal transform pair is of the form

$$\mathcal{X}[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k,n], \quad 0 \le k \le N-1$$

Analysis equation

$$x[n] = \sum_{k=0}^{N-1} \mathcal{X}[k]\psi[k,n], \quad 0 \le n \le N-1$$

Synthesis equation

- $\psi[k,n]$, called the basis sequences, are also length-N sequences

Orthogonal Transforms

- In the class of transforms to be considered in this course, the basis sequences satisfy the condition

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi[k,n]\psi^*[\ell,n]=\begin{cases}1, & \ell=k\\ 0, & \ell\neq k\end{cases}$$

- Basis sequences satisfying the above condition are said to be orthogonal to each other

Orthogonal Transforms

- To verify the inverse transform expression

$$x[n] = \sum_{k=0}^{N-1} \mathcal{X}[k]\psi[k,n], \quad 0 \le n \le N-1$$

we substitute it into

$$\mathcal{X}[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k,n], \quad 0 \le k \le N-1$$

Orthogonal Transforms

- The substitution yields

$$\sum_{n=0}^{N-1} x[n]\psi^*[\ell,n] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\psi[k,n]\right)\psi^*[\ell,n]$$

$$= \sum_{k=0}^{N-1} X[k]\left(\frac{1}{N}\sum_{n=0}^{N-1}\psi[k,n]\psi^*[\ell,n]\right) = X[\ell]$$

Orthogonal Transforms

- Energy Preservation Property-

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

- An important consequence of the orthogonality of the basis sequences
- More commonly known as the Parseval's theorem

Discrete Fourier Transform

- Definition - The simplest relation between a length-N sequence $x[n]$, defined for $0 \le n \le N-1$, and its DTFT $X(e^{j\omega})$ is obtained by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \le \omega < 2\pi$ at $\omega_k = 2\pi k / N$, $0 \le k \le N-1$
- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N},$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Note: $X[k]$ is also a length-N sequence in the frequency domain
- The sequence $X[k]$ is called the **discrete Fourier transform** (DFT) of the sequence $x[n]$
- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \; 0 \le k \le N-1$$

Discrete Fourier Transform

- The **inverse discrete Fourier transform** (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \le n \le N-1$$

- To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n = 0$ to $n = N-1$

Discrete Fourier Transform

resulting in

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right) W_N^{\ell n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X[k] W_N^{-(k-\ell)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} \right)$$

Discrete Fourier Transform

- From the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

it follows then that the only non-zero term in

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n}$$

is obtained when $k = \ell$ as $0 \le k, \ell \le N-1$

Discrete Fourier Transform

- Hence

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} \right)$$

$$= \frac{1}{N} \cdot X[\ell] \cdot N = X[\ell]$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$y[n]=\begin{cases}1, & n=m\\ 0, & 0\le n\le m-1,\ m+1\le n\le N-1\end{cases}$$

- Its N-point DFT is given by

$$Y[k]=\sum_{n=0}^{N-1} y[n]W_N^{kn}=y[m]W_N^{km}=W_N^{km}$$

$$0\le k\le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence defined for $0 \le n \le N-1$

$$g[n] = \cos(2\pi rn/N), \;\; 0 \le r \le N-1$$

- Using a trigonometric identity we can write

$$g[n] = \tfrac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \tfrac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

Discrete Fourier Transform

- The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, \quad \text{otherwise} \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N-r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k \le N-1$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \cdots\cdots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots\cdots & x[N-1] \end{bmatrix}^T$$

Matrix Relations

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

Matrix Relations

- Likewise, the IDFT relation given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

Matrix Relations

where

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

- Note:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

DFT Computation Using MATLAB

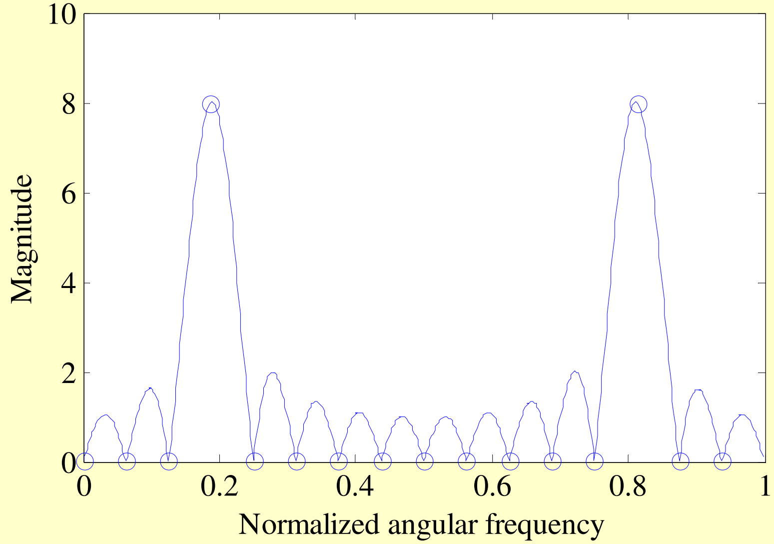
- The functions to compute the DFT and the IDFT are `fft` and `ifft`
- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation
- Programs 5_1.m and 5_2.m illustrate the use of these functions

DFT Computation Using MATLAB

- <u>Example</u> - Program 5_3.m can be used to compute the DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

as shown below



○ indicates DFT samples

DTFT from DFT by Interpolation

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points

$$\omega = \omega_k = 2\pi k/N, \quad 0 \le k \le N-1$$

- Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$

DTFT from DFT by Interpolation

- Thus

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n}}_{S}$$

DTFT from DFT by Interpolation

- To develop a compact expression for the sum S, let

$$r = e^{-j(\omega - 2\pi k/N)}$$

- Then $\mathrm{S} = \sum_{n=0}^{N-1} r^n$
- From the above

$$r\mathrm{S} = \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N-1} r^n + r^N - 1$$

$$= \sum_{n=0}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

DTFT from DFT by Interpolation

- Or, equivalently,

$$S - rS = (1 - r)S = 1 - r^N$$

- Hence

$$S = \frac{1 - r^N}{1 - r} = \frac{1 - e^{-j(\omega N - 2\pi k)}}{1 - e^{-j[\omega - (2\pi k / N)]}}$$

$$= \frac{\sin\left(\dfrac{\omega N - 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k / N)][(N-1)/2]}$$

DTFT from DFT by Interpolation

- Therefore

$$X(e^{j\omega})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

Sampling the DTFT

- Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k/N$, $0 \leq k \leq N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$

Sampling the DTFT

- Now $$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$$
- Thus $$Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$$
$$= \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\ell/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}$$
- An IDFT of $Y[k]$ yields
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}$$

Sampling the DTFT

- i.e. $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-k(n-r)} = \begin{cases} 1, & \text{for } r = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Sampling the DTFT

we arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

Sampling the DTFT

- To apply

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \;\; 0 \le n \le N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros

- Thus if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$

Sampling the DTFT

- If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$
- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

Sampling the DTFT

$$y[n] = x[n] + x[n+4] + x[n-4]\ ,\ 0 \le n \le 3$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

$\{x[n]\}$ cannot be recovered from $\{y[n]\}$

Numerical Computation of the DTFT Using the DFT

- A practical approach to the numerical computation of the DTFT of a finite-length sequence
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x[n]$
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k/M$, $0 \le k \le M-1$, where $M >> N$:

Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n]e^{-j2\pi kn/M}$$

Numerical Computation of the DTFT Using the DFT

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$
- The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2
- The function `freqz` employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$