Classification of DFT

- An N-point DFT $X[k]$ is said to be a circular conjugate-symmetric sequence if

$$X[k] = X^*[\langle -k \rangle_N] = X^*[\langle N-k \rangle_N]$$

- An N-point DFT $X[k]$ is said to be a circular conjugate-antisymmetric sequence if

$$X[k] = -X^*[\langle -k \rangle_N] = -X^*[\langle N-k \rangle_N]$$

Classification Based on Conjugate Symmetry

- A complex DFT $X[k]$ can be expressed as a sum of a circular conjugate symmetric part $X_{cs}[k]$ and a circular conjugate anti-symmetric part $X_{ca}[k]$

$$X[k] = X_{cs}[k] + X_{ca}[k], \quad 0 \le k \le N-1$$

where

$$X_{cs}[k] = \tfrac{1}{2}\left(X[k] + X^*[\langle -k \rangle_N]\right), \quad 0 \le k \le N-1$$

$$X_{ca}[k] = \tfrac{1}{2}\left(X[k] - X^*[\langle -k \rangle_N]\right), \quad 0 \le k \le N-1$$

Classification Based on Geometric Symmetry

- A length-N symmetric sequence $x[n]$ satisfies the condition

$$x[n] = x[N-1-n]$$

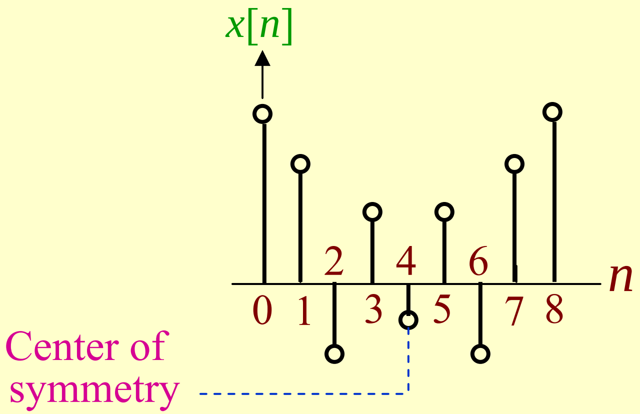
- A length-N antisymmetric sequence $x[n]$ satisfies the condition

$$x[n] = -x[N-1-n]$$

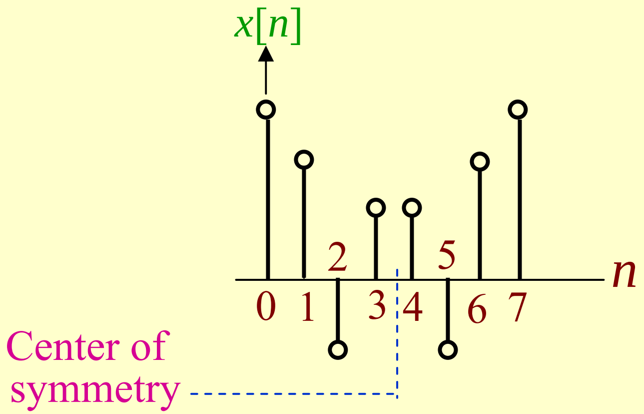
Classification Based on Geometric Symmetry

- Four types of geometric symmetry

Type 1 – Symmetric Sequence with Odd Length

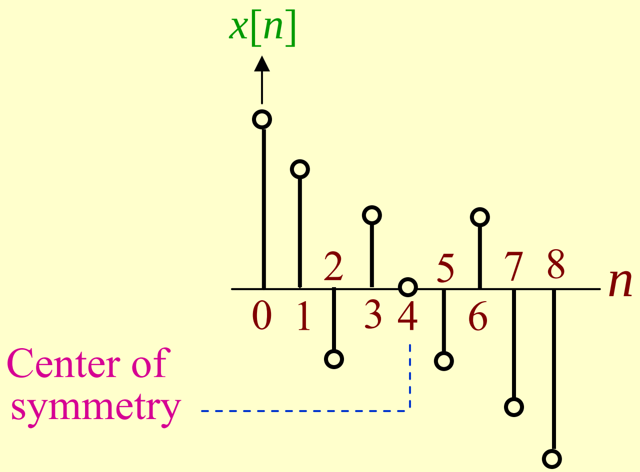


Type 2 – Symmetric Sequence with Even Length

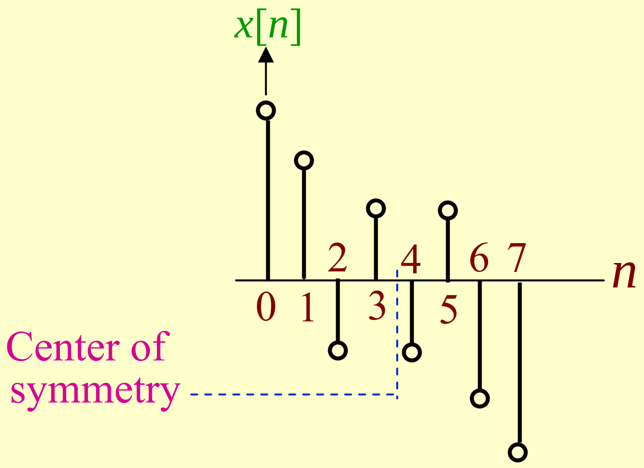


Classification Based on Geometric Symmetry

Type 3 – Antisymmetric Sequence with Odd Length



Type 4 – Antisymmetric Sequence with Even Length



Fourier Transforms of Sequences with Geometric Symmetry

Type 1: Symmetric Sequence with Odd Length

- The DTFT of a length-N symmetric sequence $x[n]$ with N odd is of the form

$$X(e^{j\omega}) = e^{j(N-1)\omega/2}\left\{ x\left[\tfrac{N-1}{2}\right] + 2\sum_{n=1}^{(N-1)/2} x\left[\tfrac{N-1}{2} - n\right]\cos(\omega n) \right\}$$

- The phase $\theta(\omega)$ is given by $\theta(\omega) = -(\tfrac{N-1}{2})\omega + \beta$ where β is either 0 or π

Fourier Transforms of Sequences with Geometric Symmetry

- The N-point DFT $X[k]$ of the Type 1 linear-phase length-N sequence $x[n]$ is obtained by uniformly sampling its DTFT $X(e^{j\omega})$ given in the previous slide at $\omega = 2\pi k/N$, $0 \le k \le N-1$

$$X[k] = e^{j(N-1)\pi k/N} \times$$

$$\left\{ x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2} - n\right]\cos\left(\frac{2\pi k n}{N}\right) \right\}$$

$$0 \le k \le N-1$$

Fourier Transforms of Sequences with Geometric Symmetry

Type 2: Symmetric Sequence with Even Length

- The DTFT of a length-N symmetric sequence $x[n]$ with N even is of the form

$$X(e^{j\omega}) = e^{j(N-1)\omega/2}\left\{2\sum_{n=1}^{N/2} x\left[\tfrac{N}{2}-n\right]\cos(\omega(n-\tfrac{1}{2}))\right\}$$

- The phase $\theta(\omega)$ is given by $\theta(\omega) = -(\frac{N-1}{2})\omega + \beta$ where β is either 0 or π

Fourier Transforms of Sequences with Geometric Symmetry

- The N-point DFT $X[k]$ of the Type 2 linear-phase length-N sequence $x[n]$ is obtained by uniformly sampling its DTFT $X(e^{j\omega})$ given in the previous slide at $\omega = 2\pi k/N$, $0 \le k \le N-1$

$$X[k] = e^{j(N-1)\pi k/N}\left\{2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right]\cos\left(\frac{\pi k(2n-1)}{N}\right)\right\}$$

$$0 \le k \le N-1$$

Fourier Transforms of Sequences with Geometric Symmetry

Type 3: Antisymmetric Sequence with Odd Length

- The DTFT of a length-N antisymmetric sequence $x[n]$ with N odd is of the form

$$X(e^{j\omega}) = je^{j(N-1)\omega/2}\left\{2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2}-n\right]\sin(\omega n)\right\}$$

- The phase $\theta(\omega)$ is given by $\theta(\omega) = -(\frac{N-1}{2})\omega + \frac{\pi}{2} + \beta$ where β is either 0 or π

Fourier Transforms of Sequences with Geometric Symmetry

- The N-point DFT $X[k]$ of the Type 1 linear-phase length-N sequence $x[n]$ is obtained by uniformly sampling its DTFT $X(e^{j\omega})$ given in the previous slide at $\omega = 2\pi k/N$, $0 \le k \le N-1$

$$X[k] = e^{j(N-1)\pi k/N} \times$$

$$\left\{ x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2} - n\right]\cos\left(\frac{2\pi k n}{N}\right)\right\}$$

$$0 \le k \le N-1$$

Fourier Transforms of Sequences with Geometric Symmetry

Type 4: Antisymmetric Sequence with Even Length

- The DTFT of a length-N antisymmetric sequence $x[n]$ with N even is of the form

$$X(e^{j\omega}) = je^{j(N-1)\omega/2}\left\{2\sum_{n=1}^{N/2} x\left[\tfrac{N}{2}-n\right]\sin(\omega(n-\tfrac{1}{2}))\right\}$$

- The phase $\theta(\omega)$ is given by $\theta(\omega) = -(\frac{N-1}{2})\omega + \frac{\pi}{2} + \beta$ where β is either 0 or π

Fourier Transforms of Sequences with Geometric Symmetry

- The N-point DFT $X[k]$ of the Type 1 linear-phase length-N sequence $x[n]$ is obtained by uniformly sampling its DTFT $X(e^{j\omega})$ given in the previous slide at $\omega = 2\pi k/N$, $0 \le k \le N-1$

$$X[k] = e^{j(N-1)\pi k/N} \times$$

$$\left\{ x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2} - n\right] \cos\left(\frac{2\pi k n}{N}\right) \right\}$$

$$0 \le k \le N-1$$

DFT Properties

- Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications
- Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different
- A summary of the DFT properties are given in tables in the following slides

Table 5.1: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

Table 5.2: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{\text{pe}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{po}}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$ $\text{Im}\,X[k] = -\text{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\text{pe}}[n]$ and $x_{\text{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

Table 5.3: DFT Theorems

Theorems	Length-N Sequence	N-point DFT
	$g[n]$	$G[k]$
	$h[n]$	$H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Circular Convolution

- This operation is analogous to linear convolution, but with a subtle difference
- Consider two length-N sequences, $g[n]$ and $h[n]$, respectively
- Their linear convolution results in a length-$(2N-1)$ sequence $y_L[n]$ given by

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \quad 0 \le n \le 2N-2$$

Circular Convolution

- In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to $2N-1$
- The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right
- The first nonzero value of $y_L[n]$ is $y_L[0]=g[0]h[0]$, and the last nonzero value is $y_L[2N-2]=g[N-1]h[N-1]$

Circular Convolution

- To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift
- Resulting operation, called a **circular convolution**, is defined by

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N], \quad 0 \le n \le N-1$$

Circular Convolution

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y[n] = g[n] \, \textcircled{\scriptsize N} \, h[n]$$

- The circular convolution is commutative, i.e.

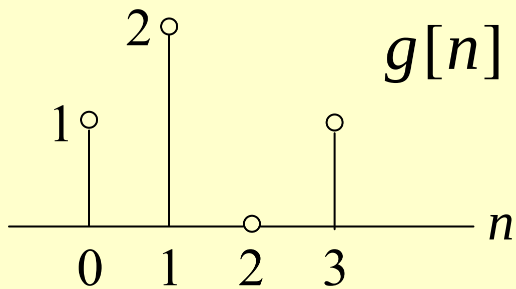
$$g[n] \, \textcircled{\scriptsize N} \, h[n] = h[n] \, \textcircled{\scriptsize N} \, g[n]$$

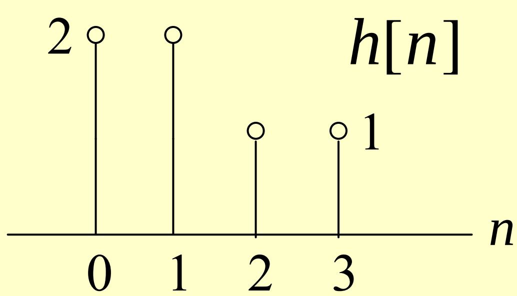
Circular Convolution

- <u>Example</u> - Determine the 4-point circular convolution of the two length-4 sequences:

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \ \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

as sketched below





Circular Convolution

- The result is a length-4 sequence $y_C[n]$ given by

$$y_C[n] = g[n] \, ④ \, h[n] = \sum_{m=0}^{3} g[m] h[\langle n-m \rangle_4],$$

$$0 \le n \le 3$$

- From the above we observe

$$y_C[0] = \sum_{m=0}^{3} g[m] h[\langle -m \rangle_4]$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$= (1 \times 2) + (2 \times 1) + (0 \times 1) + (1 \times 2) = 6$$

Circular Convolution

- Likewise $y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m\rangle_4]$

$$= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= (1\times 2) + (2\times 2) + (0\times 1) + (1\times 1) = 7$$

$$y_C[2] = \sum_{m=0}^{3} g[m]h[\langle 2-m\rangle_4]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) + (1\times 1) = 6$$

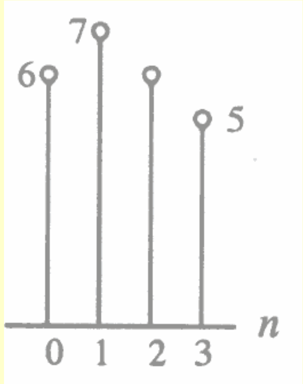
Circular Convolution

and

$$y_C[3] = \sum_{m=0}^{3} g[m]h[\langle 3-m \rangle_4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

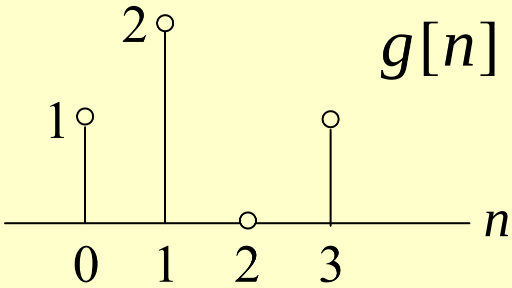


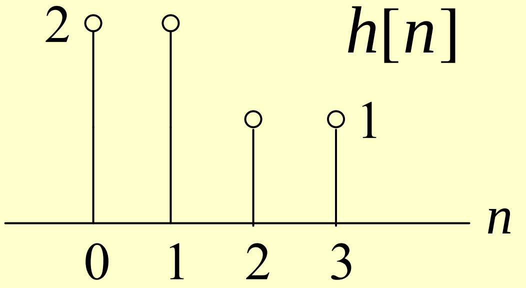
$y_C[n]$

- The circular convolution can also be computed using a DFT-based approach as indicated in Table 5.3

Circular Convolution

- Example - Consider the two length-4 sequences repeated below for convenience:





- The 4-point DFT $G[k]$ of $g[n]$ is given by

$$G[k] = g[0] + g[1]e^{-j2\pi k/4}$$
$$+ g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4}$$
$$= 1 + 2e^{-j\pi k/2} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

Circular Convolution

- Therefore $G[0] = 1 + 2 + 1 = 4,$

$$G[1] = 1 - j2 + j = 1 - j,$$

$$G[2] = 1 - 2 - 1 = -2,$$

$$G[3] = 1 + j2 - j = 1 + j$$

- Likewise,

$$H[k] = h[0] + h[1]e^{-j2\pi k/4}$$

$$+ h[2]e^{-j4\pi k/4} + h[3]e^{-j6\pi k/4}$$

$$= 2 + 2e^{-j\pi k/2} + e^{-j\pi k} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

Circular Convolution

- Hence, $H[0] = 2 + 2 + 1 + 1 = 6,$

$$H[1] = 2 - j2 - 1 + j = 1 - j,$$

$$H[2] = 2 - 2 + 1 - 1 = 0,$$

$$H[3] = 2 + j2 - 1 - j = 1 + j$$

- The two 4-point DFTs can also be computed using the matrix relation given earlier

Circular Convolution

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$\mathbf{D}_4$ is the 4-point DFT matrix

Circular Convolution

- If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then from Table 3.5 we observe

$$Y_C[k] = G[k]H[k], \;\; 0 \le k \le 3$$

- Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Circular Convolution

- A 4-point IDFT of $Y_C[k]$ yields

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\mathbf{D}_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

Circular Convolution

- Example - Now let us extended the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

Circular Convolution

- We next determine the 7-point circular convolution of $g_e[n]$ and $h_e[n]$:

$$y[n] = \sum_{m=0}^{6} g_e[m] h_e[\langle n-m \rangle_7], \quad 0 \le n \le 6$$

- From the above $y[0] = g_e[0]h_e[0] + g_e[1]h_e[6]$

$$+ g_e[3]h_e[4] + g_e[4]h_e[3] + g_e[5]h_e[2] + g_e[6]h_e[1]$$

$$= g[0]h[0] = 1 \times 2 = 2$$

Circular Convolution

- Continuing the process we arrive at

$$y[1] = g[0]h[1] + g[1]h[0] = (1 \times 2) + (2 \times 2) = 6,$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$

$$= (1 \times 1) + (2 \times 2) + (0 \times 2) = 5,$$

$$y[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1 \times 1) + (2 \times 1) + (0 \times 2) + (1 \times 2) = 5,$$

$$y[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$

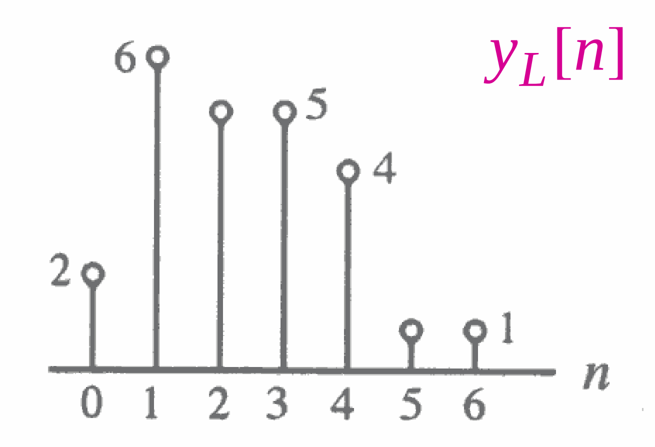
$$= (2 \times 1) + (0 \times 1) + (1 \times 2) = 4,$$

Circular Convolution

$$y[5] = g[2]h[3] + g[3]h[2] = (0\times 1) + (1\times 1) = 1,$$

$$y[6] = g[3]h[3] = (1\times 1) = 1$$

- As can be seen from the above that $y[n]$ is precisely the sequence $y_L[n]$ obtained by a linear convolution of $g[n]$ and $h[n]$



Circular Convolution

- The N-point circular convolution can be written in matrix form as

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

- Note: The elements of each diagonal of the $N \times N$ matrix are equal
- Such a matrix is called a **circulant matrix**

Circular Convolution

- Tabular Method
- We illustrate the method by an example
- Consider the evaluation of $y[n] = h[n] \circledast g[n]$ where $\{g[n]\}$ and $\{h[n]\}$ are length-4 sequences
- First, the samples of the two sequences are multiplied using the conventional multiplication method as shown on the next slide

Circular Convolution

$n:$	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$g[n]:$	$g[0]$	$g[1]$	$g[2]$	$g[3]$			
$h[n]$	$h[0]$	$h[1]$	$h[2]$	$h[3]$			
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$			
		$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$\boxed{g[3]h[1]}$		
			$g[0]h[2]$	$g[1]h[2]$	$\boxed{g[2]h[2]}$	$\boxed{g[3]h[2]}$	
				$g[0]h[3]$	$\boxed{g[1]h[3]}$	$\boxed{g[2]h[3]}$	$\boxed{g[3]h[3]}$

The partial products generated in the 2nd, 3rd, and 4th rows are circularly shifted to the left as indicated above

Circular Convolution

- The modified table after circular shifting is shown below

n:	0	1	2	3
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$
$h[n]$:	$h[0]$	$h[1]$	$h[2]$	$h[3]$
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$
	$g[3]h[1]$	$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$
	$g[2]h[2]$	$g[3]h[2]$	$g[0]h[2]$	$g[1]h[2]$
	$g[1]h[3]$	$g[2]h[3]$	$g[3]h[3]$	$g[0]h[3]$
$y_C[n]$:	$y_C[0]$	$y_C[1]$	$y_C[2]$	$y_C[3]$

- The samples of the sequence $\{y_c[n]\}$ are obtained by adding the 4 partial products in the column above of each sample

Circular Convolution

- Thus

$$y_c[0] = g[0]h[0] + g[3]h[1] + g[2]h[2] + g[1]h[3]$$

$$y_c[1] = g[1]h[0] + g[0]h[1] + g[3]h[2] + g[2]h[3]$$

$$y_c[2] = g[2]h[0] + g[1]h[1] + g[0]h[2] + g[3]h[3]$$

$$y_c[3] = g[3]h[0] + g[2]h[1] + g[1]h[2] + g[0]h[3]$$

Circular Convolution

- **Example** – Let $\{g[n]\} = \{1,\ 2,\ 0,\ 1\}$ and $\{g[n]\} = \{1,\ 2,\ 0,\ 1\}$

  We determine $\{y_{\mathrm{c}}[n]\} = \{g[n]\} ④ \{h[n]\}$ using the tabular method
- We first multiply the samples of the two sequences using the conventional multiplication method

Circular Convolution

$$
\begin{array}{c|ccccccc}
n: & 0 & 1 & 2 & 3 & \langle 4\rangle_4 & \langle 5\rangle_4 & \langle 6\rangle_4 \\
\hline
h[n]: & 1 & 2 & 0 & 1 & & & \\
h[n]: & 2 & 2 & 1 & 1 & & & \\
\hline
 & 2 & 4 & 0 & 2 & & & \\
 & & 2 & 4 & 0 & 2 & & \\
 & & & 1 & 2 & 0 & 1 & \\
 & & & & 1 & 2 & 0 & 1
\end{array}
$$

- Since $\langle 4\rangle_4 = 0$, the number 2 in line 2 at position $n = \langle 4\rangle_4$ is next moved to position $n = 0$

Circular Convolution

n:	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$h[n]$:	1	2	0	1			
$h[n]$:	2	2	1	1			
	2	4	0	2			
	2	2	4	0			
			1	2	0	1	
				1	2	0	1

- Next, the number 0 in line 3 at position $n = \langle 4\rangle_4$ is next moved to position $n = 0$ and the number 1 is moved to position $n = \langle 5\rangle_4 = 1$

Circular Convolution

n:	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$h[n]$:	1	2	0	1			
$h[n]$:	2	2	1	1			
	2	4	0	2			
	2	2	4	0			
	0	1	1	2			
				1	2	0	1

- Next, the number 2 in line 4 at position $n = \langle 4\rangle_4$ is next moved to position $n = 0$, the number 0 is moved to position $n = \langle 5\rangle_4 = 1$ and the number 1 is moved to position $n = \langle 6\rangle_4 = 2$

Circular Convolution

n:	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$g[n]$:	1	2	0	1			
$h[n]$:	2	2	1	1			
	2	4	0	2			
	2	2	4	0			
	0	1	1	2			
	2	0	1	1			
$y_c[n]$:	6	7	6	5			