Computer-Aided Design of IIR Digital Filter

- The IIR filter design algorithms discussed so far are used in applications requiring filters with a frequency-selective magnitude response having either a lowpass or a highpass or a bandpass or a bandstop characteristics
- Designing IIR filters with other types of frequency responses usually involve the use of some type of iterative optimization techniques that are used to minimize the error between the desired response and that of the computer-generated filter

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Computer-Aided Design of IIR Digital Filter

- **Basic Idea -**
- Let $G(e^{j\omega})$ denote the frequency response of the computer generated transfer function $G(z)$
- Let $D(e^{j\omega})$ denote the desired frequency response
- Objective is to design $G(z)$ so that $G(e^{j\omega})$ approximates $D(e^{j\omega})$ in some sense

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- Usually the difference between $G(e^{j\omega})$ and $D(e^{j\omega})$ specified as a weighted error function $\mathcal{E}(\omega)$:

$$\mathcal{E}(\omega) = W(e^{j\omega})[G(e^{j\omega}) - D(e^{j\omega})]$$

is minimized for all values of ω over closed subintervals of $0 \le \omega \le \pi$

where $W(e^{j\omega})$ is some user-specified positive weighting function

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- A commonly used approximation measure, called Chebyshev or minimax criterion, is to minimize the peak absolute value of $\mathcal{E}(\omega)$ given by

$$\varepsilon = \max_{\omega \in R} \left| \mathcal{E}(\omega) \right|$$

where R is the set of disjoint frequency bands in the range $0 \leq \omega \leq \pi$, on which the desired frequency response is defined

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- In filtering applications, R is composed of the desired passbands and stopbands of the filter to be designed
- For example, for a lowpass filter design, R is the disjoint union of the frequency ranges $[0, \omega_p]$ and $[\omega_s, \pi]$, where ω_p and ω_s are, respectively, the passband and stopband edges

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- Another approximation measure, called the least-p criterion, is to minimize the integral of the p-th power of $\mathcal{E}(\omega)$:

$$\varepsilon = \int_{\omega \in R} \left| W(e^{j\omega})[G(e^{j\omega}) - D(e^{j\omega})] \right|^{p} d\omega$$

over the specified frequency range R with p a positive integer

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- The least-squares criterion obtained with $p = 2$ is often used for simplicity
- It can be shown that as $p \to \infty$, the least p-th solution approaches the minimax solution
- In practice, the integral error measure is approximated by a finite sum given by

$$\varepsilon = \sum_{i=1}^{K} \left| W(e^{j\omega_i})[G(e^{j\omega_i}) - D(e^{j\omega_i})] \right|^p$$

where ω_i, $1 \le i \le K$, is a suitably chosen dense grid of digital angular frequencies

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IIR Digital Filter Design Using Optimization Techniques

- In the case of IIR digital filter design, $G(e^{j\omega})$ and $D(e^{j\omega})$ are replaced with their magnitude functions
- Consider the real rational transfer function

$$G(z) = C\frac{1 + p_1 z^{-1} + \ldots + p_M z^{-M}}{1 + d_1 z^{-1} + \ldots + d_N z^{-N}}$$

where the coefficients p_ℓ and d_ℓ are real, and the gain constant C is a positive number

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IIR Digital Filter Design Using Optimization Techniques

- Let $|G(\mathbf{x},\omega)|$ denote the magnitude response of $G(z)$ where $\mathbf{x}$ represents the column vector of the adjustable parameters consisting of the filter coefficients p_ℓ and d_ℓ, the gain constant C; that is,

$$\mathbf{x} = [p_1,\ p_2,\ \ldots,\ p_M,\ d_1,\ d_2,\ \ldots,\ d_N,\ C]$$

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IIR Digital Filter Design Using Optimization Techniques

- The approximation error at a given frequency ω is given by the weighted difference between $|G(\mathbf{x},\omega)|$ and the desired magnitude response $|D(e^{j\omega})|$:

$$\varepsilon(\mathbf{x},\omega) = W(\omega)(|G(\mathbf{x},\omega)| - |D(\omega)|)$$

with $W(\omega)$ denoting the user specified weighting function

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IIR Digital Filter Design Using Optimization Techniques

- The error vector is obtained by evaluating the error $\varepsilon(\mathbf{x},\omega)$ at a dense grid of frequency points, $\omega_1, \omega_2, \ldots, \omega_K$:

$$\mathbf{E}(\mathbf{x}) = [\varepsilon(\mathbf{x},\omega_1), \varepsilon(\mathbf{x},\omega_2), \ldots, \varepsilon(\mathbf{x},\omega_K)]$$

- The filter design method is based on the adjustment of the parameters of $\mathbf{x}$ iteratively until a set of values $\mathbf{x} = \hat{\mathbf{x}}$ is found for which $\varepsilon(\hat{\mathbf{x}},\omega_i) \approx 0$ for $i = 0, 1, \ldots, K$

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IIR Digital Filter Design Using Optimization Techniques

- A commonly used approximation measure is the least-*p*th objective function given by

$$\varepsilon(\mathbf{x}) = \hat{\mathbf{E}}(\mathbf{x})\left\{\sum_{i=1}^{K}\left[\frac{\varepsilon(\mathbf{x},\omega_i)}{\hat{\mathbf{E}}(\mathbf{x})}\right]^p\right\}^{1/p}$$

where

$$\hat{\mathbf{E}}(\mathbf{x}) = \max_{1\le i\le K}|\varepsilon(\mathbf{x},\omega_i)|$$

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IIR Digital Filter Design Using Optimization Techniques

- The MATLAB function `iirlpnorm` employs am uncostrained quasi-Newton algorithm to minimize the objective function $\varepsilon(\mathbf{x})$
- If at any stage of the iteration process one or more poles and/or zeros of $G(z)$ lie outside the unit circle, they are reflected back to inside the unit circle, which does not change the magnitude function $|G(e^{j\omega})|$

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Group Delay Equalization of IIR Digital Filter

- For a distortion-free transmission of an input signal in a prescribed frequency range through a digital filter, the transfer function of the filter should exhibit a unity magnitude response and a linear-phase response, i.e., a constant group delay, in the frequency band of interest

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- The IIR digital filter design methods discussed so far lead to transfer functions with nonlinear phase responses
- Thus, to arrive at a frequency selective IIR digital filter with a constant group delay, a practical approach is to cascade the IIR digital filter meeting the magnitude response specifications with an allpass filter so that the overall group delay is a constant group delay in the band of interest

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Group Delay Equalization of IIR Digital Filter

- The allpass delay equalizer is usually designed using a computer-aided optimization method
- We outline one such method next
- Let $H(z)$ be the IIR transfer function expressed in the factored form:

$$H(z) = \frac{p_0}{d_0} \frac{\prod_{\ell=1}^{N_1/2}(1 + p_{1\ell} z^{-1} + p_{2\ell} z^{-2})}{\prod_{\ell=1}^{N_2/2}(1 + d_{1\ell} z^{-1} + d_{2\ell} z^{-2})}$$

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- Without any loss of generality, assume that the degrees N_1 and N_2 of the numerator and denominator polynomials are even
- Its group delay $\tau_g^H(\omega)$ is given by

$$\tau_g^H(\omega)$$

$$= -\sum_{\ell=1}^{N_1/2} \frac{1 - p_{2\ell}^2 + p_{1\ell}(1 - p_{2\ell})\cos\omega}{p_{2\ell}^2 + p_{1\ell}^2 + 1 + 2p_{2\ell}(2\cos^2\omega - 1) + 2p_{1\ell}(1 + p_{2\ell})\cos\omega}$$

$$+ \sum_{\ell=1}^{N_2/2} \frac{1 - d_{2\ell}^2 + d_{1\ell}(1 - d_{2\ell})\cos\omega}{d_{2\ell}^2 + d_{1\ell}^2 + 1 + 2d_{2\ell}(2\cos^2\omega - 1) + 2d_{1\ell}(1 + d_{2\ell})\cos\omega}$$

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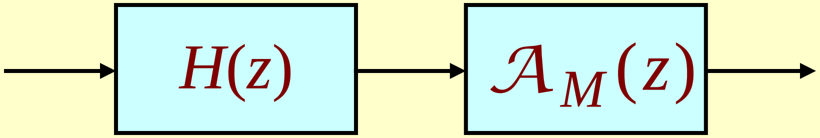
- Our objective is to design a stable allpass section $\mathcal{A}_M(z)$ so that the overall group delay $\tau_g^{\mathcal{A}}(\omega)$ of the cascaded system



is approximately constant in the passband $\omega_L \leq \omega \leq \omega_H$ of $H(z)$

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- Let the transfer function of the allpass section be given by

$$\mathcal{A}_M(z) = \prod_{\ell=1}^{M/2} \frac{a_{2\ell} + a_{1\ell} z^{-1} + z^{-2}}{1 + a_{1\ell} z^{-1} + a_{2\ell} z^{-2}}$$

where we have assumed the order M to be even

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- The group delay $\tau_g^{\mathcal{A}}(\omega)$ of the allpass section is given by

$$\tau_g^{\mathcal{A}}(\omega)$$

$$= \sum_{\ell=1}^{M/2} \frac{1 - a_{2\ell}^2 + a_{1\ell}(1 - a_{2\ell})\cos\omega}{a_{2\ell}^2 + a_{1\ell}^2 + 1 + 2a_{2\ell}(2\cos^2\omega - 1) + 2a_{1\ell}(1 + d_{2\ell})\cos\omega}$$

- Hence, the overall group delay of the cascade of $H(z)$ and $\mathcal{A}_M(z)$ is given by

$$\tau_g^{H\mathcal{A}}(\omega) = \tau_g^{H}(\omega) + \tau_g^{\mathcal{A}}(\omega)$$

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- We need to determine the coefficients $a_{1\ell}$ and $a_{2\ell}$ of the allpass section so that the overall group delay $\tau_g^{H\mathcal{A}}(\omega)$ is approximately a constant τ_o in the passband of $H(z)$
- Moreover, to guarantee the stability of the allpass section we need to ensure that $|a_{2\ell}|<1$, $|a_{1\ell}|<1+a_{2\ell}$ for $1\le \ell \le M/2$

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- The approximation error at a given frequency ω is given by the difference between the overall group delay $\tau_g^{H\mathcal{A}}(\omega)$ and the constant group delay τ_o :

$$\varepsilon(\mathbf{x},\omega) = \tau_g^{H\mathcal{A}}(\omega) - \tau_o$$

where $\mathbf{x}$ is the column vector of the allpass section coefficients and τ_o:

$$\mathbf{x} = [a_{11},\ a_{21},\ \ldots,\ a_{1,M/2},\ \ a_{2,M/2},\ \tau_o]$$

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- The error vector is obtained by evaluating the error $\varepsilon(\mathbf{x},\omega)$ at a dense grid of frequencies $\omega_1, \omega_2, \ldots, \omega_K$ in the passband of $H(z)$:

$$\mathbf{E}(\mathbf{x}) = [\,\varepsilon(\mathbf{x},\omega_1),\ \ \varepsilon(\mathbf{x},\omega_2),\ \ldots,\ \ \varepsilon(\mathbf{x},\omega_K),]^t$$

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- The allpass delay equalizer design can be formulated as a minimax optimization problem in which we minimize the peak value of the error by adjusting the parameters of $\mathbf{x}$ in the passband of $H(z)$

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- A measure of the goodness of the final design has been defined as

$$Q = \frac{100(\bar{\tau}_g^{H\mathcal{A}} - \hat{\tau}_g^{H\mathcal{A}})}{2\tilde{\tau}_g^{H\mathcal{A}}}$$

where

$$\bar{\tau}_g^{H\mathcal{A}}(\omega) = \max_{\omega_L \le \omega \le \omega_H} \tau_g^{H\mathcal{A}}(\omega)$$

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$$\hat{\tau}_g^{H\mathcal{A}}(\omega) = \min_{\omega_L \le \omega \le \omega_H} \tau_g^{H\mathcal{A}}(\omega)$$

and

$$\tilde{\tau}_g^{H\mathcal{A}}(\omega) = \frac{1}{2}\left(\bar{\tau}_g^{H\mathcal{A}}(\omega) + \hat{\tau}_g^{H\mathcal{A}}(\omega)\right)$$

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Group Delay Equalization of IIR Digital Filter

- The optimization method may lead to an unstable equalizer
- However, replacing poles outside the unit circle with their mirror-images inside the unit circle will not work as the process also changes the group delay response of the equalizer

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- One possible solution is to start with a single second-order equalizer section and carry out the optimization until a stable design is reached
- Then add another second-order equalizer section and adjust its coefficients to arrive at a stable equalier

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- Continue this process until a desriable value of the quality factor Q is reached
- The M-file `iirgrpdelay` can be used to design the allpass delay equalizer which is availble with several versions
- We illustrate its use next

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- **Example** – We design an 8-th order allpass equalizer to equalize the group delay of a 4-th order elliptic lowpass filter with a passband edge at 0.3π, passband ripple of 1 dB and a minimum stopband attenuation of 30 dB using Program 9_4.m
- The group delays of the lowpass filter and the overall cascade are shown on the next slide

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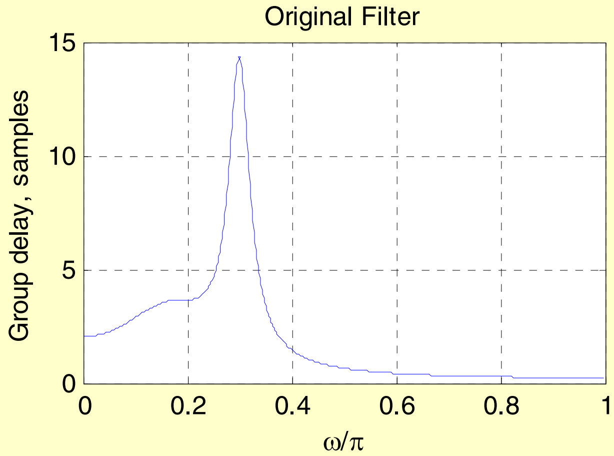

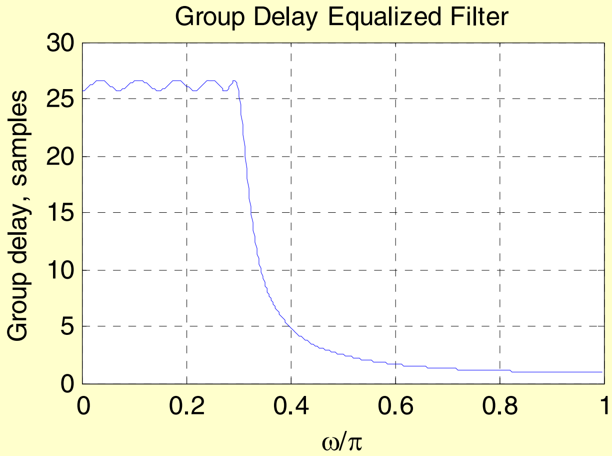

It can be shown that the designed allpass filter is stable

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