# Time-Domain Characterization of LTI Discrete-Time System

- **Input-Output Relationship -**
  A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely characterized by its impulse response
- ⇨ Knowing the impulse response one can compute the output of the system for any arbitrary input



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# Time-Domain Characterization of LTI Discrete-Time System

- Let $h[n]$ denote the impulse response of a LTI discrete-time system
- We compute its output $y[n]$ for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

- As the system is linear, we can compute its outputs for each member of the input separately and add the individual outputs to determine $y[n]$



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# Time-Domain Characterization of LTI Discrete-Time System

- Since the system is time-invariant

input output

$$\delta[n+2] \rightarrow h[n+2]$$

$$\delta[n-1] \rightarrow h[n-1]$$

$$\delta[n-2] \rightarrow h[n-2]$$

$$\delta[n-5] \rightarrow h[n-5]$$



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# Time-Domain Characterization of LTI Discrete-Time System

- Likewise, as the system is linear

input output

$$0.5\delta[n+2] \rightarrow 0.5h[n+2]$$

$$1.5\delta[n-1] \rightarrow 1.5h[n-1]$$

$$-\delta[n-2] \rightarrow -h[n-2]$$

$$0.75\delta[n-5] \rightarrow 0.75h[n-5]$$

- Hence because of the linearity property we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$



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# Time-Domain Characterization of LTI Discrete-Time System

- Now, any arbitrary input sequence $x[n]$ can be expressed as a linear combination of delayed and advanced unit sample sequences in the form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The response of the LTI system to an input $x[k]\delta[n-k]$ will be $x[k]h[n-k]$



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# Time-Domain Characterization of LTI Discrete-Time System

- Hence, the response $y[n]$ to an input

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which can be alternately written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$



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# Convolution Sum

- The summation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[n]$$

is thus the **convolution sum** of the sequences $x[n]$ and $h[n]$ and represented compactly as

$$y[n] = x[n] \circledast h[n]$$



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# Convolution Sum

- **Example** – Consider an LTI discrete-time system with an impulse response $h[n]$ generating an output $y[n]$ for a input $x[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \circledast h[n]$$

- We determine the output $y_1[n]$ of an LTI discrete-time system with an impulse response $h[n-m]$ for the same input $x[n]$



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# Convolution Sum

- Now

$$y_1[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-m-k] = x[n] \circledast h[n-m]$$

- Hence,

$$y_1[n] = y[n-m]$$



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# Convolution Sum

- **Properties -**
- Commutative property:

$$x[n] \circledast h[n] = h[n] \circledast x[n]$$

- Associative property :

$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

- Distributive property :

$$x[n] \circledast (h[n] + y[n]) = x[n] \circledast h[n] + x[n] \circledast y[n]$$



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# Convolution Sum

- **Interpretation -**
- 1) Time-reverse $h[k]$ to form $h[-k]$
- 2) Shift $h[-k]$ to the right by $n$ sampling periods if $n > 0$ or shift to the left by $n$ sampling periods if $n < 0$ to form $h[n-k]$
- 3) Form the product $v[k] = x[k]h[n-k]$
- 4) Sum all samples of $v[k]$ to develop the $n$-th sample of $y[n]$ of the convolution sum



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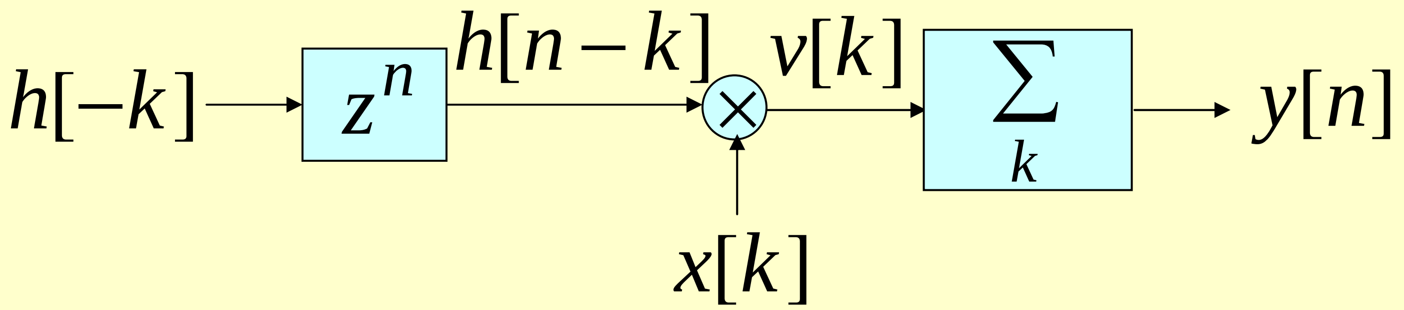
# Convolution Sum

- **Schematic Representation -**




- The computation of an output sample using the convolution sum is simply a sum of products
- Involves fairly simple operations such as additions, multiplications, and delays



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# Time-Domain Characterization of LTI Discrete-Time System

- In practice, if either the input or the impulse response is of finite length, the convolution sum can be used to compute the output sample as it involves a finite sum of products
- If both the input sequence and the impulse response sequence are of finite length, the output sequence is also of finite length



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# Time-Domain Characterization of LTI Discrete-Time System

- If both the input sequence and the impulse response sequence are of infinite length, convolution sum cannot be used to compute the output
- For systems characterized by an infinite impulse response sequence, an alternate time-domain description involving a finite sum of products will be considered


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# Tabular Method of Convolution Sum Computation

- Can be used to convolve two finite-length sequences
- Consider the convolution of $\{g[n]\}, 0 \le n \le 3$, with $\{h[n]\}, 0 \le n \le 2$, generating the sequence $y[n] = g[n] \circledast h[n]$
- Samples of $\{g[n]\}$ and $\{h[n]\}$ are then multiplied using the conventional multiplication method without any carry operation



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# Tabular Method of Convolution Sum Computation

| $n$: | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $g[n]$: | $g[0]$ | $g[1]$ | $g[2]$ | $g[3]$ | | |
| $h[n]$: | $h[0]$ | $h[1]$ | $h[2]$ | | | |
| | $g[0]h[0]$ | $g[1]h[0]$ | $g[2]h[0]$ | $g[3]h[0]$ | | |
| | | $g[0]h[1]$ | $g[1]h[1]$ | $g[2]h[1]$ | $g[3]h[1]$ | |
| | | | $g[0]h[2]$ | $g[1]h[2]$ | $g[2]h[2]$ | $g[3]h[2]$ |
| $y[n]$: | $y[0]$ | $y[1]$ | $y[2]$ | $y[3]$ | $y[4]$ | $y[5]$ |

- The samples $y[n]$ generated by the convolution sum are obtained by adding the entries in the column above each sample



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# Tabular Method of Convolution Sum Computation

- The samples of $\{y[n]\}$ are given by

$$y[0] = g[0]h[0]$$

$$y[1] = g[1]h[0] + g[0]h[1]$$

$$y[2] = g[2]h[0] + g[1]h[1] + g[0]h[2]$$

$$y[3] = g[3]h[0] + g[2]h[1] + g[1]h[2]$$

$$y[4] = g[3]h[1] + g[2]h[2]$$

$$y[5] = g[3]h[2]$$


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# Tabular Method of Convolution Sum Computation

- The method can also be applied to convolve two finite-length two-sided sequences
- In this case, a decimal point is first placed to the right of the sample with the time index $n = 0$ for each sequence
- Next, convolution is computed ignoring the location of the decimal point



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# Tabular Method of Convolution Sum Computation

- Finally, the decimal point is inserted according to the rules of conventional multiplication
- The sample immediately to the left of the decimal point is then located at the time index $n = 0$



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# Convolution Using MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences
- If $\mathtt{a} = [-2 \quad 0 \quad 1 \quad -1 \quad 3]$

  $\mathtt{b} = [1 \quad 2 \quad 0 \quad -1]$

  then `conv(a,b)` yields

  $$[-2 \quad -4 \quad 1 \quad 3 \quad 1 \quad 5 \quad 1 \quad -3]$$



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# Stability Condition of an LTI Discrete-Time System

- **BIBO Stability Condition** - A discrete-time is BIBO stable if and only if the output sequence $\{y[n]\}$ remains bounded for all bounded input sequence $\{x[n]\}$
- An LTI discrete-time system is BIBO stable if and only if its impulse response sequence $\{h[n]\}$ is absolutely summable, i.e.

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$



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# Stability Condition of an LTI Discrete-Time System

- Proof: Assume $h[n]$ is a real sequence
- Since the input sequence $x[n]$ is bounded we have

$$|x[n]| \leq B_x < \infty$$

- Therefore

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\leq B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x S$$



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# Stability Condition of an LTI Discrete-Time System

- Thus, $S < \infty$ implies $|y[n]| \le B_y < \infty$ indicating that $y[n]$ is also bounded
- To prove the converse, assume $y[n]$ is bounded, i.e., $|y[n]| \le B_y$
- Consider the input given by

$$x[n] = \begin{cases} \mathrm{sgn}(h[-n]), & \text{if } h[-n] \neq 0 \\ K, & \text{if } h[-n] = 0 \end{cases}$$



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# Stability Condition of an LTI Discrete-Time System

where $\mathrm{sgn}(c) = +1$ if $c > 0$ and $\mathrm{sgn}(c) = -1$ if $c < 0$ and $|K| \leq 1$

- Note: Since $|x[n]| \leq 1$, $\{x[n]\}$ is obviously bounded

- For this input, $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} \mathrm{sgn}(h[k])h[k] = S \leq B_y < \infty$$

- Therefore, $|y[n]| \leq B_y$ implies $S < \infty$



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# Stability Condition of an LTI Discrete-Time System

- Example - Consider an LTI discrete-time system with an impulse response

$$h[n] = (\alpha)^n \mu[n]$$

- For this system

$$S = \sum_{n=-\infty}^{\infty} \left|\alpha^n\right| \mu[n] = \sum_{n=0}^{\infty} \left|\alpha\right|^n = \frac{1}{1-\left|\alpha\right|} \quad \text{if } \left|\alpha\right| < 1$$

- Therefore $S < \infty$ if $\left|\alpha\right| < 1$ for which the system is BIBO stable
- If $\left|\alpha\right| = 1$, the system is not BIBO stable


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# Causality Condition of an LTI Discrete-Time System

- Let $x_1[n]$ and $x_2[n]$ be two input sequences with

$$x_1[n] = x_2[n] \quad \text{for} \quad n \leq n_o$$

$$x_1[n] \neq x_2[n] \quad \text{for} \quad n > n_o$$

- The corresponding output samples at $n = n_o$ of an LTI system with an impulse response $\{h[n]\}$ are then given by



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# Causality Condition of an LTI Discrete-Time System

$$y_1[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_1[n_o - k] = \sum_{k=0}^{\infty} h[k]x_1[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_1[n_o - k]$$

$$y_2[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_2[n_o - k] = \sum_{k=0}^{\infty} h[k]x_2[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$



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# Causality Condition of an LTI Discrete-Time System

- If the LTI system is also causal, then

$$y_1[n_o] = y_2[n_o]$$

- As $x_1[n] = x_2[n]$ for $n \le n_o$

$$\sum_{k=0}^{\infty} h[k]x_1[n_o - k] = \sum_{k=0}^{\infty} h[k]x_2[n_o - k]$$

- This implies

$$\sum_{k=-\infty}^{-1} h[k]x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$



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# Causality Condition of an LTI Discrete-Time System

- As $x_1[n] \neq x_2[n]$ for $n > n_o$ the only way the condition

$$\sum_{k=-\infty}^{-1} h[k] x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k] x_2[n_o - k]$$

will hold if both sums are equal to zero, which is satisfied if

$$h[k] = 0 \quad \text{for } k < 0$$



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# Causality Condition of an LTI Discrete-Time System

- An LTI discrete-time system is **causal** if and only if its impulse response $\{h[n]\}$ is a causal sequence
- Example - The discrete-time system defined by

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

is a causal system as it has a causal impulse response $\{h[n]\} = \{\underset{\uparrow}{\alpha_1} \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\}$



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# Causality Condition of an LTI Discrete-Time System

- Example - The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$



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# Causality Condition of an LTI Discrete-Time System

- <u>Example</u> - The factor-of-2 interpolator defined by

$$y[n] = x_u[n] + \tfrac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

is noncausal as it has a noncausal impulse response given by

$$\{h[n]\} = \{0.5 \quad \underset{\uparrow}{1} \quad 0.5\}$$



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# Causality Condition of an LTI Discrete-Time System

- Note: A noncausal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay
- For example, a causal version of the factor-of-2 interpolator is obtained by delaying the input by one sample period:

$$y[n] = x_u[n-1] + \tfrac{1}{2}\left(x_u[n-2] + x_u[n]\right)$$



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# Simple Interconnection Schemes

- Two simple interconnection schemes are:
- Cascade Connection
- Parallel Connection



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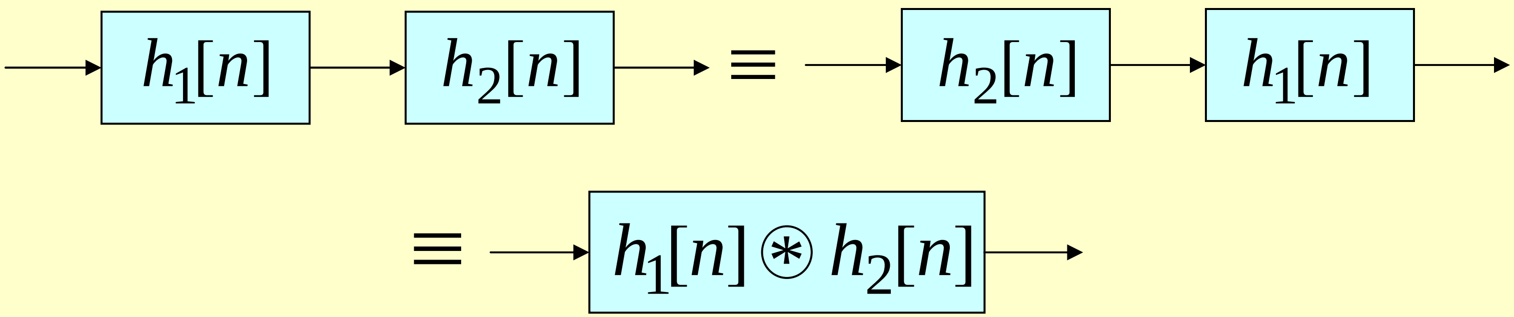
# Cascade Connection




- Impulse response $h[n]$ of the cascade of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] \circledast h_2[n]$$



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# Cascade Connection

- Note: The ordering of the systems in the cascade has no effect on the overall impulse response because of the commutative property of convolution
- A cascade connection of two stable systems is stable
- A cascade connection of two passive (lossless) systems is passive (lossless)



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# Cascade Connection

- An application is in the development of an **inverse system**
- If the cascade connection satisfies the relation

$$h_1[n] \circledast h_2[n] = \delta[n]$$

then the LTI system $h_1[n]$ is said to be the inverse of $h_2[n]$ and vice-versa



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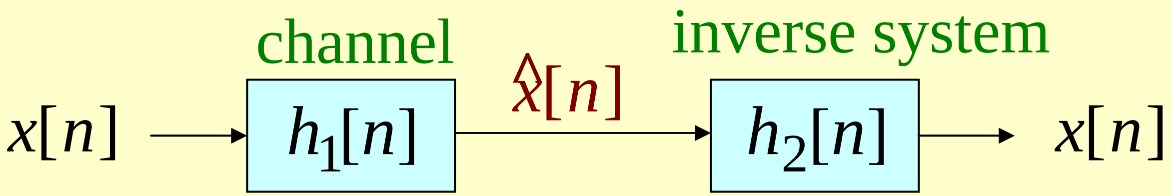
# Cascade Connection

- An application of the inverse system concept is in the recovery of a signal $x[n]$ from its distorted version $\hat{x}[n]$ appearing at the output of a transmission channel
- If the impulse response of the channel is known, then $x[n]$ can be recovered by designing an inverse system of the channel




$$h_1[n] \circledast h_2[n] = \delta[n]$$



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# Cascade Connection

- Example - Consider the discrete-time accumulator with an impulse response $\mu[n]$
- Its inverse system satisfy the condition

$$\mu[n] \circledast h_2[n] = \delta[n]$$

- It follows from the above that $h_2[n] = 0$ for $n < 0$ and

$$h_2[0] = 1$$

$$\sum_{\ell=0}^{n} h_2[\ell] = 0 \quad \text{for} \quad n \geq 1$$



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# Cascade Connection

- Thus the impulse response of the inverse system of the discrete-time accumulator is given by

$$h_2[n] = \delta[n] - \delta[n-1]$$

which is called a **backward difference system**



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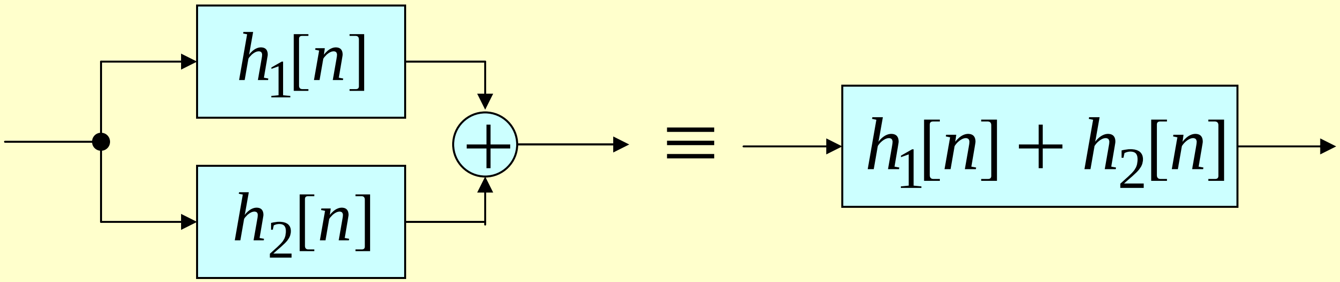
# Parallel Connection




- Impulse response $h[n]$ of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] + h_2[n]$$



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# Simple Interconnection Schemes

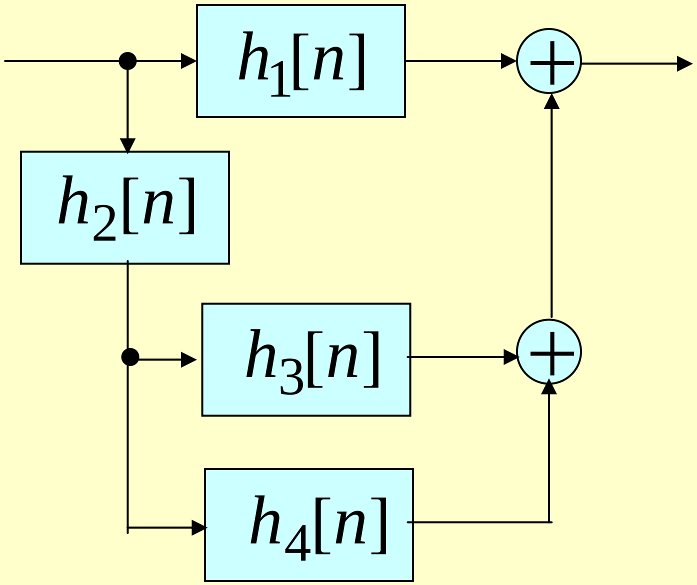
- Consider the discrete-time system where

$$h_1[n] = \delta[n] + 0.5\delta[n-1],$$

$$h_2[n] = 0.5\delta[n] - 0.25\delta[n-1],$$

$$h_3[n] = 2\delta[n],$$

$$h_4[n] = -2(0.5)^n \mu[n]$$



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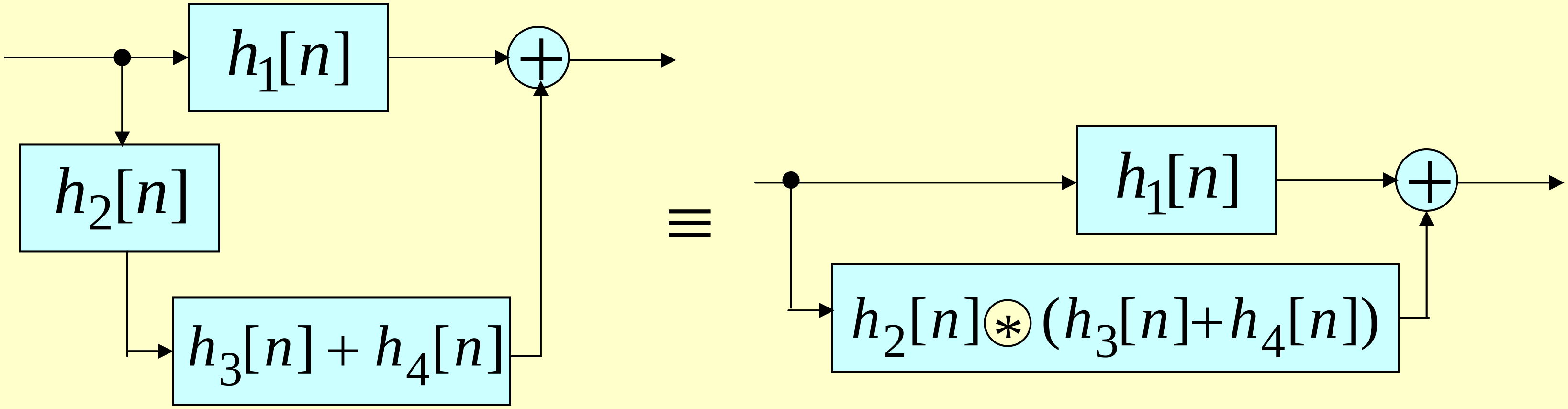
# Simple Interconnection Schemes

- Simplifying the block-diagram we obtain



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# Simple Interconnection Schemes

- Overall impulse response $h[n]$ is given by

$$h[n] = h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n])$$
$$= h_1[n] + h_2[n] \circledast h_3[n] + h_2[n] \circledast h_4[n]$$

- Now,

$$h_2[n] \circledast h_3[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]) \circledast 2\delta[n]$$
$$= \delta[n] - \tfrac{1}{2}\delta[n-1]$$



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# Simple Interconnection Schemes

$$h_2[n] \circledast h_4[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]) \circledast \left(-2(\tfrac{1}{2})^n \mu[n]\right)$$

$$= -(\tfrac{1}{2})^n \mu[n] + \tfrac{1}{2}(\tfrac{1}{2})^{n-1} \mu[n-1]$$

$$= -(\tfrac{1}{2})^n \mu[n] + (\tfrac{1}{2})^n \mu[n-1]$$

$$= -(\tfrac{1}{2})^n \delta[n] = -\delta[n]$$

- Therefore

$$h[n] = \delta[n] + \tfrac{1}{2}\delta[n-1] + \delta[n] - \tfrac{1}{2}\delta[n-1] - \delta[n] = \delta[n]$$



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# Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- $x[n]$ and $y[n]$ are, respectively, the input and the output of the system
- $\{d_k\}$ and $\{p_k\}$ are constants characterizing the system



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# Finite-Dimensional LTI Discrete-Time Systems

- The **order** of the system is given by $\max(N,M)$, which is the order of the difference equation
- It is possible to implement an LTI system characterized by a constant coefficient difference equation as here the computation involves two finite sums of products



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# Finite-Dimensional LTI Discrete-Time Systems

- If we assume the system to be causal, then the output $y[n]$ can be recursively computed using

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

provided $d_0 \neq 0$

- $y[n]$ can be computed for all $n \geq n_o$ , knowing $x[n]$ and the initial conditions

$$y[n_o - 1], y[n_o - 2], \ldots, y[n_o - N]$$



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# Classification of LTI Discrete-Time Systems

**Based on Impulse Response Length -**

- If the impulse response $h[n]$ is of finite length, i.e.,

$$h[n] = 0 \;\text{ for } n < N_1 \text{ and } n > N_2, \quad N_1 < N_2$$

then it is known as a **finite impulse response** (FIR) discrete-time system

- The convolution sum description here is

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$



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# Classification of LTI Discrete-Time Systems

- The output $y[n]$ of an FIR LTI discrete-time system can be computed directly from the convolution sum as it is a finite sum of products
- Examples of FIR LTI discrete-time systems are the moving-average system and the linear interpolators



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# Classification of LTI Discrete-Time Systems

- If the impulse response is of infinite length, then it is known as an **infinite impulse response** (IIR) discrete-time system
- The class of IIR systems we are concerned with in this course are characterized by linear constant coefficient difference equations



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# Classification of LTI Discrete-Time Systems

- Example - The discrete-time accumulator defined by

$$y[n] = y[n-1] + x[n]$$

is seen to be an IIR system



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# Classification of LTI Discrete-Time Systems

- <u>Example</u> - The familiar numerical integration formulas that are used to numerically solve integrals of the form

$$y(t) = \int_{0}^{t} x(\tau) d\tau$$

can be shown to be characterized by linear constant coefficient difference equations, and hence, are examples of IIR systems



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# Classification of LTI Discrete-Time Systems

- If we divide the interval of integration into $n$ equal parts of length $T$, then the previous integral can be rewritten as

$$y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau$$

where we have set $t = nT$ and used the notation

$$y(nT) = \int_{0}^{nT} x(\tau)d\tau$$



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# Classification of LTI Discrete-Time Systems

- Using the trapezoidal method we can write

$$\int_{(n-1)T}^{nT} x(\tau)d\tau = \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

- Hence, a numerical representation of the definite integral is given by

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$



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# Classification of LTI Discrete-Time Systems

- Let $y[n] = y(nT)$ and $x[n] = x(nT)$
- Then

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

reduces to

$$y[n] = y[n-1] + \frac{T}{2}\{x[n] + x[n-1]\}$$

which is recognized as the difference equation representation of a first-order IIR discrete-time system



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# Classification of LTI Discrete-Time Systems

**Based on the Output Calculation Process**

- **Nonrecursive System** - Here the output can be calculated sequentially, knowing only the present and past input samples
- **Recursive System** - Here the output computation involves past output samples in addition to the present and past input samples



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# Classification of LTI Discrete-Time Systems

**Based on the Coefficients -**

- **Real Discrete-Time System** - The impulse response samples are real valued
- **Complex Discrete-Time System** - The impulse response samples are complex valued



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