Digital Processing of Continuous-Time Signals

- Digital processing of a continuous-time signal involves the following basic steps:

  (1) Conversion of the continuous-time signal into a discrete-time signal,

  (2) Processing of the discrete-time signal,

  (3) Conversion of the processed discrete-time signal back into a continuous-time signal

Digital Processing of Continuous-Time Signals

- Conversion of a continuous-time signal into digital form is carried out by an **analog-to-digital (A/D) converter**
- The reverse operation of converting a digital signal into a continuous-time signal is performed by a **digital-to-analog (D/A) converter**

Digital Processing of Continuous-Time Signals

- Since the A/D conversion takes a finite amount of time, a **sample-and-hold** (**S/H**) **circuit** is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation

Digital Processing of Continuos-Time Signals

- To prevent aliasing, an analog **anti-aliasing filter** is employed before the S/H circuit
- To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog **reconstruction filter** is used

Digital Processing of Continuous-Time Signals

Complete block-diagram

- Since both the anti-aliasing filter and the reconstruction filter are analog lowpass filters, we review first the theory behind the design of such filters
- Also, the most widely used IIR digital filter design method is based on the conversion of an analog lowpass prototype

Sampling of Continuous-Time Signals

- As indicated earlier, discrete-time signals in many applications are generated by sampling continuous-time signals
- We have seen earlier that identical discrete-time signals may result from the sampling of more than one distinct continuous-time function

Sampling of Continuous-Time Signals

- In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal
- However, under certain conditions, it is possible to relate a unique continuous-time signal to a given discrete-time signal

Sampling of Continuous-Time Signals

- If these conditions hold, then it is possible to recover the original continuous-time signal from its sampled values
- We next develop this correspondence and the associated conditions

Effect of Sampling in the Frequency Domain

- Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g[n]$ where

$$g[n] = g_a(nT), \quad -\infty < n < \infty$$

with T being the **sampling period**

- The reciprocal of T is called the **sampling frequency** F_T, i.e.,

$$F_T = \frac{1}{T}$$

Effect of Sampling in the Frequency Domain

- Now, the frequency-domain representation of $g_a(t)$ is given by its continuos-time Fourier transform (CTFT):

$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

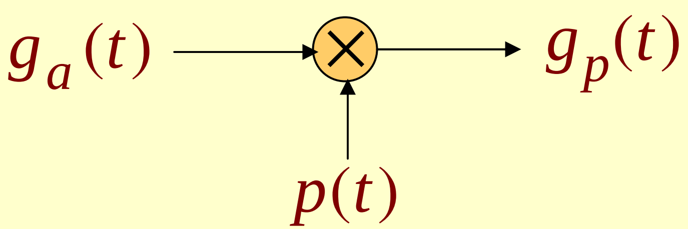
- The frequency-domain representation of $g[n]$ is given by its discrete-time Fourier transform (DTFT):

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

Effect of Sampling in the Frequency Domain

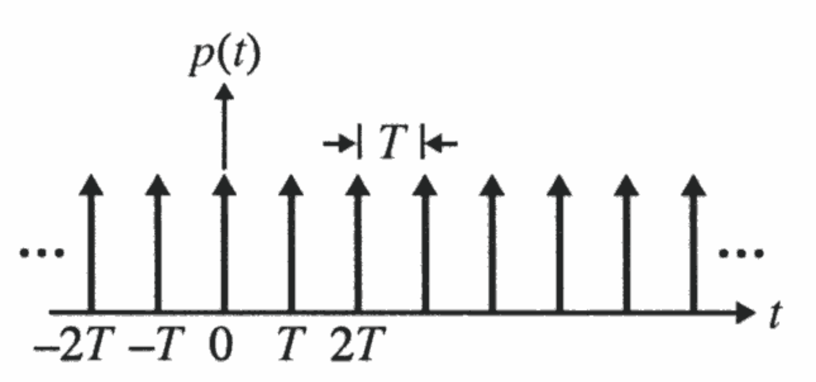
- To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a multiplication of $g_a(t)$ by a **periodic impulse train** $p(t)$:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Effect of Sampling in the Frequency Domain

- $p(t)$ consists of a train of ideal impulses with a period T as shown below

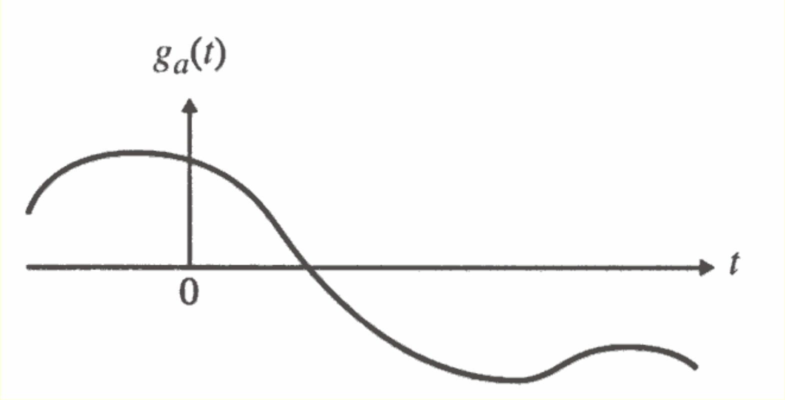


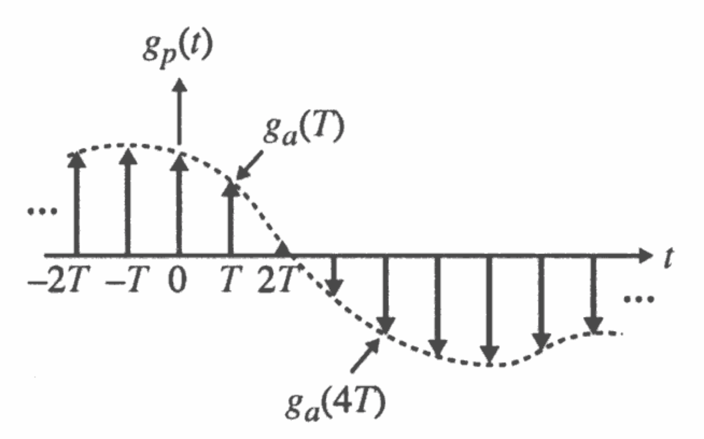
- The multiplication operation yields an impulse train:

$$g_p(t) = g_a(t)\,p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\,\delta(t - nT)$$

Effect of Sampling in the Frequency Domain

- $g_p(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant





Effect of Sampling in the Frequency Domain

- There are two different forms of $G_p(j\Omega)$:
- One form is given by the weighted sum of the CTFTs of $\delta(t-nT)$:

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

- To derive the second form, we make use of the Poisson's formula:

$$\sum_{n=-\infty}^{\infty} \phi(t+nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)e^{jk\Omega_T t}$$

where $\Omega_T = 2\pi / T$ and $\Phi(j\Omega)$ is the CTFT of $\phi(t)$

Effect of Sampling in the Frequency Domain

- For $t = 0$

$$\sum_{n=-\infty}^{\infty} \phi(t+nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)e^{jk\Omega_T t}$$

reduces to

$$\sum_{n=-\infty}^{\infty} \phi(nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)$$

- Now, from the frequency shifting property of the CTFT, the CTFT of $g_a(t)e^{-j\Psi t}$ is given by $G_a(j(\Omega+\Psi))$

Effect of Sampling in the Frequency Domain

- Substituting $\phi(t) = g_a(t)e^{-j\Psi t}$ in

$$\sum_{n=-\infty}^{\infty} \phi(nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \Phi(jk\Omega_T)$$

we arrive at

$$\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Psi nT} = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(k\Omega_T + \Psi))$$

- By replacing Ψ with Ω in the above equation we arrive at the alternative form of the CTFT $G_p(j\Omega)$ of $g_p(t)$

Effect of Sampling in the Frequency Domain

- The alternative form of the CTFT of $g_p(t)$ is given by

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

- Therefore, $G_p(j\Omega)$ is a periodic function of Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by $\frac{1}{T}$

Effect of Sampling in the Frequency Domain

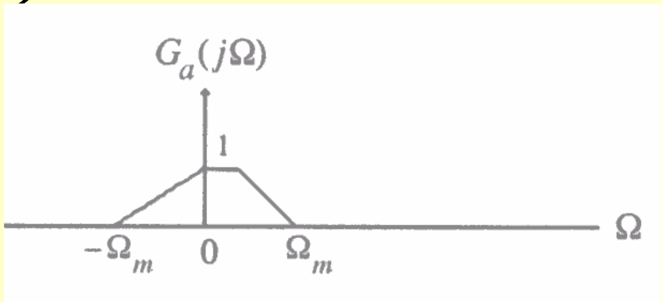
- The term on the RHS of the previous equation for $k = 0$ is the **baseband** portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$
- The frequency range

$$-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$$

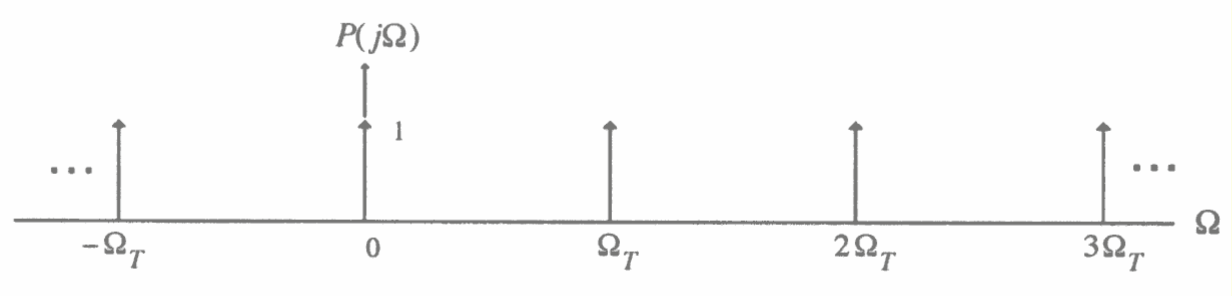
- is called the **baseband** or **Nyquist band**

Effect of Sampling in the Frequency Domain

- Assume $g_a(t)$ is a band-limited signal with a CTFT $G_a(j\Omega)$ as shown below

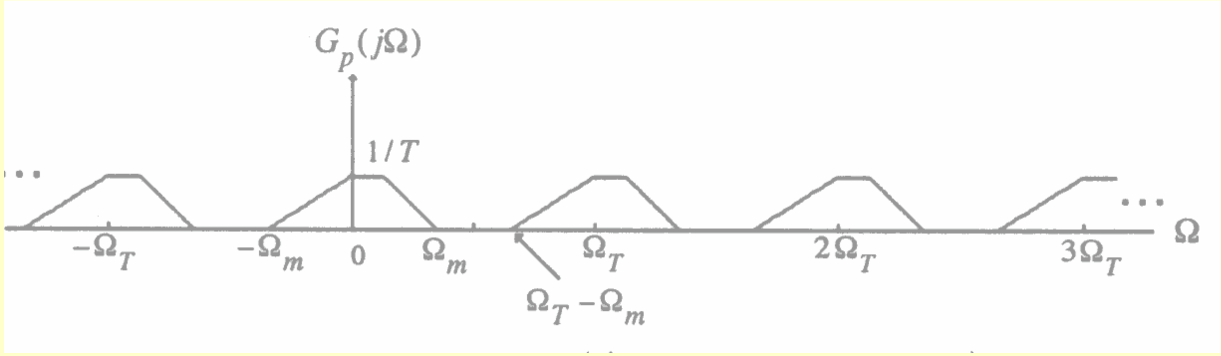


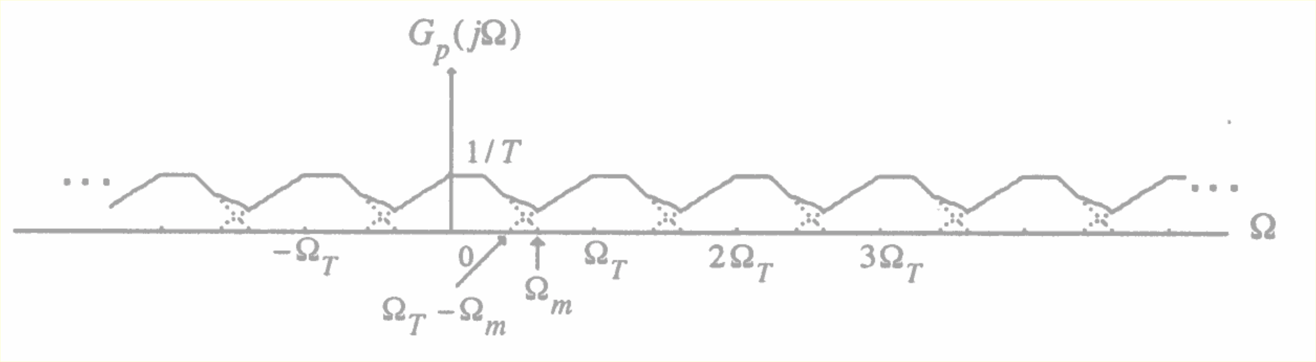
- The spectrum $P(j\Omega)$ of $p(t)$ having a sampling period $T = \dfrac{2\pi}{\Omega_T}$ is indicated below



Effect of Sampling in the Frequency Domain

- Two possible spectra of $G_p(j\Omega)$ are shown below



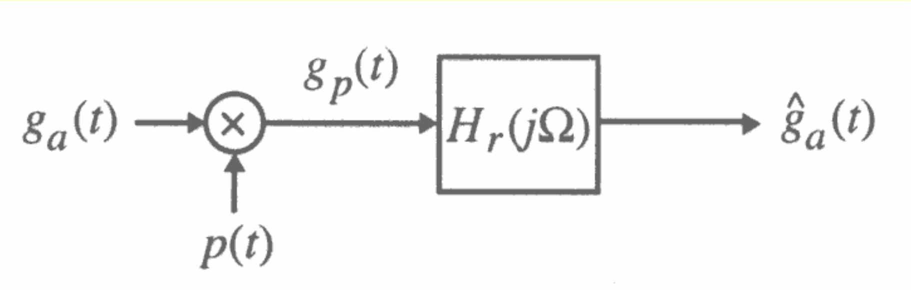


Effect of Sampling in the Frequency Domain

- It is evident from the top figure on the previous slide that if $\Omega_T > 2\Omega_m$, there is no overlap between the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$
- On the other hand, as indicated by the figure on the bottom, if $\Omega_T < 2\Omega_m$, there is an overlap of the spectra of the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$

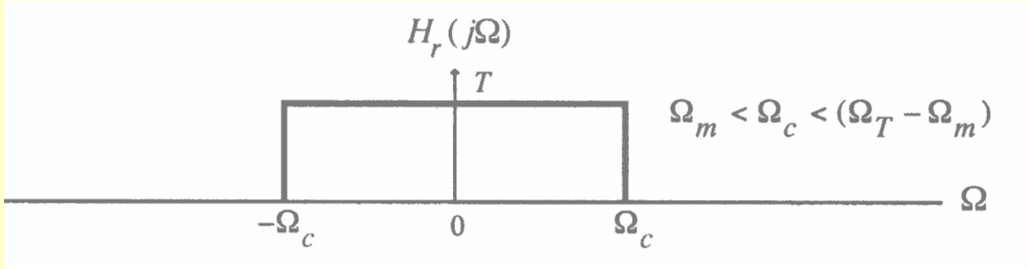
Effect of Sampling in the Frequency Domain

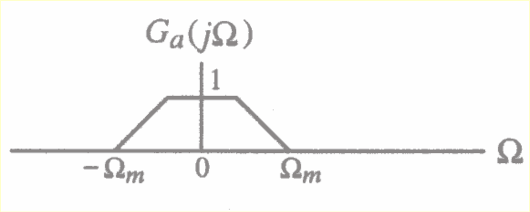
If $\Omega_T > 2\Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as shown below

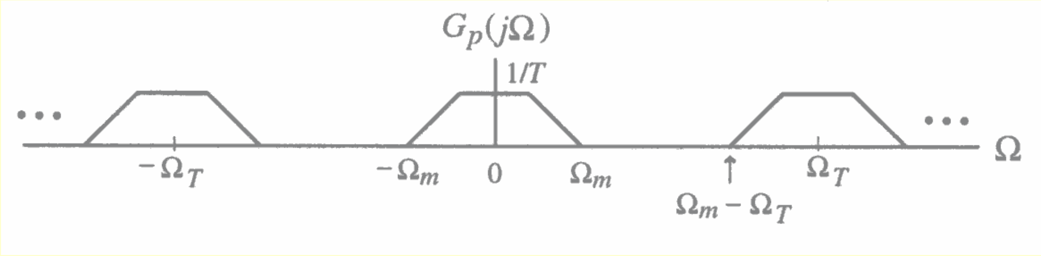


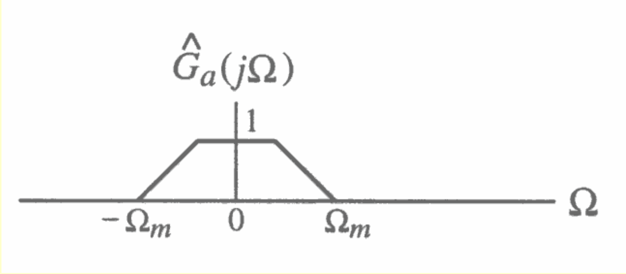
Effect of Sampling in the Frequency Domain

- The spectra of the filter and pertinent signals are shown below









Effect of Sampling in the Frequency Domain

- On the other hand, if $\Omega_T < 2\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_a(j\Omega)$ cannot be separated by filtering to recover $G_a(j\Omega)$ because of the distortion caused by a part of the replicas immediately outside the baseband folded back or **aliased** into the baseband

Effect of Sampling in the Frequency Domain

Sampling theorem - Let $g_a(t)$ be a band-limited signal with CTFT $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$

- Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty \leq n \leq \infty$ if

$$\Omega_T \geq 2\Omega_m$$

where $\Omega_T = 2\pi / T$

Effect of Sampling in the Frequency Domain

- The condition $\Omega_T \geq 2\Omega_m$ is often referred to as the **Nyquist condition**
- The frequency $\dfrac{\Omega_T}{2}$ is usually referred to as the **folding frequency**

Effect of Sampling in the Frequency Domain

- Given $\{g_a(nT)\}$, we can recover exactly $g_a(t)$ by generating an impulse train

$$g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$

and then passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c satisfying

$$\Omega_m < \Omega_c < (\Omega_T - \Omega_m)$$

Effect of Sampling in the Frequency Domain

- The highest frequency Ω_m contained in $g_a(t)$ is usually called the **Nyquist frequency** since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$ that must be used to fully recover $g_a(t)$ from its sampled version
- The frequency $2\Omega_m$ is called the **Nyquist rate**

Effect of Sampling in the Frequency Domain

- **Oversampling** - The sampling frequency is higher than the Nyquist rate
- **Undersampling** - The sampling frequency is lower than the Nyquist rate
- **Critical sampling** - The sampling frequency is equal to the Nyquist rate
- Note: A pure sinusoid may not be recoverable from its critically sampled version

Effect of Sampling in the Frequency Domain

- In digital telephony, a 3.4 kHz signal bandwidth is acceptable for telephone conversation
- Here, a sampling rate of 8 kHz, which is greater than twice the signal bandwidth, is used

Effect of Sampling in the Frequency Domain

- In high-quality analog music signal processing, a bandwidth of 20 kHz has been determined to preserve the fidelity
- Hence, in compact disc (CD) music systems, a sampling rate of 44.1 kHz, which is slightly higher than twice the signal bandwidth, is used

Effect of Sampling in the Frequency Domain

- Example - Consider the three continuous-time sinusoidal signals:

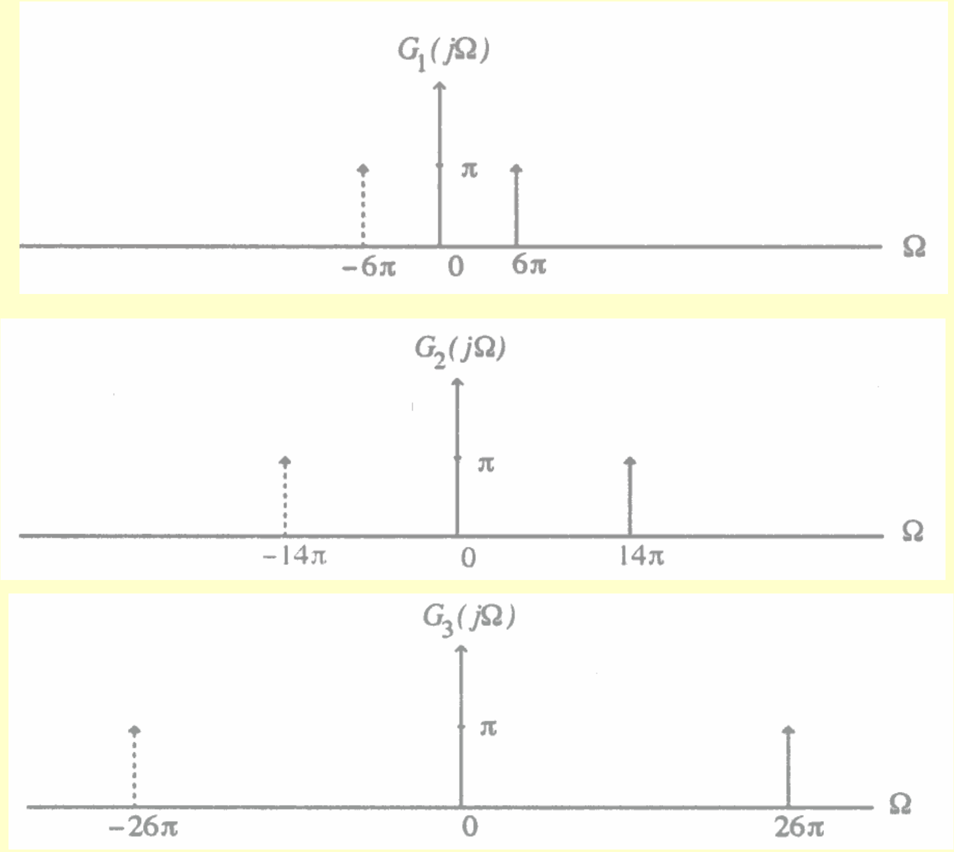
$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

- Their corresponding CTFTs are:

$$G_1(j\Omega) = \pi[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)]$$
$$G_2(j\Omega) = \pi[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)]$$
$$G_3(j\Omega) = \pi[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)]$$

Effect of Sampling in the Frequency Domain

- These three transforms are plotted below



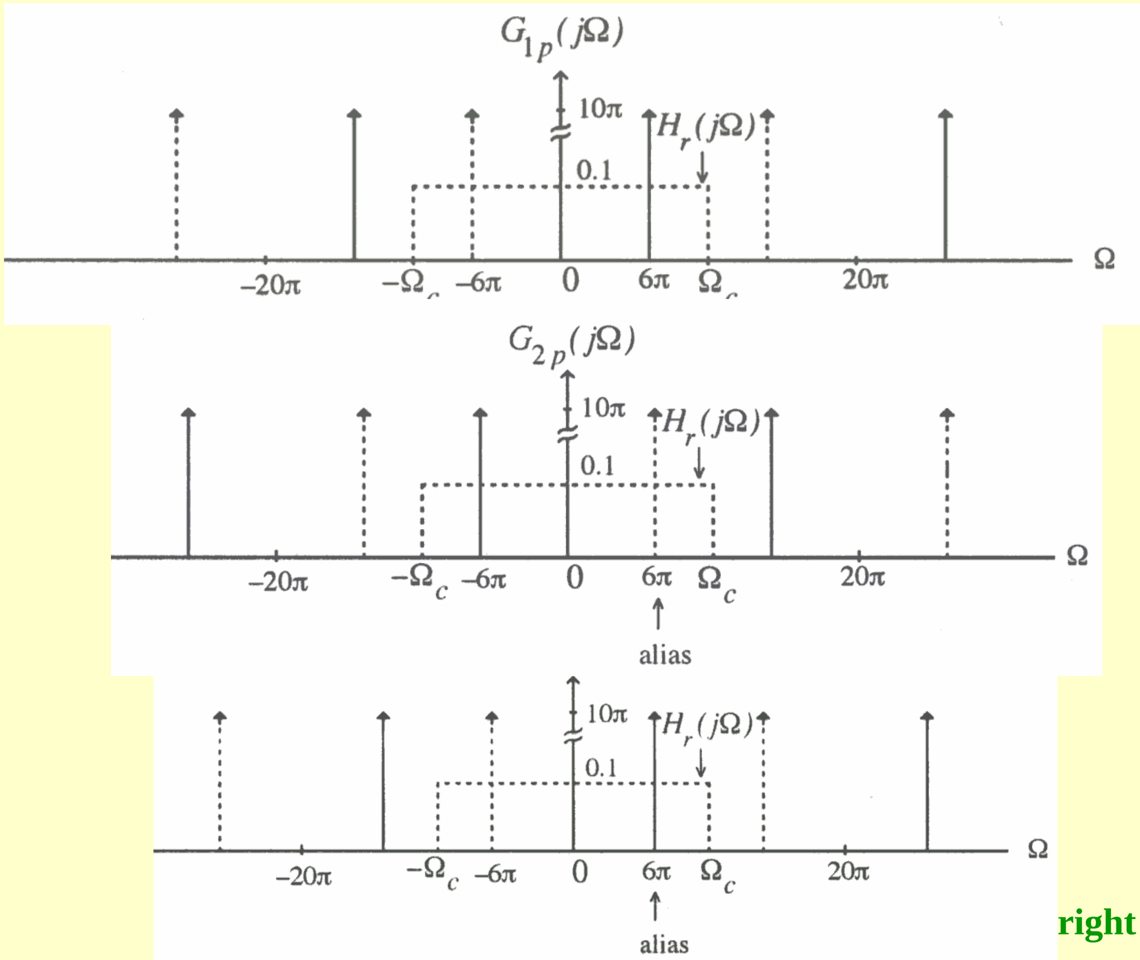
Effect of Sampling in the Frequency Domain

- These continuous-time signals sampled at a rate of $T = 0.1$ sec, i.e., with a sampling frequency $\Omega_T = 20\pi$ rad/sec
- The sampling process generates the continuous-time impulse trains, $g_{1p}(t)$, $g_{2p}(t)$, and $g_{3p}(t)$
- Their corresponding CTFTs are given by

$$G_{\ell p}(j\Omega) = 10\sum_{k=-\infty}^{\infty} G_{\ell}\big(j(\Omega - k\Omega_T)\big), \quad 1 \le \ell \le 3$$

Effect of Sampling in the Frequency Domain

- Plots of the 3 CTFTs are shown below



Effect of Sampling in the Frequency Domain

- These figures also indicate by dotted lines the frequency response of an ideal lowpass filter with a cutoff at $\Omega_c = \Omega_T / 2 = 10\pi$ and a gain $T = 0.1$
- The CTFTs of the lowpass filter output are also shown in these three figures
- In the case of $g_1(t)$, the sampling rate satisfies the Nyquist condition, hence no aliasing

Effect of Sampling in the Frequency Domain

- Moreover, the reconstructed output is precisely the original continuous-time signal
- In the other two cases, the sampling rate does not satisfy the Nyquist condition, resulting in aliasing and the filter outputs are all equal to $\cos(6\pi t)$

Effect of Sampling in the Frequency Domain

- Note: In the figure at the bottom, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_2(j\Omega)$ at $\Omega = -14\pi$
- Likewise, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_3(j\Omega)$ at $\Omega = 26\pi$

Effect of Sampling in the Frequency Domain

- We now derive the relation between the DTFT of $g[n]$ and the CTFT of $g_p(t)$
- To this end we compare

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

with

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

and make use of $g[n] = g_a(nT), \; -\infty < n < \infty$

Effect of Sampling in the Frequency Domain

- Observation: We have

$$G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\omega/T}$$

or, equivalently,

$$G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}$$

- From the above observation and

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

Effect of Sampling in the Frequency Domain

we arrive at the desired result given by

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j\Omega - jk\Omega_T)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j\tfrac{\omega}{T} - jk\Omega_T)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j\tfrac{\omega}{T} - j\tfrac{2\pi k}{T})$$

Effect of Sampling in the Frequency Domain

- The relation derived on the previous slide can be alternately expressed as

$$G(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j\Omega - jk\Omega_T)$$

- From

$$G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\omega/T}$$

or from

$$G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}$$

it follows that $G(e^{j\omega})$ is obtained from $G_p(j\Omega)$ by applying the mapping $\Omega = \frac{\omega}{T}$

Effect of Sampling in the Frequency Domain

- Now, the CTFT $G_p(j\Omega)$ is a periodic function of Ω with a period $\Omega_T = 2\pi/T$
- Because of the mapping, the DTFT $G(e^{j\omega})$ is a periodic function of ω with a period 2π

Recovery of the Analog Signal

- We now derive the expression for the output $\hat{g}_a(t)$ of the ideal lowpass reconstruction filter $H_r(j\Omega)$ as a function of the samples $g[n]$
- The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse DTFT of $H_r(j\Omega)$:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

Recovery of the Analog Signal

- Thus, the impulse response is given by

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_T t/2}, \qquad -\infty \leq t \leq \infty$$

- The input to the lowpass filter is the impulse train $g_p(t)$:

$$g_p(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t-nT)$$

Recovery of the Analog Signal

- Therefore, the output $\hat{g}_a(t)$ of the ideal lowpass filter is given by:

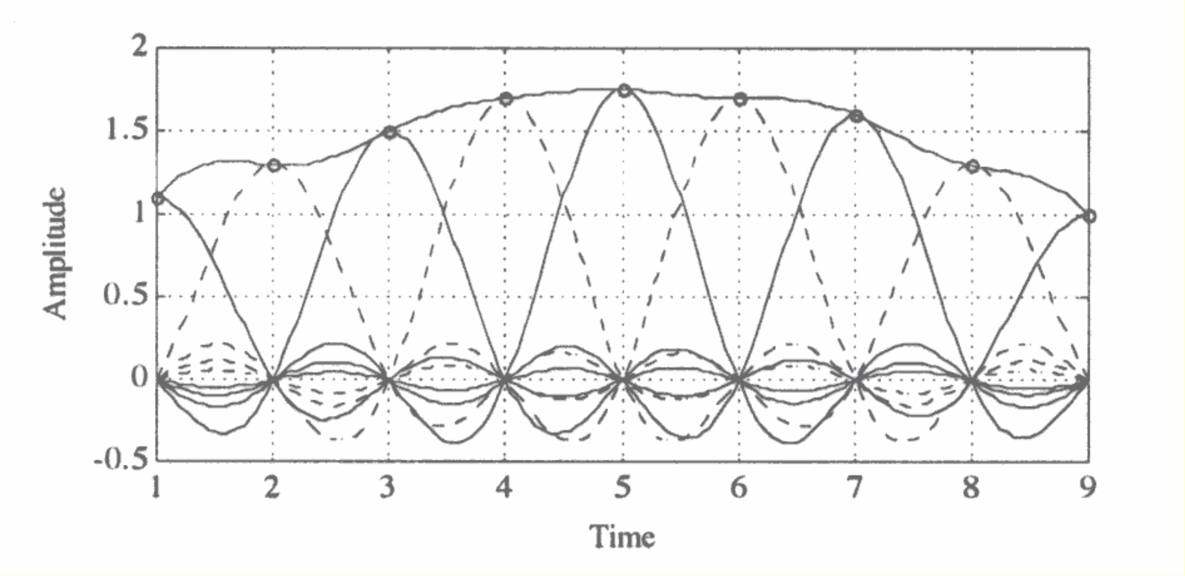
$$\hat{g}_a(t) = h_r(t) \circledast g_p(t) = \sum_{n=-\infty}^{\infty} g[n] h_r(t - nT)$$

- Substituting $h_r(t) = \sin(\Omega_c t)/(\Omega_T t/2)$ in the above and assuming for simplicity $\Omega_c = \Omega_T/2 = \pi/T$, we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

Recovery of the Analog Signal

- The ideal bandlimited interpolation process is illustrated below



Recovery of the Analog Signal

- It can be shown that when $\Omega_c = \Omega_T/2$ in

$$h_r(t) = \frac{\sin(\Omega_c t)}{\Omega_T t/2}$$

$h_r(0) = 1$ and $h_r(nT) = 0$ for $n \neq 0$

- As a result, from

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

we observe

$$\hat{g}_a(rT) = g[r] = g_a(rT)$$

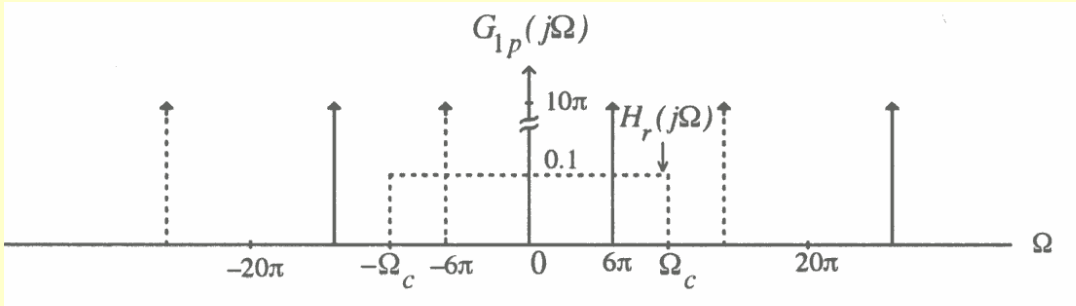
for all integer values of r in the range $-\infty < r < \infty$

Recovery of the Analog Signal

- The relation
$$\hat{g}_a(rT) = g[r] = g_a(rT)$$
holds whether or not the condition of the sampling theorem is satisfied
- However, $\hat{g}_a(rT) = g_a(rT)$ for all values of t only if the sampling frequency Ω_T satisfies the condition of the sampling theorem

Implication of the Sampling Process

- Consider again the three continuous-time signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$
- The plot of the CTFT $G_{1p}(j\Omega)$ of the sampled version $g_{1p}(t)$ of $g_1(t)$ is shown below



Implication of the Sampling Process

- From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos[(20k \pm 6)\pi t]$ outside the baseband by passing $g_{1p}(t)$ through an ideal analog bandpass filter with a passband centered at $\Omega = (20k \pm 6)\pi$

Implication of the Sampling Process

- For example, to recover the signal $\cos(34\pi t)$, it will be necessary to employ a bandpass filter with a frequency response

$$H_r(j\Omega) = \begin{cases} 0.1, & (34-\Delta)\pi \le |\Omega| \le (34+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

where Δ is a small number

Implication of the Sampling Process

- Likewise, we can recover the aliased baseband component $\cos(6\pi t)$ from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response:

$$H_r(j\Omega) = \begin{cases} 0.1, & (6-\Delta)\pi \le |\Omega| \le (6+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

Implication of the Sampling Process

- There is no aliasing distortion unless the original continuous-time signal also contains the component $\cos(6\pi t)$
- Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $g_2(t)$ or $g_3(t)$ as the case may be, by employing suitable filters

Sampling of Bandpass Signals

- The conditions developed earlier for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the continuous-time signal is bandlimited in the frequency range from dc to some frequency Ω_m
- Such a continuous-time signal is commonly referred to as a **lowpass signal**

Sampling of Bandpass Signals

- There are applications where the continuous-time signal is bandlimited to a higher frequency range $\Omega_L \le |\Omega| \le \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the **bandpass signal**
- To prevent aliasing a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring

$$\Omega_T \ge 2\Omega_H$$

Sampling of Bandpass Signals

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if Ω_H is very large, the sampling rate also has to be very large which may not be practical in some situations

Sampling of Bandpass Signals

- A more practical approach is to use under-sampling
- Let $\Delta\Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency Ω_H contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M(\Delta\Omega)$$

Sampling of Bandpass Signals

- We choose the sampling frequency Ω_T to satisfy the condition

$$\Omega_T = 2(\Delta\Omega) = \frac{2\Omega_H}{M}$$

which is smaller than $2\Omega_H$, the Nyquist rate

- Substitute the above expression for Ω_T in

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

Sampling of Bandpass Signals

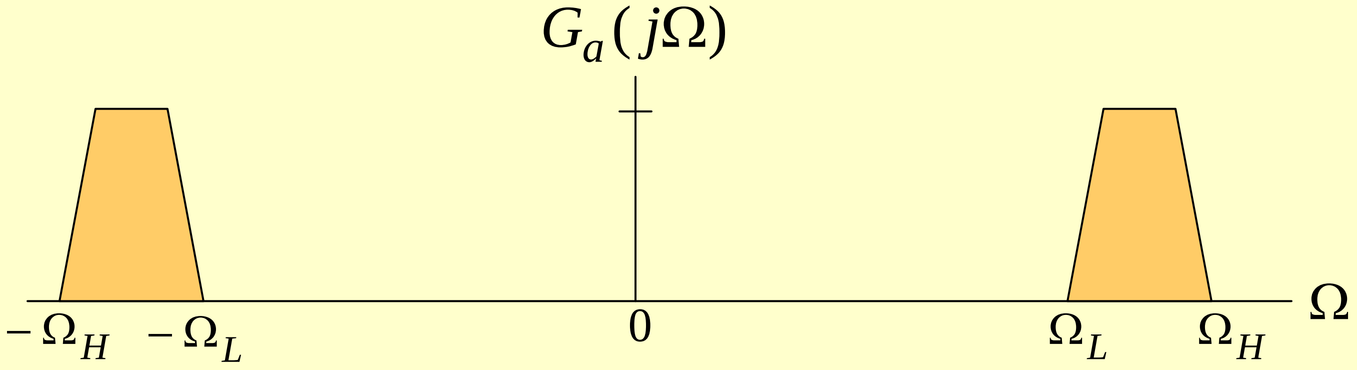
- This leads to

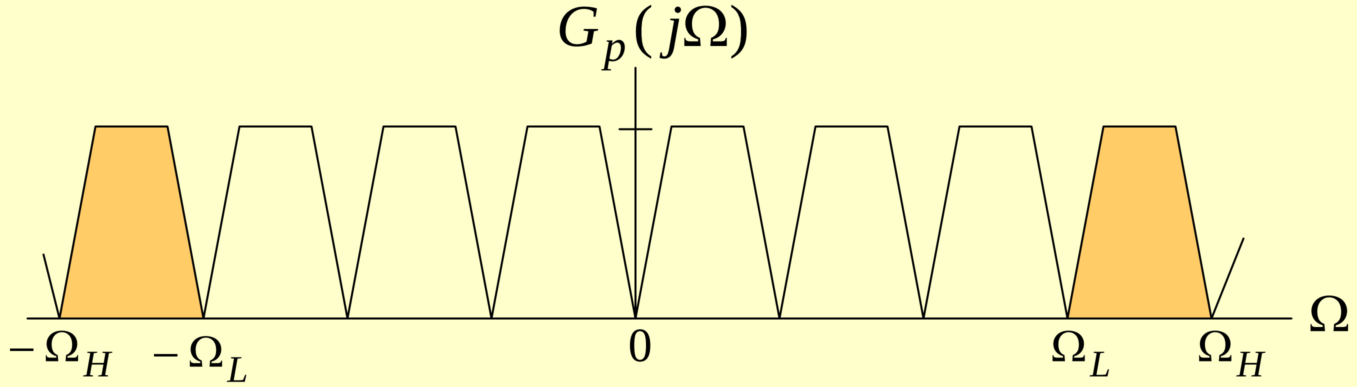
$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j\Omega - j2k(\Delta\Omega)\big)$$

- As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$ and scaled by $1/T$
- The amount of shift for each value of k ensures that there will be no overlap between all shifted replicas $\Rightarrow$ no aliasing

Sampling of Bandpass Signals

- Figure below illustrate the idea behind





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- As can be seen, $g_a(t)$ can be recovered from $g_p(t)$ by passing it through an ideal bandpass filter with a passband given by $\Omega_L \leq |\Omega| \leq \Omega_H$ and a gain of T
- Note: Any of the replicas in the lower frequency bands can be retained by passing $g_p(t)$ through bandpass filters with passbands $\Omega_L - k(\Delta\Omega) \leq |\Omega| \leq \Omega_H - k(\Delta\Omega)$, $1 \leq k \leq M-1$ providing a translation to lower frequency ranges