Discrete-Time Signals in the Frequency Domain

- The frequency-domain representation of a discrete-time sequence is the discrete-time Fourier transform (DTFT)
- This transform maps a time-domain sequence into a continuous function of the frequency variable ω
- We first review briefly the continuous-time Fourier transform (CTFT)

Continuous-Time Fourier Transform

- **Definition** – The CTFT of a continuous-time signal $x_a(t)$ is given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the Fourier spectrum or simply the spectrum of the continuous-time signal

Continuous-Time Fourier Transform

- **Definition** – The inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}\,d\Omega$$

- Often referred to as the Fourier integral
- A CTFT pair will be denoted as

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

Continuous-Time Fourier Transform

- Ω is real and denotes the continuous-time angular frequency variable in radians/sec if the unit of the independent variable t is in seconds
- In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$
- It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

Continuous-Time Fourier Transform

- The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum
- Both spectrums are real functions of Ω
- In general, the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet conditions given on the next slide

Continuous-Time Fourier Transform

Dirichlet Conditions

- (a) The signal $x_a(t)$ has a finite number of discontinuities and a finite number of maxima and minima in any finite interval
- (b) The signal is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} \left| x_a(t) \right| dt < \infty$$

Continuous-Time Fourier Transform

- If the Dirichlet conditions are satisfied, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}\,d\Omega$$

converges to $x_a(t)$ at values of t except at values of t where $x_a(t)$ has discontinuities

- It can be shown that if $x_a(t)$ is absolutely integrable, then $|X_a(j\Omega)| < \infty$ proving the existence of the CTFT

Energy Density Spectrum

- The total energy $\mathcal{E}_x$ of a finite energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} \left| x_a(t) \right|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t)\, dt$$

- The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

Energy Density Spectrum

- Interchanging the order of the integration we get

$$\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)\left[\int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt\right]d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)X_a(j\Omega)d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X_a(j\Omega)\right|^2 d\Omega$$

Energy Density Spectrum

- Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

- The above relation is more commonly known as the Parseval's theorem for finite-energy continuous-time signals

Energy Density Spectrum

- The quantity $\left|X_a(j\Omega)\right|^2$ is called the energy density spectrum of $x_a(t)$ and usually denoted as

$$S_{xx}(\Omega) = \left|X_a(j\Omega)\right|^2$$

- The energy over a specified range of frequencies $\Omega_a \le \Omega \le \Omega_b$ can be computed using

$$\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\, d\Omega$$

Band-limited Continuous-Time Signals

- A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$
- A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty < \Omega < \infty$

Band-limited Continuous-Time Signals

- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$, that is

$$X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_a \\ 0, & \Omega_b < |\Omega| < \infty \end{cases}$$

- However, an ideal band-limited signal cannot be generated in practice

Band-limited Continuous-Time Signals

- Band-limited signals are classified according to the frequency range where most of the signal's is concentrated
- A lowpass, continuous-time signal has a spectrum occupying the frequency range $|\Omega| \leq \Omega_p < \infty$ where Ω_p is called the bandwidth of the signal

Band-limited Continuous-Time Signals

- A highpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \le |\Omega| < \infty$ where the bandwidth of the signal is from Ω_p to ∞
- A bandpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \le |\Omega| \le \Omega_H < \infty$ where $\Omega_H - \Omega_L$ is the bandwidth

Discrete-Time Fourier Transform

- Definition - The **discrete-time Fourier transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j\,X_{im}(e^{j\omega})$$

Discrete-Time Fourier Transform

- $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω
- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

Discrete-Time Fourier Transform

- $\left|X(e^{j\omega})\right|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

Discrete-Time Fourier Transform

- For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{re}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{im}(e^{j\omega})$ are odd functions of ω
- Note: $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega+2\pi k)}$

$$= \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

for any integer k

- ⇒ The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the **principal value**

Discrete-Time Fourier Transform

- The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses
- An alternate type of phase function that is a continuous function of ω is often used
- It is derived from the original phase function by removing the discontinuities of 2π

Discrete-Time Fourier Transform

- The process of removing the discontinuities is called “**unwrapping**”
- The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$
- In some cases, discontinuities of π may be present after unwrapping

Discrete-Time Fourier Transform

- Example - The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

- Example - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

Discrete-Time Fourier Transform

- Its DTFT is given by

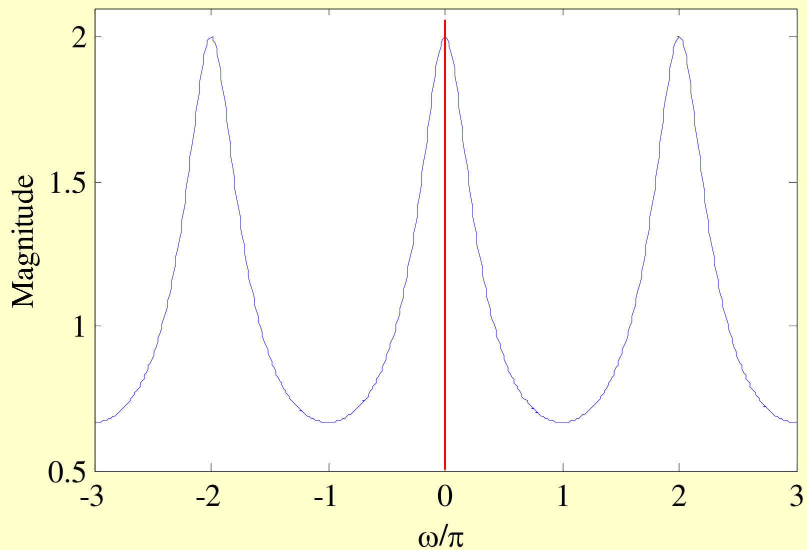
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

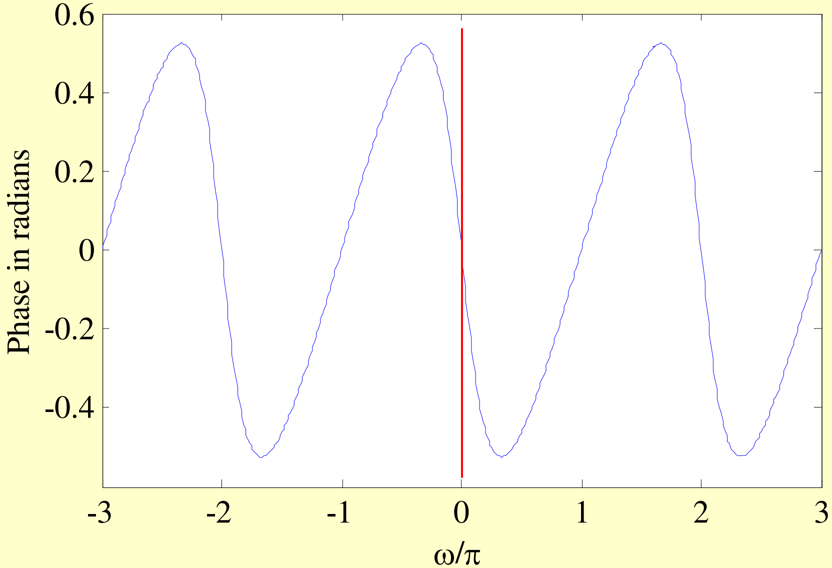
as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

Discrete-Time Fourier Transform

- The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$ are shown below



$$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$$



$$\theta(\omega) = -\theta(-\omega)$$

Discrete-Time Fourier Transform

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π:

$$X(e^{j(\omega_o+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega_o+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n}e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n} = X(e^{j\omega_o})$$

Discrete-Time Fourier Transform

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete-Time Fourier Transform

- **Inverse discrete-time Fourier transform**:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Proof:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell} \right) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists
- Then $\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}$$

Discrete-Time Fourier Transform

- Now $$\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases}$$

$$= \delta[n-\ell]$$

- Hence

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] = x[n]$$

Discrete-Time Fourier Transform

- **Convergence Condition -** An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

may or may not converge

- Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Then for uniform convergence of $X(e^{j\omega})$,

$$\lim_{K\to\infty}\left|X(e^{j\omega})-X_K(e^{j\omega})\right|=0$$

- Now, if $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|<\infty$$

Discrete-Time Fourier Transform

- Then

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

for all values of ω

- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Example - The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} \left|\alpha^n\right| \mu[n] = \sum_{n=0}^{\infty} \left|\alpha^n\right| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT $X(e^{j\omega})$ therefore converges to $1/(1-\alpha e^{-j\omega})$ uniformly

Discrete-Time Fourier Transform

- Since

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 \leq \left(\sum_{n=-\infty}^{\infty} \left| x[n] \right| \right)^2,$$

an absolutely summable sequence has always a finite energy

- However, a finite-energy sequence is not necessarily absolutely summable

Discrete-Time Fourier Transform

- Example - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^2 = \frac{\pi^2}{6}$$

- But, $x[n]$ is not absolutely summable

Discrete-Time Fourier Transform

- To represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a **mean-square convergence** of $X(e^{j\omega})$:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

must approach zero at each value of ω as K goes to ∞

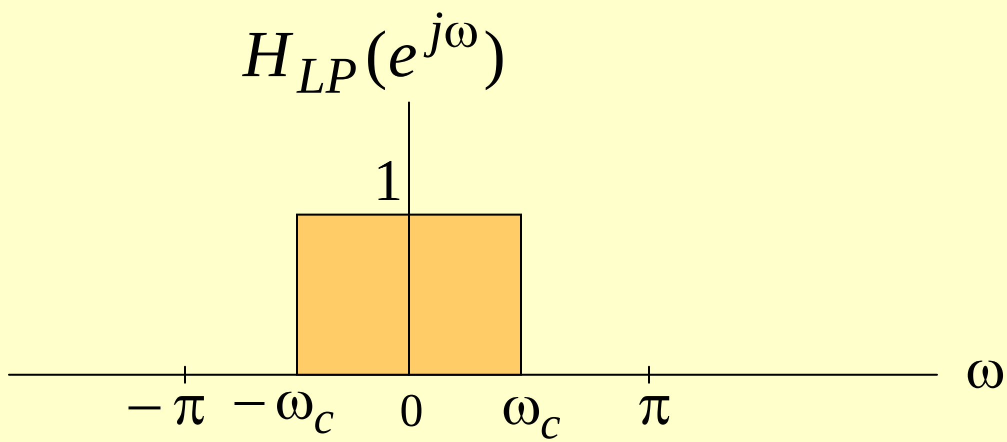
- In such a case, the absolute value of the error $\left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|$ may not go to zero as K goes to ∞ and the DTFT is no longer bounded

Discrete-Time Fourier Transform

- Example - Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

shown below



Discrete-Time Fourier Transform

- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin\omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by ω_c/π
- $\Rightarrow$ $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable

Discrete-Time Fourier Transform

- As a result

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense

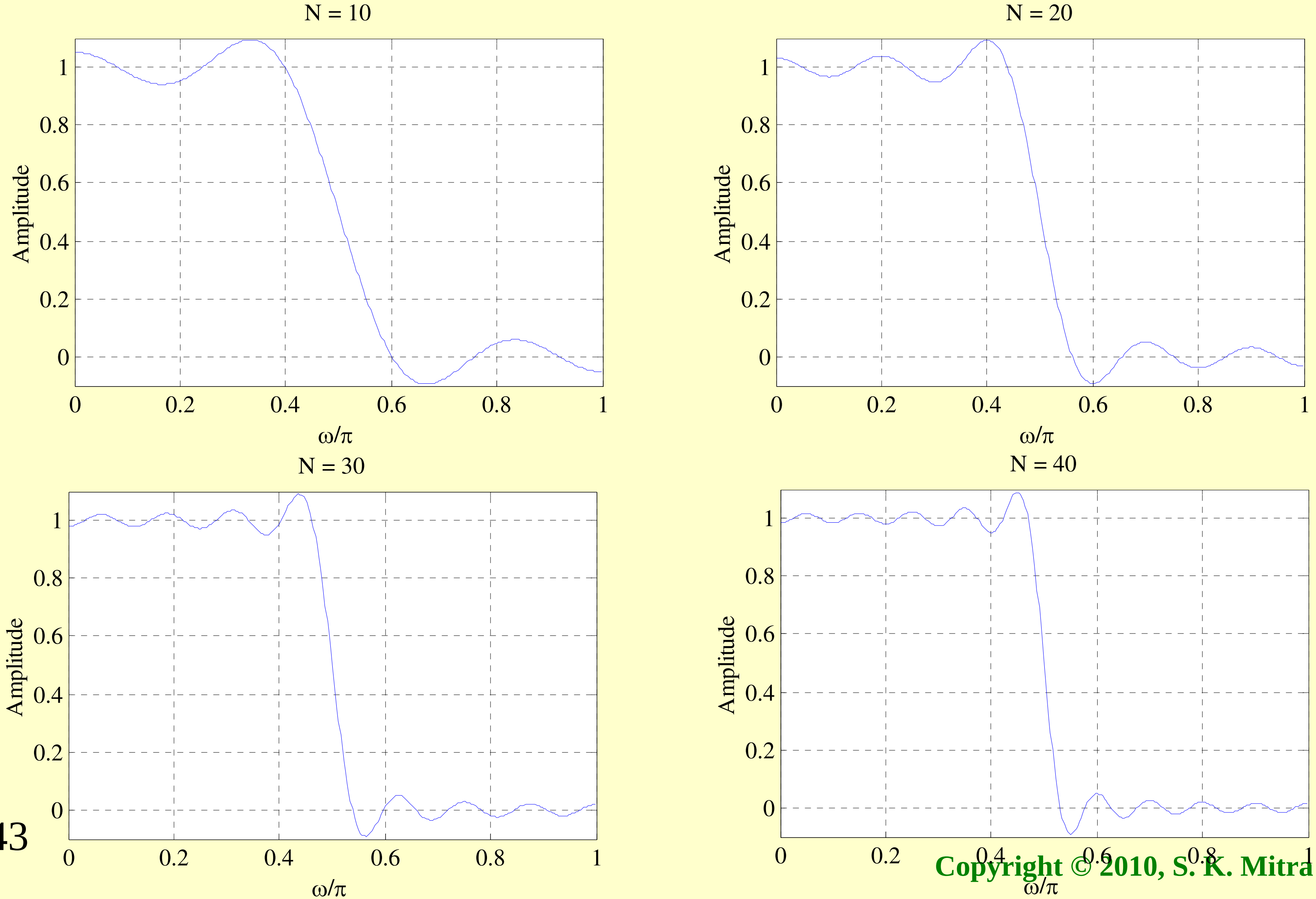
Discrete-Time Fourier Transform

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next

Discrete-Time Fourier Transform



Discrete-Time Fourier Transform

- As can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K

Discrete-Time Fourier Transform

- As K goes to infinity, the condition

$$\lim_{K\to\infty}\int_{-\pi}^{\pi}\left|H_{LP}(e^{j\omega})-H_{LP,K}(e^{j\omega})\right|^2 d\omega=0$$

  holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$
- The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**

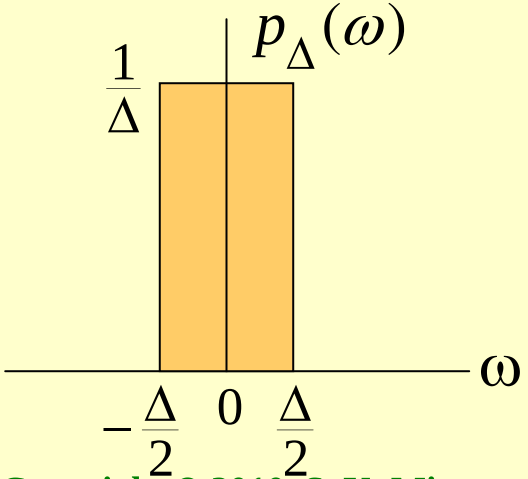
Discrete-Time Fourier Transform

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_o n + \phi)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function** $\delta(\omega)$

Discrete-Time Fourier Transform

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_\Delta(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$



Discrete-Time Fourier Transform

- Example - Consider the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \le \omega_o \le \pi$$

Discrete-Time Fourier Transform

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

  is a periodic function of ω with a period 2π and is called a **periodic impulse train**
- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_o n}$ we compute the inverse DTFT of $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Thus

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_o) e^{j\omega n} d\omega = e^{j\omega_o n}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Sequence		**DTFT**
$\delta[n]$	$\leftrightarrow$	1
1	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$e^{j\omega_o n}$	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_o+2\pi k)$
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
$\alpha^n\mu[n],\ (\lvert\alpha\rvert<1)$	$\leftrightarrow$	$\dfrac{1}{1-\alpha e^{-j\omega}}$

DTFT Properties and Theorems

- There are a number of important properties and theorems of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table 3.1: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

$x[n]$: A complex sequence

Table 3.2: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

$x[n]$: A real sequence

Table 3.4 DTFT Theorems

Theorems	Sequence	DTFT
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\, d\theta$
Parseval's relation	$\displaystyle\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\, d\omega$	