Linear System Theory
Homework 9

Due date: 12/28/2023

You can use Matlab to help you solve the following problems.

1. Consider the 2-input 2-output system (with 2 actuators and 2 sensors)

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -3 & -3 \\ -2 & -1 & -2 \\ 2 & 3 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \mathbf{x}$$

   (a) Compute the eigenvalues of the system.

   (b) Verify that the system is controllable, observable and unstable.

   (c) Is it possible to stabilize the system by state feedback using one actuator only? Explain.

   (d) Design a state feedback so that the closed-loop system has triple eigenvalues at $\lambda = -1$.

   (e) Is it possible to stabilize the system with dynamic output feedback (e.g. an observer-based controller) using one sensor only? Explain.

2. Assume that the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ is observable. Show that if for all $\mathbf{x}(0)$ the corresponding output $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$, then the system is stable.

3. Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{y} = \begin{bmatrix} c_{11} & c_{12} & 0 & \omega \\ c_{21} & c_{22} & -\omega & 0 \end{bmatrix} \mathbf{x}$$

   where $\omega > 0$. Is the system observable? Of course your answer may depend on the parameters c_{ij}, μ and ω.

4. (a) Given the $n \times n$ matrix $\mathbf{Q} = \mathbf{Q}^T \geq 0$ with rank$(\mathbf{Q}) = r$, show that there is a full rank $r \times n$ matrix $\mathbf{C}$ such that $\mathbf{C}^T\mathbf{C} = \mathbf{Q}$. Such a $\mathbf{C}$ is called a square root of $\mathbf{Q}$.

   (b) Show that if $\mathbf{C}$ is a square root of $\mathbf{Q}$ and $\mathbf{O}$ is an $r \times r$ orthogonal matrix, that is, $\mathbf{O}^T\mathbf{O} = \mathbf{I}$, then $\mathbf{OC}$ is also a square root of $\mathbf{Q}$. Thus $\mathbf{Q}$ has many square roots.

   (c) Show that if $\mathbf{C}_1$ and $\mathbf{C}_2$ are both full rank square roots of $\mathbf{Q}$, then there is an $r \times r$ orthogonal matrix $\mathbf{O}$ such that $\mathbf{C}_1 = \mathbf{O}\mathbf{C}_2$.

(d) Suppose $\mathbf{C}_0$ is a square root of $\mathbf{Q}$ and that $(\mathbf{C}_0, \mathbf{A})$ is observable. Show that if $\mathbf{C}$ is any square roots of $\mathbf{Q}$, then $(\mathbf{C}, \mathbf{A})$ is observable. Thus the phrase '$(\sqrt{\mathbf{Q}}, A)$ is observable' makes sense, even though there are many square roots of $\mathbf{Q}$.

5. Suppose the continuous-time system $(\mathbf{A}, \mathbf{B})$ is controllable. Let $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ be the optimal state feedback gain matrix obtained by solving the continuous-time LQR problem, with weighting matrices $\mathbf{Q} \geq 0$ and $\mathbf{R} > 0$. Show that $\mathbf{A} - \mathbf{BK}$ is stable provided that $(\sqrt{\mathbf{Q}}, \mathbf{A})$ is observable.