Linear System Theory
Homework 8

Due date: 12/21/2023

1. Consider the continuous-time system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & -3 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = [\mathbf{b}_1,\ \mathbf{b}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

   (a) Let $u_1 = 0$ and $u_2 = -\mathbf{k}_2\mathbf{x}$, where $\mathbf{k}_2 \in \mathbb{R}^{1\times 3}$. Find $\mathbf{k}_2$ such that the closed-loop eigenvalues are located at $-1, -1, -1$.

   (b) Let $u_1 = -\mathbf{k}_1\mathbf{x}$, where $\mathbf{k}_1 \in \mathbb{R}^{1\times 3}$. Find $\mathbf{k}_1$ such that $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is controllable. Specify the reason for your choice.

2. Consider an n-degree-of-freedom mass-spring-damper system as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{u}$$

   where $\mathbf{q}, \mathbf{u} \in \mathbb{R}^n$ are the position and force, respectively. $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n\times n}$ denote mass, damping, and spring matrices, respectively, and $\mathbf{M} > 0$. Define $\mathbf{x}_1 = \mathbf{q}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$, and $\mathbf{x} = [\mathbf{x}_1^T,\ \mathbf{x}_2^T]^T$. Then the state space representation of the mass-spring-damper system is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}$$

   (a) Show that $(\mathbf{A}, \mathbf{B})$ is controllable.

   (b) Suppose that a PID controller is used to regulate the position to zero, i.e. applying

$$\mathbf{u}(t) = -\left(\mathbf{K}_P\mathbf{q}(t) + \mathbf{K}_D\dot{\mathbf{q}}(t) + \mathbf{K}_I\int_0^t \mathbf{q}(\tau)d\tau\right)$$

   such that $\|\mathbf{q}\|$ is as small as possible. $\mathbf{K}_P, \mathbf{K}_D, \mathbf{K}_I \in \mathbb{R}^{n\times n}$ are PID gains to be determined. Formulate PID gain-tuning as an LQR problem and then find the optimal gains $\mathbf{K}_P, \mathbf{K}_D, \mathbf{K}_I$.

3. An angular position control system is described by the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

   Choose $R = 1$ and $\mathbf{Q} = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$, $q > 0$, as the weighting matrices in the LQR cost. Show that the optimal control is

$$u(t) = -\sqrt{q}x_1(t) - \left(\sqrt{100 + 2\sqrt{q}} - 10\right)x_2(t)$$

   and the closed-loop characteristic equation is $s^2 + \sqrt{100 + 2\sqrt{q}}s + \sqrt{q} = 0$.

4. Suppose that $\mathbf{u} = -\mathbf{K}\mathbf{x}$ is the input which minimizes

$$J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt$$

for the controllable pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{Q} > 0$ and $\mathbf{R} > 0$. Prove that with $\mathbf{u} = -\alpha\mathbf{K}\mathbf{x}$ and $\alpha \geq \frac{1}{2}$, the closed-loop system is stable. In control terms, this means that the LQR system has an infinite positive gain margin and at least a 6dB negative gain margin.

Hint: Let $\mathbf{P}$ be the cost-to-go matrix for the LQR system and show that $(\mathbf{A} - \alpha\mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \alpha\mathbf{B}\mathbf{K}) < 0$ for $\alpha \geq \frac{1}{2}$.

5. (Use Matlab to solve this problem) A DC motor driven by field current is described by the following differential equation

$$\ddot{\theta}(t) + 3\dot{\theta}(t) = u(t)$$

where $\theta(t)$ is the shaft angle and $u(t)$ is the field current at time t.

(a) Write state equation for the system using θ and $\dot{\theta}$ as state variables.

(b) Find LQR optimal control law (state feedback) with

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad R = 1$$

With the optimal control, plot the state and input trajectory for initial condition $\theta(0) = 1$ and $\dot{\theta}(0) = 0$. (Matlab command `lqr`)

(c) Discretize the system with sampling interval $h = 0.01$.

(d) For the discretized system, find LQR optimal control law (state feedback) with

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad \mathbf{Q}_f = \mathbf{0}, \quad R = 1, \quad \text{and} \quad N = 100$$

Plot the optimal feedback gains. With the optimal control, plot the state and input trajectory for initial condition $\theta(0) = 1$ and $\dot{\theta}(0) = 0$.