Linear System Theory
Solution to Homework 4

1. (a) Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 \mathbf{u} + \alpha_2 \mathbf{v} + \alpha_3 \mathbf{w} = \mathbf{0} \tag{1}$$

Rewrite (1) as

$$\alpha_1 \mathbf{u} + \frac{1}{2}\big(\alpha_2 - j\alpha_3\big)\big(\mathbf{v} + j\mathbf{w}\big) + \frac{1}{2}\big(\alpha_2 + j\alpha_3\big)\big(\mathbf{v} - j\mathbf{w}\big) = \mathbf{0}$$

Since $\mathbf{u}$ and $\mathbf{v} \pm j\mathbf{w}$ are eigenvectors associated with distinct eigenvalues λ, and $\sigma \pm j\omega$, respectively, they are linearly independent. Hence $\alpha_1 = 0$ and $\alpha_2 \pm j\alpha_3 = 0$. In other words, (1) implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, and $\mathbf{S} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$ is invertible.

(b) Since $\mathbf{v} + j\mathbf{w}$ is an eigenvector associated with the eigenvalue $\sigma + j\omega$, we have

$$\mathbf{A}(\mathbf{v} + j\mathbf{w}) = (\sigma + j\omega)(\mathbf{v} + j\mathbf{w})$$

Then

$$\begin{aligned} \mathbf{A}\mathbf{v} &= \sigma\mathbf{v} - \omega\mathbf{w} \\ \mathbf{A}\mathbf{w} &= \sigma\mathbf{w} + \omega\mathbf{v} \\ \Rightarrow \mathbf{A}\big[\mathbf{v}, \mathbf{w}\big] &= \big[\mathbf{v}, \mathbf{w}\big]\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \end{aligned}$$

Moreover, $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. Hence

$$\mathbf{A}\big[\mathbf{u}, \mathbf{v}, \mathbf{w}\big] = \big[\mathbf{u}, \mathbf{v}, \mathbf{w}\big]\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix}$$

In other words,

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix}$$

2. (a) If $\mathbf{y} \in \mathcal{R}(\mathbf{R}_k)$, then $\exists \mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = \mathbf{R}_k\mathbf{x} = \mathbf{p}_k\mathbf{q}_k^T\mathbf{x} = (\mathbf{q}_k^T\mathbf{x})\mathbf{p}_k$. Therefore, $\mathcal{R}(\mathbf{R}_k) = \text{span}\{\mathbf{p}_k\}$ and $\text{rank}(\mathbf{R}_k) = 1$.

On the other hand, if $\mathbf{x} \in \mathcal{N}(\mathbf{R}_k)$, then $\mathbf{R}_k\mathbf{x} = \mathbf{p}_k\mathbf{q}_k^T\mathbf{x} = \mathbf{0}$, which indicates that $\mathbf{x}$ is orthogonal to $\mathbf{q}_k$. Since $\mathbf{p}_k$ and $\mathbf{q}_k$ are the right and left eigenvectors of $\mathbf{A}$, respectively, and $\mathbf{q}_k^T\mathbf{p}_i = 0$, for $i \neq k$, we conclude that $\mathcal{N}(\mathbf{R}_k) = \text{span}\{\mathbf{p}_1, \cdots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \cdots, \mathbf{p}_n\}$.

(b) Since $\mathbf{p}_i$ and $\mathbf{q}_i$ are the right and left eigenvectors associated with λ_i, respectively, we have $\mathbf{p}_i^T\mathbf{q}_j = 0$ for $i \neq j$ and $\mathbf{p}_i^T\mathbf{q}_j = 1$ for $i = j$. Hence $\mathbf{R}_i\mathbf{R}_j = \mathbf{p}_i\mathbf{q}_i^T\mathbf{p}_j\mathbf{q}_j^T = (\mathbf{q}_i^T\mathbf{p}_j)\mathbf{p}_i\mathbf{q}_j^T = \mathbf{0}$ for $i \neq j$ and $\mathbf{R}_i^2 = \mathbf{p}_i\mathbf{q}_i^T\mathbf{p}_i\mathbf{q}_i^T = (\mathbf{q}_i^T\mathbf{p}_i)\mathbf{p}_i\mathbf{q}_i^T = \mathbf{R}_i$.

(c) Let $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Then $\mathbf{A} = \mathbf{P\Lambda Q}$ and $\mathbf{PQ} = \mathbf{I}$. Hence

$$\begin{aligned}
(s\mathbf{I} - \mathbf{A})^{-1} &= (s\mathbf{PQ} - \mathbf{P\Lambda Q})^{-1} = \mathbf{P}(s\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{Q} \\
&= \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \\
&= \sum_{k=1}^{n} \frac{\mathbf{R}_k}{s - \lambda_k}
\end{aligned}$$

(d)

$$\mathbf{I} = \mathbf{PQ} = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} = \mathbf{R}_1 + \cdots + \mathbf{R}_n$$

3. (a) Let $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ be the state transition matrices of $\dot{\mathbf{x}} = \mathbf{Ax}$ and $\dot{\mathbf{y}} = -\mathbf{A}^T\mathbf{y}$, respectively. Then $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ and $\mathbf{\Psi}(t) = e^{-\mathbf{A}^Tt}$. Consequently, $\mathbf{\Psi}^T(t)\mathbf{\Phi}(t) = e^{-\mathbf{A}t}e^{\mathbf{A}t} = \mathbf{I}$, or equivalently, $\mathbf{\Phi}^{-1}(t) = \mathbf{\Psi}^T(t)$.

(b)

$$\mathbf{x}^T(t)\mathbf{y}(t) = (\mathbf{\Phi}(t)\mathbf{x}(0))^T\mathbf{\Psi}(t)\mathbf{y}(0) = \mathbf{x}^T(0)\mathbf{\Phi}^T(t)\mathbf{\Psi}(t)\mathbf{y}(0) = \mathbf{x}^T(0)\mathbf{y}(0), \quad \forall t$$

4. (a) Note that $\mathbf{x}(t_2) = e^{\mathbf{A}(t_2-t_1)}\mathbf{x}(t_1)$ for $t_1, t_2 \geq 0$. Hence $\mathbf{x}(1) = e^{\mathbf{A}}\mathbf{x}(0)$ and $\mathbf{x}(2) = e^{\mathbf{A}}\mathbf{x}(1)$. Or equivalently,

$$\begin{bmatrix} \mathbf{x}(2) & \mathbf{x}(1) \end{bmatrix} = e^{\mathbf{A}} \begin{bmatrix} \mathbf{x}(1) & \mathbf{x}(0) \end{bmatrix} \Rightarrow \begin{bmatrix} -11 & -4 \\ 23 & 10 \end{bmatrix} = e^{\mathbf{A}} \begin{bmatrix} -4 & 1 \\ 10 & 2 \end{bmatrix}$$

Hence

$$e^{\mathbf{A}} = \begin{bmatrix} -11 & -4 \\ 23 & 10 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 10 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1.5 \\ 3 & 3.5 \end{bmatrix}$$

(b) The characteristic equation of $e^{\mathbf{A}}$ is

$$\det(s\mathbf{I} - e^{\mathbf{A}}) = \det \begin{bmatrix} s+1 & 1.5 \\ -3 & s-3.5 \end{bmatrix} = (s+1)(s-3.5) + 4.5 = s^2 - 2.5s + 1 = 0$$

Hence the eigenvalues of $e^{\mathbf{A}}$ are $s_1 = 0.5$ and $s_2 = 2$.
Let the eigenvalues of $\mathbf{A}$ be λ_1 and λ_2. By spectral mapping theorem, $s_1 = e^{\lambda_1}$ and $s_2 = e^{\lambda_2}$. Hence

$$\lambda_1 = \ln(s_1) = -0.6931, \quad \lambda_2 = \ln(s_2) = 0.6931$$

(c) Note that $\mathbf{A}$ has one stable mode ($\lambda_1 < 0$) and one unstable mode ($\lambda_2 > 0$). For the state to converge to zero, the state should stay in the subspace spanned by the eigenvector associated with the stable mode. Note that an eigenvector of $\mathbf{A}$ associated with λ_1 is also an eigenvector of $e^{\mathbf{A}}$ associated with s_1. Hence we find the eigenvector associated with s_1 as follows:

$$(s_1\mathbf{I} - e^{\mathbf{A}})\mathbf{v} = \begin{bmatrix} 1.5 & 1.5 \\ -3 & -3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence if $\mathbf{x}_0 = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}[1, -1]^T$, then $\mathbf{x}(t) \to 0$ as $t \to \infty$.

5. (a) Suppose that $\mathbf{x} \in \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$, i.e. $(\lambda\mathbf{I} - \mathbf{A})^i\mathbf{x} = \mathbf{0}$. Then

$$(\lambda\mathbf{I} - \mathbf{A})^{i+1}\mathbf{x} = (\lambda\mathbf{I} - \mathbf{A})(\lambda\mathbf{I} - \mathbf{A})^i\mathbf{x} = 0 \Rightarrow \mathbf{x} \in \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^{i+1}\big)$$

This shows that $\mathcal{N}\big((\lambda_1\mathbf{I} - \mathbf{A})^i\big) \subseteq \mathcal{N}\big((\lambda_1\mathbf{I} - \mathbf{A})^{i+1}\big)$ for all $i \in \mathbb{N}$.
To show the subset is proper, let $\mathbf{v}_1$ be the eigenvector associated with λ, and $\mathbf{v}_k$, $k = 2, \cdots, m$ be the generalized eigenvectors associated with λ, i.e.

$$\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1, \quad \mathbf{A}\mathbf{v}_k = \lambda\mathbf{v}_k + \mathbf{v}_{k-1}, \; k = 2, \cdots, m$$

For $k = 2, \cdots, m$,

$$\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I})^{k-1}\mathbf{v}_k &= (\mathbf{A} - \lambda\mathbf{I})^{k-2}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_k \\
&= (\mathbf{A} - \lambda\mathbf{I})^{k-2}\mathbf{v}_{k-1} \\
&= (\mathbf{A} - \lambda\mathbf{I})^{k-3}\mathbf{v}_{k-2} \\
&\cdots \\
&= (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \\
&= \mathbf{v}_1 \neq \mathbf{0} \\
(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v}_k &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}
\end{aligned}$$

This implies that $\mathbf{v}_k \in \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^k\big)$ but $\mathbf{v}_k \notin \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^{k-1}\big)$, for $k = 2, \cdots, m$. In other words,

$$\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^{k-1}\big) \subset \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^k\big), \quad k = 2, \cdots, m$$

(b) For $i \in \mathbb{N}$, suppose that $\mathbf{x} \in \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$. Then

$$(\lambda\mathbf{I} - \mathbf{A})^i\mathbf{A}\mathbf{x} = \underbrace{(\lambda\mathbf{I} - \mathbf{A})\cdots(\lambda\mathbf{I} - \mathbf{A})}_{i \text{ terms}}\mathbf{A}\mathbf{x} = \mathbf{A}\underbrace{(\lambda\mathbf{I} - \mathbf{A})\cdots(\lambda\mathbf{I} - \mathbf{A})}_{i \text{ terms}}\mathbf{x} = \mathbf{0}$$

This means that $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$ is $\mathbf{A}$-invariant.

(c) The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}
\det(\lambda\mathbf{I} - \mathbf{A}) &= \det\begin{bmatrix} \lambda + 2 & -1 & -1 \\ 0 & \lambda + 1 & 0 \\ 1 & -1 & \lambda \end{bmatrix} = \lambda(\lambda + 1)(\lambda + 2) + (\lambda + 1) \\
&= (\lambda + 1)(\lambda^2 + 2\lambda + 1) = (\lambda + 1)^3
\end{aligned}$$

Hence the eigenvalues are $\lambda = -1, -1, -1$. The corresponding eigenvectors are

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. We can see that $\lambda = -1$ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity 3. However, $\lambda = -1$ has only two linearly independent eigenvectors $[1,\ 1,\ 0]^T$ and $[1,\ 0,\ 1]^T$. Hence we look for the generalized eigenvector $[w_1,\ w_2,\ w_3]^T$ which satisfies

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

Notice that the generalized eigenvector is generalized from the eigenvector $[1,\ 0,\ 1]^T$, not from $[1,\ 1,\ 0]^T$, because (2) does not hold for $[1,\ 1,\ 0]^T$.

Based on the results in parts (a) and (b), we can decompose $\mathbb{R}^3$ as $\mathbb{R}^3 = N_1 + N_2$, where

$$N_1 = \mathcal{N}\big((-\mathbf{I} - \mathbf{A})^2\big) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad N_2 = \mathcal{N}(-\mathbf{I} - \mathbf{A}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$