Linear System Theory
Homework 3

Due date: 10/26/2023

1. Suppose that $\mathbf{A} \in \mathbb{R}^{2\times 2}$. Given that $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, suppose

   (a) $\mathbf{x}(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$ if $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, and

   (b) $\mathbf{x}(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$ if $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   Determine the matrix $e^{\mathbf{A}t}$ and the matrix $\mathbf{A}$.

2. Consider the linear autonomous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$. You can use Matlab to do the following calculation.

   (a) Find $e^{\mathbf{A}}$.

   (b) Let $\mathbf{x}(t) = [x_1(t),\, x_2(t)]^T$. Suppose $x_1(0) = 1$ and $x_2(1) = 2$. Find $\mathbf{x}(2)$.

3. Suppose $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $\dot{\mathbf{z}} = \sigma\mathbf{z} + \mathbf{A}\mathbf{z} = (\mathbf{A} + \sigma\mathbf{I})\mathbf{z}$, where $\sigma \in \mathbb{R}$ and $\mathbf{x}(0) = \mathbf{z}(0)$. How are $\mathbf{x}(t)$ and $\mathbf{z}(t)$ related? Find the simplest possible expression for $\mathbf{z}(t)$ in terms of $\mathbf{x}(t)$.

4. Consider a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix. Suppose that $\mathbf{A}$ has a pair of complex eigenvalues $\sigma \pm j\omega$, where $\sigma, \omega \in \mathbb{R}$, and the corresponding eigenvectors are $\mathbf{u} \pm j\mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Define $\mathbf{T} = [\mathbf{u}, \mathbf{v}]$, and $\mathbf{x}(t) = \mathbf{T}\mathbf{z}(t)$.

   (a) Given $\mathbf{z}(0)$, find $\mathbf{z}(t)$, $t \geq 0$, in terms of σ and ω.

   (b) Let $\mathbf{u} = [1, -1]^T$ and $\mathbf{v} = [0, 1]^T$. Suppose that $\mathbf{x}(0) = [1, 0]^T$ and $\mathbf{x}(1) = [-\frac{1}{2}, 1]^T$. Find the eigenvalues of $\mathbf{A}$.

   (c) Find $e^{\mathbf{A}}$.

5. Consider the following discrete-time system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k), \quad \mathbf{x}(0) = \mathbf{x}_0$$

   where $\mathbf{A} \in \mathbb{R}^{3\times 3}$ with triple eigenvalues at $\lambda = 0.5$. Express $\mathbf{A}^{-1}$ as a polynomial of $\mathbf{A}$.