Linear System Theory
Solution to Homework 9

1. (a) The eigenvalues of the system are $-1, 1, 2$.

   (b) The controllability and observability matrices are full rank. Hence the system is controllable and observable. The system is unstable because two eigenvalues are on the right-half plane.

   (c) Let $\mathbf{b}_1 = [-1,\ -1,\ 2]^T$, $\mathbf{b}_2 = [0,\ 1,\ 0]^T$, $\lambda_1 = 1$, and $\lambda_2 = 2$. Note that λ_1 and λ_2 are the unstable modes of the system. Then

$$\begin{aligned} \operatorname{rank}\,[\lambda_i \mathbf{I} - \mathbf{A},\ \mathbf{b}_1] &= 3, \quad i = 1, 2 \\ \operatorname{rank}\,[\lambda_1 \mathbf{I} - \mathbf{A},\ \mathbf{b}_2] &= 2 \\ \operatorname{rank}\,[\lambda_2 \mathbf{I} - \mathbf{A},\ \mathbf{b}_2] &= 3 \end{aligned}$$

   By PBH rank test, the system can be stabilized by the first input only, but cannot be stabilized by the 2nd one, because the unstable mode λ_1 cannot be changed by the 2nd input along.

   (d) We can use the first input to do the state feedback; however, the mode $\lambda = -1$ is not controllable with respect to the first input. Hence we transform the system into the modal form:

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \operatorname{diag}(-1, 1, 2), \quad \tilde{\mathbf{b}}_1 = \mathbf{V}^{-1}\mathbf{b}_1$$

   where $\mathbf{V} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ whose columns are eigenvectors of $\mathbf{A}$ associated with eigenvalues $-1, 1, 2$, respectively.
   Notice that $\tilde{\mathbf{b}}_1 = [0,\ 1,\ 1]^T$. Let $\mathbf{k}_m = [k_1,\ k_2,\ k_3]^T$. Then

$$\mathbf{\Lambda} - \tilde{\mathbf{b}}_1 \mathbf{k}_m = \begin{bmatrix} -1 & 0 & 0 \\ -k_1 & 1 - k_2 & -k_3 \\ -k_1 & -k_2 & 2 - k_3 \end{bmatrix}$$

   The characteristic polynomial of $\mathbf{\Lambda} - \tilde{\mathbf{b}}_1 \mathbf{k}_m$ is

$$(s+1)\big((s - 1 + k_2)(s - 2 + k_3) - k_2 k_3\big) = (s+1)^3$$

   Then $k_2 = -4$ and $k_3 = 9$. k_1 can be assigned arbitrarily, and we choose $k_1 = 0$. Then $\mathbf{K} = \begin{bmatrix} \mathbf{k}_m \mathbf{V}^{-1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} -4 & 9 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

   (e) From either output, the system is not detectable. Hence the observer-based controller with only one sensor feedback cannot stabilize the system.

2. For all $\mathbf{x}(0) \in \mathbb{R}^n$,

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) \to \mathbf{0}, \quad \text{as } t \to \infty$$

Suppose that $\mathbf{A}$ is unstable and λ is one of the unstable eigenvalues of $\mathbf{A}$. Let $\mathbf{v}$ be the eigenvector of $\mathbf{A}$ associated with λ. Then choose $\mathbf{x}(0) = \mathbf{v}$ and we have $\mathbf{y}(t) = e^{\lambda t}\mathbf{C}\mathbf{v} \to \mathbf{0}$, as $t \to \infty$. Since λ is unstable, we have $\mathbf{C}\mathbf{v} = \mathbf{0}$. By PBH test, the system is unobservable, which is a contradiction.

3. The eigenvalues of the system are μ (with multiplicity 2) and $\pm j\omega$. The corresponding eigenvectors are $\mathbf{v}_1 = [1, 0, 0, 0]^T$ (for μ) and $\mathbf{v}_2 = [0, 0, \pm j, 1]^T$ (for $\pm j\omega$). For the system to be observable, $\mathbf{C}\mathbf{v}_1 = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This implies c_{11} and c_{21} cannot be zero simultaneously. On the other hand, $\mathbf{C}\mathbf{v}_2 = \omega \begin{bmatrix} 1 \\ \pm j \end{bmatrix} \neq \mathbf{0}$, because $\omega > 0$. Hence the system is observable if c_{11} and c_{21} are not zero simultaneously.

4. (a) Since $\mathbf{Q}$ is symmetric positive semidefinite with rank r, it has r positive eigenvalues $\lambda_1, \cdots, \lambda_r$ and $n - r$ zero eigenvalues. Moreover, $\mathbf{Q}$ can be diagonalized by orthogonal matrix as follows:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^T \\ \mathbf{Q}_2^T \end{bmatrix} = \mathbf{Q}_1 \mathbf{\Lambda} \mathbf{Q}_1^T$$

where $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \cdots, \lambda_r)$, $\mathbf{Q}_1 \in \mathbb{R}^{n\times r}$, and $\mathbf{Q}_2 \in \mathbb{R}^{n\times(n-r)}$. Define $\mathbf{C} = \operatorname{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_r})\mathbf{Q}_1^T$. Then $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$.

(b)

$$(\mathbf{OC})^T\mathbf{OC} = \mathbf{C}^T\mathbf{O}^T\mathbf{OC} = \mathbf{C}^T\mathbf{C} = \mathbf{Q}$$

Hence $\mathbf{OC}$ is also a square root of $\mathbf{Q}$.

(c) We have

$$\mathbf{C}_1^T\mathbf{C}_1 = \mathbf{C}_2^T\mathbf{C}_2 \tag{1}$$

Because $\mathbf{C}_1^T \in \mathbb{R}^{n\times r}$ has r linearly independent columns, it has a left inverse $\left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\mathbf{C}_1$. Multiplying both sides of (1) by $\left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\mathbf{C}_1$ leads to

$$\mathbf{C}_1 = \left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\mathbf{C}_1\mathbf{C}_2^T\mathbf{C}_2 = \mathbf{O}\mathbf{C}_2$$

where $\mathbf{O} = \left(\mathbf{C}_1\mathbf{C}_1^T\right)^{-1}\mathbf{C}_1\mathbf{C}_2^T$. Then we need to show that $\mathbf{O} \in \mathbb{R}^{r\times r}$ is orthogonal, i.e $\mathbf{O}^T\mathbf{O} = \mathbf{I}$. Notice that (1) becomes

$$\mathbf{C}_1^T\mathbf{C}_1 = \left(\mathbf{O}\mathbf{C}_2\right)^T\mathbf{O}\mathbf{C}_2 = \mathbf{C}_2^T\mathbf{O}^T\mathbf{O}\mathbf{C}_2 = \mathbf{C}_2^T\mathbf{C}_2 \tag{2}$$

Since $\mathbf{C}_2^T$ also has full column rank, it has a left inverse $\mathbf{C}_2^\dagger = \left(\mathbf{C}_2\mathbf{C}_2^T\right)^{-1}\mathbf{C}_2$. Multiplying $\mathbf{C}_2^\dagger$ from the left of the third equality of (2) and $\mathbf{C}_2^{\dagger T}$ from the right yields $\mathbf{O}^T\mathbf{O} = \mathbf{I}$. Therefore, $\mathbf{O}$ is an orthogonal matrix.

(d) There is an orthogonal matrix $\mathbf{O}$ so that $\mathbf{C} = \mathbf{O}\mathbf{C}_0$. Then

$$\begin{bmatrix} \mathbf{C} \\ \lambda\mathbf{I} - \mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}_0 \\ \lambda\mathbf{I} - \mathbf{A} \end{bmatrix}$$

Since $\mathbf{O}$ is invertible,

$$\operatorname{rank}\begin{bmatrix} \mathbf{C} \\ \lambda\mathbf{I}-\mathbf{A} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathbf{C}_0 \\ \lambda\mathbf{I}-\mathbf{A} \end{bmatrix}$$

Thus $(\mathbf{C},\mathbf{A})$ is observable.

5. Let $\mathbf{C}=\sqrt{\mathbf{Q}}$ and $\mathbf{y}=\mathbf{C}\mathbf{x}$. The cost function can then be expressed as

$$J=\int_0^\infty \big(\|\mathbf{y}(t)\|^2+\mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\big)dt$$

Let $\mathbf{u}^*$ denote the optimal control that minimizes J and let $\mathbf{x}^*$ and $\mathbf{y}^*$ be the corresponding state and output under the optimal control. Then $\mathbf{u}^*(t)=-\mathbf{K}\mathbf{x}^*(t)$, and the optimal cost $J^*<\infty$ since $(\mathbf{A},\mathbf{B})$ is assumed controllable. It then follows that $\mathbf{y}^*(t)\to\mathbf{0}$ and $\mathbf{u}^*(t)\to\mathbf{0}$ as $t\to\infty$, since $\mathbf{R}>0$. Since both $\mathbf{u}^*$ and $\mathbf{y}^*$ are solutions of a linear system, each component of which is a linear combination of exponential functions (or products of polynomial and exponential functions). The derivatives, of all orders, of $\mathbf{y}^*$ and $\mathbf{u}^*$ must also converge to zero. It then follows that

$$\begin{aligned} \mathbf{C}\mathbf{x}^* &= \mathbf{y}^*\to\mathbf{0} \text{ as } t\to\infty \\ \mathbf{C}\mathbf{A}\mathbf{x}^* &= \dot{\mathbf{y}}^*(t)-\mathbf{C}\mathbf{B}\mathbf{u}^*(t)\to\mathbf{0} \text{ as } t\to\infty \\ &\ \vdots \end{aligned}$$

Then we conclude that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}\mathbf{x}^*(t)\to\mathbf{0} \text{ as } t\to\infty$$

By assumption, the observability matrix has full column rank, it follows that $\mathbf{x}^*(t)\to\mathbf{0}$ as $t\to\infty$ and the closed-loop system is stable.