# Linear System Theory
# Solution to Homework 7

1. (a) Since $\mathbf{u}(t) = \mathbf{0}$ for $t_a < t \le t_f$ , we have

$$\mathbf{x}(t_f) = e^{\mathbf{A}(t_f - t_a)}\mathbf{x}(t_a)$$

To achieve $\mathbf{x}(t_f) = \mathbf{x}_{des}$ we then should make

$$\mathbf{x}(t_a) = e^{\mathbf{A}(t_a - t_f)}\mathbf{x}_{des}$$

So $\mathbf{u}^*$ is the minimum energy input that moves the state from $\mathbf{0}$ to $\mathbf{x}(t_a) = e^{\mathbf{A}(t_a - t_f)}\mathbf{x}_{des}$. Therefore

$$\begin{aligned}
\mathbf{u}(t) &= \mathbf{B}^T e^{\mathbf{A}^T(t_a - t)}\mathbf{W}_r^{-1}(0, t_a)\mathbf{x}(t_a) \\
&= \mathbf{B}^T e^{\mathbf{A}^T(t_a - t)}\left(\int_0^{t_a} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\right)^{-1}\mathbf{x}(t_a) \\
&= \mathbf{B}^T e^{\mathbf{A}^T(t_a - t)}\left(\int_0^{t_a} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\right)^{-1} e^{\mathbf{A}(t_a - t_f)}\mathbf{x}_{des}
\end{aligned}$$

(b) The minimum control energy required is

$$\begin{aligned}
E_{t_a} &= \mathbf{x}^T(t_a)\mathbf{W}_r^{-1}(0, t_a)\mathbf{x}(t_a) \\
&= \mathbf{x}^T(t_a)\left(\int_0^{t_a} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\right)^{-1}\mathbf{x}(t_a) \\
&= \mathbf{x}_{des}^T e^{\mathbf{A}^T(t_a - t_f)}\left(\int_0^{t_a} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\right)^{-1} e^{\mathbf{A}(t_a - t_f)}\mathbf{x}_{des} \\
&= \mathbf{x}_{des}^T\left[e^{\mathbf{A}(t_f - t_a)}\left(\int_0^{t_a} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\right)e^{\mathbf{A}^T(t_f - t_a)}\right]^{-1}\mathbf{x}_{des} \\
&= \mathbf{x}_{des}^T\left[\int_0^{t_a} e^{\mathbf{A}(t_f - t_a + \tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_f - t_a + \tau)} d\tau\right]^{-1}\mathbf{x}_{des} \\
&= \mathbf{x}_{des}^T\left[\int_{t_f - t_a}^{t_f} e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt\right]^{-1}\mathbf{x}_{des}, \quad (t = t_f - t_a + \tau)
\end{aligned}$$

As $t_a$ increases, the interval of integration (in the last equation) increases while the matrix function inside the integral remains the same and positive semi-definite; hence the matrix inside the parenthesis becomes more positive definite and the inverse of it becomes less positive definite. We thus conclude that $E_{t_a}$ decreases as $t_a$ increase. Note also that as $t_a \to 0$, the minimum energy $\to \infty$.

2. The following equality holds for all $\mathbf{K} \in \mathbb{R}^{m\times n}$ and all $\lambda$:

$$\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} + \mathbf{BK} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{I} \end{bmatrix}$$

Since the matrix $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{I} \end{bmatrix}$ is invertible, it follows that $\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} + \mathbf{BK} & \mathbf{B} \end{bmatrix}$ and $\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix}$ have the same rank for all $\lambda$. The assertion follows from PBH rank test for controllability.

3. (a) The controllability matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which has rank 1; hence the system is not controllable.

   (b) With the state feedback the closed-loop system becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} -2-k_1 & 3-k_2 \\ -k_1 & 1-k_2 \end{bmatrix} \mathbf{x}$$

   Then the closed-loop characteristic equation is

$$\big(\lambda+2+k_1\big)\big(\lambda-1+k_2\big) + k_1(3-k_2) = \big(\lambda+2\big)\big(\lambda-1+k_1+k_2\big) = 0$$

   Hence the closed-loop eigenvalues are $-2$ and $1-k_1-k-2$. For $\lambda=-1$ to be one of the closed-loop eigenvalues, the condition is $k_1+k_2=2$.

   (c) Given that $k_1+k_2=2$,

$$\|\mathbf{K}\|^2 = k_1^2 + k_2^2 = (2-k_2)^2 + k_2^2 = 2k_2^2 - 4k_2 + 4$$

   Hence the minimum value of $\|\mathbf{K}\|^2$ is 2 when $k_2 = 1$ and $k_1 = 1$, i.e. $\mathbf{K} = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

4. Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$. The discrete-time system satisfies

$$\mathbf{A}_d = e^{\mathbf{A}h}, \quad \mathbf{B}_d = \int_0^h e^{\mathbf{A}\tau} d\tau \mathbf{B}$$

Let $\mathbf{w}$ be a left eigenvector of $\mathbf{A}$ associated with the eigenvalue $\lambda$, i.e. $\mathbf{w}^T\mathbf{A} = \lambda\mathbf{w}^T$. Then for any $t \in \mathbb{R}$,

$$\mathbf{w}^T e^{\mathbf{A}t} = \mathbf{w}^T\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2} + \cdots\right) = \mathbf{w}^T\left(1 + \lambda t + \frac{\lambda^2t^2}{2} + \cdots\right) = e^{\lambda t}\mathbf{w}^T$$

In particular, $\mathbf{w}^T e^{\mathbf{A}h} = e^{\lambda h}\mathbf{w}^T$. This implies that if $(\lambda, \mathbf{w})$ is a pair of eigenvalue and left eigenvector of $\mathbf{A}$, then $(e^{\lambda h}, \mathbf{w})$ is a pair of eigenvalue and left eigenvector of $\mathbf{A}_d$.

Let $\lambda_1, \cdots, \lambda_n$ be distinct eigenvalues of $\mathbf{A}$ associated with linearly independent left eigenvectors $\mathbf{w}_1, \cdots, \mathbf{w}_n$. Then any left eigenvector of $\mathbf{A}$ must be aligned with one of $\mathbf{w}_i$, i.e. if $\mathbf{w}$ is a left eigenvector, then $\mathbf{w} = \alpha\mathbf{w}_i$ for some $i = 1, \cdots, n$ and $\alpha \in \mathbb{R}$.

Because $(\mathbf{A}, \mathbf{B})$ is controllable, each left eigenvector $\mathbf{w}$ of $\mathbf{A}$ satisfies $\mathbf{w}^T\mathbf{B} \neq \mathbf{0}$. This is equivalent to $\mathbf{w}_i^T\mathbf{B} \neq \mathbf{0}$ for $i = 1, \cdots, n$. Then

$$\mathbf{w}_i^T\mathbf{B}_d = \mathbf{w}_i^T \int_0^h e^{\mathbf{A}\tau} d\tau \mathbf{B} = \int_0^h e^{\lambda_i\tau} d\tau \mathbf{w}_i^T\mathbf{B} \neq 0$$

If $\mathbf{A}_d$ has distinct eigenvalues, then we conclude that for any left eigenvector $\mathbf{w}$ of $\mathbf{A}_d$, $\mathbf{w}^T\mathbf{B}_d \neq \mathbf{0}$. Therefore $(\mathbf{A}_d, \mathbf{B}_d)$ is controllable.

However, the problem gets more complicated if the eigenvalues of $\mathbf{A}_d$ are not distinct. For example, if $\lambda_i - \lambda_l = j\frac{2\pi q}{h}$, $q \in \mathbb{Z}$, then $e^{\lambda_i h} = e^{\lambda_l h}$. Consequently, any $\mathbf{w} \in \text{span}\{\mathbf{w}_i, \mathbf{w}_l\}$ is a left eigenvector of $\mathbf{A}_d$. Note that $\mathbf{w}_i^T\mathbf{B}_d \neq \mathbf{0}$ and $\mathbf{w}_l^T\mathbf{B}_d \neq \mathbf{0}$ does not guarantee that $\mathbf{w}^T\mathbf{B}_d \neq \mathbf{0}$ for all $\mathbf{w} \in \text{span}\{\mathbf{w}_i, \mathbf{w}_l\}$.

For example, let $\mathbf{A} = \text{diag}(\lambda_1, \lambda_1 + j\frac{2\pi}{h}, \bar{\lambda}_1, \bar{\lambda}_1 - j\frac{2\pi}{h})$ and $\mathbf{B} = [1,\ 1,\ 1,\ 1]^T$. By PBH test, $(\mathbf{A}, \mathbf{B})$ is controllable, but the discretized system is not. This is because $\mathbf{A}_d = \text{diag}(e^{\lambda_1 h}, e^{\lambda_1 h}, e^{\bar{\lambda}_1 h}, e^{\bar{\lambda}_1 h})$ and $\begin{bmatrix} s\mathbf{I} - \mathbf{A}_d & \mathbf{B}_d \end{bmatrix}$ loses rank for $s = e^{\lambda_1 h}$.

5. (a) Note that $(\mathbf{A}, \mathbf{b}_2)$ is uncontrollable, and $\lambda = -1$ is the uncontrollable mode by $\mathbf{b}_2$. Hence the multiple eigenvalues that can be assigned by $\mathbf{b}_2$ are $\lambda = -1, -1$. Let $\mathbf{k}_1 = [k_{11},\ k_{12}]$. Since

$$\mathbf{A} - \mathbf{b}_2\mathbf{k}_1 = \begin{bmatrix} -2 - k_{11} & 1 - k_{12} \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of $\mathbf{A} - \mathbf{b}_2\mathbf{k}_1$ are $\lambda = -2 - k_{11}, -1$. Let $k_{11} = -1$, and choose $k_{12}$ arbitrarily, say $k_{12} = 0$. In other words, if you choose $\mathbf{k}_1 = [-1,\ 0]$, then $\mathbf{A} - \mathbf{b}_2\mathbf{k}_1$ has multiple eigenvalues at $-1, -1$.

(b) The statement is false. Let $\mathbf{k}_2 = [0,\ 1]$. Then

$$\mathbf{A} - \mathbf{b}_1\mathbf{k}_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

and $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_2, \mathbf{b}_2)$ is uncontrollable.