Linear System Theory
Homework 5

Due date: 11/30/2023

1. Consider the linear time-varying system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, $\mathbf{x}(0) = [x_{01}, x_{02}]^T$, where $\mathbf{A}(t) = \begin{bmatrix} -2 & \sin t \\ 0 & -1 \end{bmatrix}$.

   (a) Find the state transition matrix $\mathbf{\Phi}(t, 0)$.

   (b) Is the system stable or not? why?

   (c) It has been shown in class that we can find $\mathbf{P}(t) \in \mathbb{R}^{2\times 2}$, nonsingular for all t, such that if $\mathbf{x}(t) = \mathbf{P}(t)\mathbf{z}(t)$, then $\dot{\mathbf{z}}(t) = \mathbf{B}\mathbf{z}(t)$. Find $\mathbf{B}$ and its eigenvalues.

2. Consider the system described by the 2nd-order differential equation

$$\ddot{y}(t) + g(y)\dot{y}(t) + y(t) = 0$$

   where $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $g(0) > 0$.

   (a) Choose the state variables $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ and write the state equation of the system. Show that $(0, 0)$ is the only equilibrium point of the system.

   (b) Show that the system is stable near $(0, 0)$.

3. Consider the following linear time-varying system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, where $\mathbf{x}(t) \in \mathbf{R}^n$. Suppose that $\mathbf{\Phi}(t, s)$ is the state transition matrix that maps the state from time s to time t, i.e. $\mathbf{x}(t) = \mathbf{\Phi}(t, s)\mathbf{x}(s)$, $\forall t, s$.

   (a) Consider that s is also a variable. Show that

$$\frac{\partial \mathbf{\Phi}(t, s)}{\partial s} = -\mathbf{\Phi}(t, s)\mathbf{A}(s)$$

   (b) Consider the linear time-varying system $\dot{\mathbf{z}}(t) = -\mathbf{A}^T(t)\mathbf{z}(t)$ with initial condition $\mathbf{z}(t_0) = \mathbf{z}_0 \in \mathbb{R}^n$. Show that the solution is $\mathbf{z}(t) = \mathbf{\Phi}^T(t_0, t)\mathbf{z}_0$ for $t \geq t_0$.

4. Consider the following *switched linear system*:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{\sigma(t)}\mathbf{x}(t), \quad \mathbf{A}_{\sigma(t)} \in \{\mathbf{A}_1, \mathbf{A}_2\}$$

   where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n\times n}$. This means that the system switches between two LTI systems:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t), \quad \dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t)$$

   while the switching time is indicated by $\sigma(t)$, which can be arbitrary. Suppose that all eigenvalues of $\mathbf{A}_1, \mathbf{A}_2$ have negative real parts.

(a) Show that given a symmetric positive definite matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, if there exists $\mathbf{P} \in \mathbb{R}^{n\times n}$, symmetric positive definite, such that

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i < -\mathbf{Q}, \quad i = 1, 2 \tag{1}$$

then the switched linear system is stable.

(b) Suppose that $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$. Show that the following procedure can find $\mathbf{P}$ and $\mathbf{Q}$ that satisfy (1).

   i. Find $\mathbf{P}_1$ such that $\mathbf{A}_1^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A}_1 = -\mathbf{I}$.

   ii. Find $\mathbf{P}$ such that $\mathbf{A}_2^T\mathbf{P} + \mathbf{P}\mathbf{A}_2 = -\mathbf{P}_1$.

   iii. Choose $\alpha = \frac{1}{2}\min\{\lambda_{min}(\mathbf{P}_1), \lambda_{min}(\mathbf{Q}_1)\}$, where $\mathbf{Q}_1 = \int_0^\infty e^{\mathbf{A}_2^T t}e^{\mathbf{A}_2 t}dt$ and $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue. Then choose $\mathbf{Q} = \alpha\mathbf{I}$.

   Hint: $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ *implies* $e^{\mathbf{A}_1}e^{\mathbf{A}_2} = e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_2}e^{\mathbf{A}_1}$.