Linear System Theory
Homework 4

Due date: 11/2/2023

1. Let λ and $\sigma + j\omega$ be the real and complex eigenvalues of $\mathbf{A} \in \mathbb{R}^{3\times 3}$ with eigenvectors $\mathbf{u}$ and $\mathbf{v} \pm j\mathbf{w}$, respectively, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are real vectors, and let $\mathbf{S} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$.

   (a) Show that $\mathbf{S}$ is invertible.

   (b) Show that
   $$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix}$$

2. Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $\mathbf{p}_1, \cdots, \mathbf{p}_n$, $\|\mathbf{p}_i\| = 1$, $i = 1, \cdots, n$, with associated eigenvalues λ_i. Let $\mathbf{P} = [\mathbf{p}_1, \cdots, \mathbf{p}_n]$, $\mathbf{Q} = \mathbf{P}^{-1}$, and $\mathbf{q}_i^T$ be the ith row of $\mathbf{Q}$.

   (a) Let $\mathbf{R}_k = \mathbf{p}_k\mathbf{q}_k^T$. What is the rank of $\mathbf{R}_k$? Find linearly independent vectors that span $\mathcal{R}(\mathbf{R}_k)$. Similarly, find linearly independent vectors that span $\mathcal{N}(\mathbf{R}_k)$

   (b) Show that $\mathbf{R}_i\mathbf{R}_j = \mathbf{0}$ for $i \neq j$ and $\mathbf{R}_i^2 = \mathbf{R}_i$.

   (c) Show that
   $$(s\mathbf{I} - \mathbf{A})^{-1} = \sum_{i=1}^{n} \frac{\mathbf{R}_k}{s - \lambda_k}$$
   Notice that this is a partial fraction expansion of $(s\mathbf{I} - \mathbf{A})^{-1}$. For this reason, the $\mathbf{R}_i$'s are called the residue matrices of $\mathbf{A}$.

   (d) Show that $\mathbf{R}_1 + \cdots + \mathbf{R}_n = \mathbf{I}$.

3. The *adjoint* system associated with the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\dot{\mathbf{y}} = -\mathbf{A}^T\mathbf{y}$.

   (a) How are the state transition matrices of the system and the adjoint system related?

   (b) Show that for all t, $\mathbf{x}^T(t)\mathbf{y}(t) = \mathbf{x}^T(0)\mathbf{y}(0)$.

4. Consider the autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{x}(t) \in \mathbb{R}^2$ and $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix. Suppose that $\mathbf{x}(0) = [1, 2]^T$, $\mathbf{x}(1) = [-4, 10]^T$, and $\mathbf{x}(2) = [-11, 23]^T$.

   (a) Find $e^{\mathbf{A}}$.

   (b) Find the eigenvalues of $\mathbf{A}$.

   (c) Find an initial state $\mathbf{x}_0$, $\|\mathbf{x}_0\| = 1$, such that if $\mathbf{x}(0) = \mathbf{x}_0$, then $\mathbf{x}(t) \to 0$ as $t \to \infty$.

5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathcal{N}(\mathbf{A})$ be the null space of $\mathbf{A}$. Suppose that λ is an eigenvalue of $\mathbf{A}$ with geometric multiplicity m.

   (a) Show that

   $$\mathcal{N}(\lambda\mathbf{I} - \mathbf{A}) \subset \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^2\big) \subset \cdots \subset \mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^m\big)$$

   Note: You have to show that $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$ is a proper subset of $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^{i+1}\big)$ for $i = 1, 2, \cdots, m-1$, i.e. $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$ is contained in but not equal to $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^{i+1}\big)$.

   (b) Given $\mathbf{A} \in \mathbb{R}^{n\times n}$, a subspace $S \subseteq \mathbb{R}^n$ is said to be $\mathbf{A}$-*invariant* if $\mathbf{x} \in S$ implies that $\mathbf{A}\mathbf{x} \in S$. Show that $\mathcal{N}\big((\lambda\mathbf{I} - \mathbf{A})^i\big)$ is $\mathbf{A}$-invariant for all $i \in \mathbb{N}$.

   (c) Let

   $$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

   Decompose $\mathbb{R}^3$ into a sum of $\mathbf{A}$-invariant subspaces, i.e. $\mathbb{R}^3 = N_1 + \cdots + N_r$, where N_i is an $\mathbf{A}$-invariant subspace for $i = 1, \cdots, r$. Find a basis of each N_i.