Linear System Theory
Solution to Homework 8

1. (a) The closed-loop system is $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}_2\mathbf{k}_2)\mathbf{x}$. Let $\mathbf{k}_2 = [k_{21},\ k_{22},\ k_{23}]$. Then

$$\det \begin{bmatrix} s+1 & 0 & 0 \\ -2 & s+2 & 3 \\ 2+k_{21} & k_{22} & s-1+k_{23} \end{bmatrix} = (s{+}1)\big((s{+}2)(s{-}1{+}k_{23}){-}3k_{22}\big) = (s{+}1)^3$$

Hence

$$s^2 + (1+k_{23})s - 2(1-k_{23}) - 3k_{22} = s^2 + 2s + 1 \Rightarrow k_{23} = 1,\ k_{22} = \frac{-1}{3}$$

k_{21} can be arbitrary. We choose $k_2 = [0,\ -\frac{1}{3},\ 1]$.

(b) By PBH test, $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is uncontrollable if and only if $\exists\eta \in \mathbb{C}^3$ and $\exists\lambda \in \mathbb{C}$ such that $\eta^T(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1) = \lambda\eta^T$ and $\eta^T\mathbf{b}_2 = 0$. Because all the vectors that are orthogonal to $\mathbf{b}_2$ can be expressed as a linear combination of $\eta_1 = [1,\ 0,\ 0]^T$ and $\eta_2 = [0,\ 1,\ 0]^T$, controllability of $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is assured if $\alpha\eta_1 + \beta\eta_2$ is not a left eigenvector of $\mathbf{A} - \mathbf{b}_1\mathbf{k}_1$ for any $\alpha, \beta \in \mathbb{R}$. Hence

$$\begin{aligned} & (\alpha\eta_1 + \beta\eta_2)^T(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1) \\ = \ & \alpha\begin{bmatrix} -1 & 0 & 0 \end{bmatrix} - \mathbf{k}_1 + \beta\begin{bmatrix} 2 & -2 & -3 \end{bmatrix} \\ = \ & \begin{bmatrix} -\alpha + 2\beta & -2\beta & -3\beta \end{bmatrix} - \mathbf{k}_1 \\ \neq \ & \lambda(\alpha\eta_1 + \beta\eta_2)^T \\ = \ & \lambda\begin{bmatrix} \alpha & \beta & 0 \end{bmatrix} \end{aligned}$$

We can choose $\mathbf{k}_1 = [0,\ 1,\ 0]$, since $-2\beta - 1 = \lambda\beta$ and $-3\beta = 0$ cannot hold simultaneously. With such a $\mathbf{k}_1$, $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is controllable.

2. (a) Suppose that $(\mathbf{A}, \mathbf{B})$ is uncontrollable. Then by PBH test, there exist a nonzero vector $\eta \in \mathbb{C}^{2n}$ and $\lambda \in \mathbb{C}$ such that $\eta^T\mathbf{A} = \lambda\eta^T$, and $\eta^T\mathbf{B} = \mathbf{0}$. Let $\eta = [\eta_1^T,\ \eta_2^T]^T$, where $\eta_1, \eta_2 \in \mathbb{C}^n$. Then

$$\begin{aligned} \begin{bmatrix} \eta_1^T & \eta_2^T \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} &= \eta_2^T\mathbf{M}^{-1} = 0 \Rightarrow \eta_2 = \mathbf{0} \\ \begin{bmatrix} \eta_1^T & \eta_2^T \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \eta_1^T \end{bmatrix} = \lambda\begin{bmatrix} \eta_1^T & \mathbf{0} \end{bmatrix} \Rightarrow \eta_1 = \mathbf{0} \end{aligned}$$

However, this is a contradiction because η_1 and η_2 cannot vanish simultaneously. Hence $(\mathbf{A}, \mathbf{B})$ is controllable.

(b) Define $\mathbf{x}_3 = \ddot{\mathbf{q}}$ and $\mathbf{v} = \dot{\mathbf{u}}$. Expand the model as follows:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}\mathbf{v}$$

Let

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}, \quad \bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$$

Define the cost function as

$$J = \int_0^\infty (\mathbf{q}^T\mathbf{Q}\mathbf{q} + \mathbf{v}^T\mathbf{R}\mathbf{v})dt = \int_0^\infty \left(\bar{\mathbf{x}}^T\bar{\mathbf{Q}}\bar{\mathbf{x}} + \mathbf{v}^T\mathbf{R}\mathbf{v}\right)dt, \quad \bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{n\times n}$ are symmetric positive definite. Then the optimal state feedback control law that minimizes J is

$$\mathbf{v} = -\mathbf{R}^{-1}\bar{\mathbf{B}}^T\bar{\mathbf{P}}\mathbf{x}$$

where $\bar{\mathbf{P}}$ is the solution to the following algebraic Riccati equation (ARE)

$$\bar{\mathbf{A}}^T\bar{\mathbf{P}} + \bar{\mathbf{P}}\bar{\mathbf{A}} - \bar{\mathbf{P}}\bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}^T\bar{\mathbf{P}} + \bar{\mathbf{Q}} = \mathbf{0}$$

Let $\mathbf{R}^{-1}\bar{\mathbf{B}}^T\bar{\mathbf{P}} = \bar{\mathbf{K}} = [\mathbf{K}_1,\ \mathbf{K}_2,\ \mathbf{K}_3]$, where $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathbb{R}^{n\times n}$. Then

$$\mathbf{v} = -\bar{\mathbf{K}}\bar{\mathbf{x}} = -\left(\mathbf{K}_1\mathbf{x}_1 + \mathbf{K}_2\mathbf{x}_2 + \mathbf{K}_3\mathbf{x}_3\right) = -\left(\mathbf{K}_1\mathbf{q} + \mathbf{K}_2\dot{\mathbf{q}} + \mathbf{K}_3\ddot{\mathbf{q}}\right)$$

Thus the control law is

$$\mathbf{u}(t) = \int_0^t \mathbf{v}(\tau)d\tau = -\left(\mathbf{K}_1\int_0^t \mathbf{q}(\tau)d\tau + \mathbf{K}_2\mathbf{q}(t) + \mathbf{K}_3\dot{\mathbf{q}}(t)\right)$$

Clearly, this is a PID controller with $\mathbf{K}_P = \mathbf{K}_2$, $\mathbf{K}_D = \mathbf{K}_3$, and $\mathbf{K}_I = \mathbf{K}_1$.

3. Let $\mathbf{P} = \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}$, where $p_1 > 0$ and $p_1p_2 - p_3^2 > 0$. The ARE $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$ yields

$$\begin{aligned} -p_3^2 + q &= 0 \\ p_1 - 10p_3 - p_2p_3 &= 0 \\ p_2^2 + 20p_2 - 2p_3 &= 0 \end{aligned}$$

Hence $p_3 = \sqrt{q}$, $p_2 = -10 + \sqrt{100 + 2\sqrt{q}}$, $p_1 = \sqrt{q}\sqrt{100 + 2\sqrt{q}}$.
Note: The solution $p_3 = -\sqrt{q}$ leads to $p_2 < 0$, which violates the requirement that $\mathbf{P} > 0$.

Then LQR control law

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} = -(p_3x_1 + p_2x_2) = -\sqrt{q}x_1 - \left(\sqrt{100 + 2\sqrt{q}} - 10\right)x_2$$

The closed-loop system is $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -\sqrt{q} & -\sqrt{100 + 2\sqrt{q}} \end{bmatrix}\mathbf{x}$.
Hence the closed-loop characteristic equation is

$$s^2 + \sqrt{100 + 2\sqrt{q}}s + \sqrt{q} = 0$$

4. The optimal feedback gain is given by $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$, where $\mathbf{P} > 0$ satisfies ARE:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

$\mathbf{Q} > 0$ and $\mathbf{R} > 0$. To show $\mathbf{A} - \alpha\mathbf{B}\mathbf{K}$ is stable, it suffices to show that

$$\mathbf{Q}_\alpha = \left(\mathbf{A} - \alpha\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\right)^T\mathbf{P} + \mathbf{P}\left(\mathbf{A} - \alpha\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\right) < 0$$

By ARE we have

$$-\mathbf{Q}_\alpha = \mathbf{Q} + (2\alpha - 1)\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

Since $\mathbf{Q} > 0$ and $\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \geq 0$, we have $-\mathbf{Q}_\alpha > 0$ for $\alpha \geq \frac{1}{2}$.

5. (a) Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then the state equation is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) Use the Matlab command `lqr` to find the optimal feedback gain $\mathbf{K} = [1,\ 0.3181]$. The states and the control input are shown in Figure 1

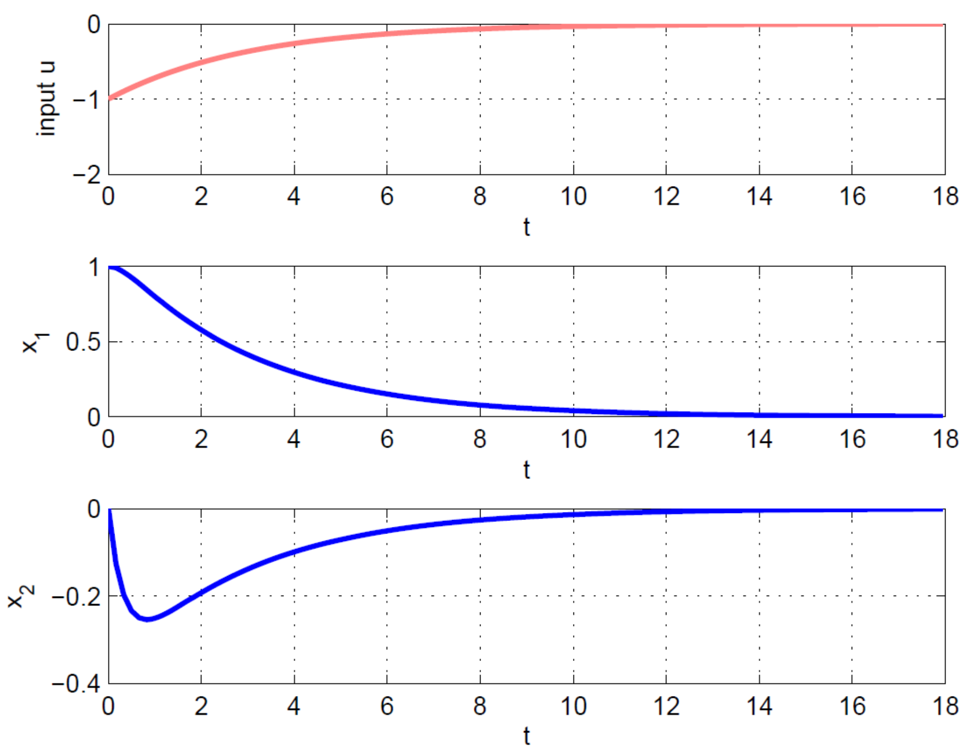

Figure 1: Control Input and States for Problem 5(b)

(c) The discretized system is $\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d u(k)$. Use Matlab command `c2d` to find $\mathbf{A}_d = \begin{bmatrix} 1 & 0.009851 \\ 0 & 0.9704 \end{bmatrix}$ and $\mathbf{B}_d = \begin{bmatrix} 4.95 \times 10^{-5} \\ 0.009851 \end{bmatrix}$.

(d) Solve the difference Riccati equation and find the feedback gain $\mathbf{K}(k)$ for each time step k. The gain is shown in Figure 2. The states and the control input are shown in Figure 3.

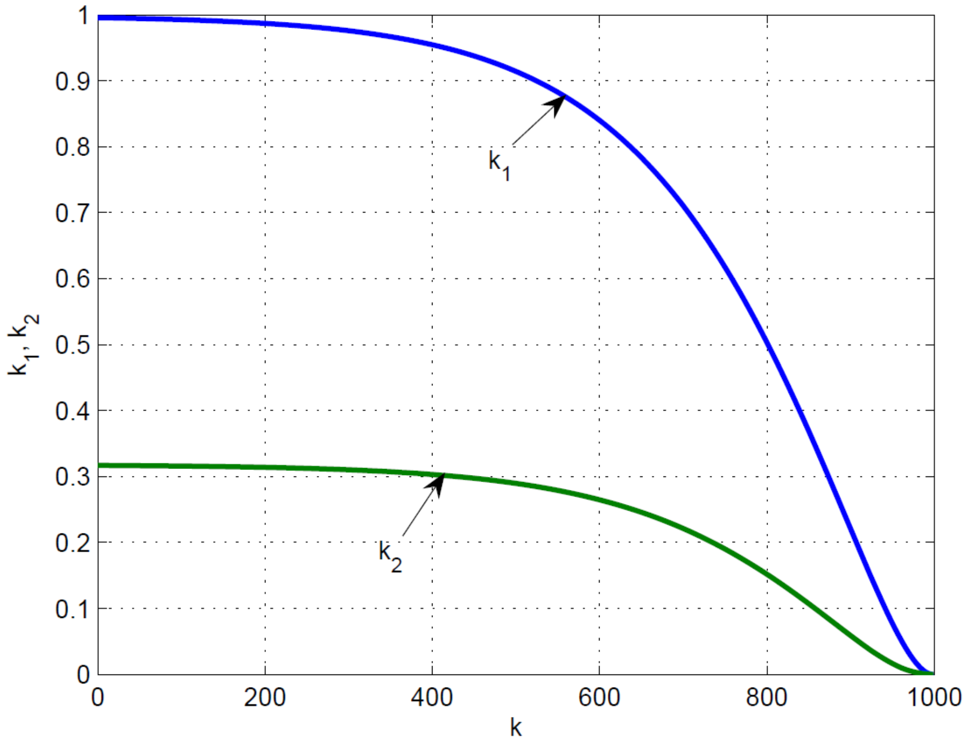

Figure 2: The feedback gain for Problem 5(d)

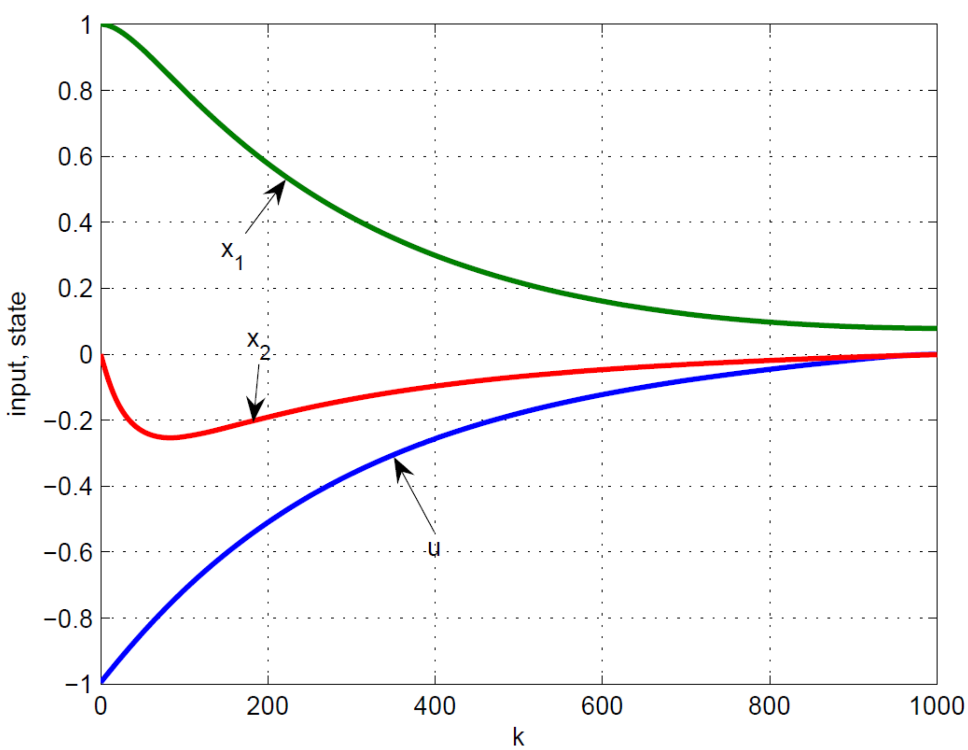

Figure 3: Control Input and States for Problem 5(d)