Linear System Theory
Homework 7

Due date: 12/14/2023

1. We consider the controllable system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{x}(0) = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$. You are to determine an input that results in $\mathbf{x}(t_f) = \mathbf{x}_{des}$, where t_f and $\mathbf{x}_{des}$ are given. You are also given t_a where $0 < t_a \le t_f$, and have the constraint that $\mathbf{u}(t) = \mathbf{0}$ for $t > t_a$. Roughly speaking, you are allowed to apply a (nonzero) input during the "controlled portion" of the trajectory, that is, from $t = 0$ until $t = t_a$; from $t = t_a$ until the final time t_f, the system "coasts" or "drifts" with $\mathbf{u}(t) = \mathbf{0}$. Among all $\mathbf{u}$ that satisfy these specifications, $\mathbf{u}^*$ will denote the one that minimizes the energy

$$\int_0^{t_a} \|\mathbf{u}(t)\|^2 dt$$

   (a) Give an explicit formula for $\mathbf{u}^*$.

   (b) Show that the minimum control energy decreases as $t_a < t_f$ is increased.

2. Given the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$. Suppose the input $\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{K}\mathbf{x}(t)$, a state feedback plus an external input. The closed-loop system, with input $\mathbf{v}(t)$, becomes $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t)$. Show that if $(\mathbf{A}, \mathbf{B})$ is controllable, then for any $\mathbf{K}$, $(\mathbf{A} - \mathbf{B}\mathbf{K}, \mathbf{B})$ is also controllable.

   Hint: PBH rank test

3. Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

   (a) Is the system controllable?

   (b) Find the set of all state feedback gain matrices $\mathbf{K} = [k_1, k_2]$ with which one of the closed-loop eigenvalues is $\lambda = -1$.

   (c) Among the gain matrices found in (b), find one that has the smallest 2-norm.

4. Suppose the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable and $\mathbf{A}$ has distinct eigenvalues. Suppose we apply piecewise constant input and sample the system with sampling period h to get the discretized system

$$\mathbf{x}_d(k+1) = \mathbf{A}_d\mathbf{x}_d(k) + \mathbf{B}_d\mathbf{u}_d(k)$$

   where $\mathbf{u}(t) = \mathbf{u}(kh)$ for $kh \le t < (k+1)h$ and $\mathbf{x}_d(k) = \mathbf{x}(kh)$, $\mathbf{u}_d(k) = \mathbf{u}(kh)$. Show that the discrete-time system is controllable unless $\mathbf{A}$ has a pair of eigenvalues λ_i and λ_l such that

$$\lambda_i - \lambda_l = j\frac{2\pi q}{h}$$

   for some nonzero integer q.

5. Consider an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$\mathbf{b}_1 = [1,\ -1]^T$, and $\mathbf{b}_2 = [1,\ 0]^T$ are the 1st and 2nd columns of $\mathbf{B}$, respectively.

(a) Find $\mathbf{k}_1 \in \mathbb{R}^{1\times 2}$ such that $\mathbf{A} - \mathbf{b}_2\mathbf{k}_1$ has multiple eigenvalues.

(b) Since $(\mathbf{A}, \mathbf{b}_1)$ is controllable, someone claims that $(\mathbf{A} - \mathbf{b}_1\mathbf{k}_2, \mathbf{b}_2)$ is always controllable for any $\mathbf{k}_2 \in \mathbf{R}^{1\times 2}$, $\mathbf{k}_2 \neq \mathbf{0}$.. If you agree his/her argument, prove it; otherwise, give a counterexample.