Linear System Theory
Homework 2

Due date: 10/19/2023

1. Suppose $E \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{n\times n}$ are invertible matrices. Show that for $A \in \mathbb{R}^{m\times n}$,
$$\mathcal{N}(EA) = \mathcal{N}(A), \quad \mathcal{R}(AF) = \mathcal{R}(A)$$

2. A linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(x) = Ax$ is *one-to-one* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; it is *onto* if for every $y \in \mathbb{R}^m$, there exists $x \in \mathbb{R}^n$ such that $f(x) = y$. Show that
   (a) f is one-to-one if and only if $\mathcal{N}(A) = \{0\}$.
   (b) f is onto if and only if $\mathcal{R}(A) = \mathbb{R}^m$.

3. Let $x, y \in \mathbb{R}^n$ and $x \neq 0$, $y \neq 0$. We say $p \in \mathbb{R}^n$ is the projection of y onto x if p is the point on the subspace spanned by x that is closet to y.
   (a) Show that $p = \frac{x^T y}{x^T x} x$.
   (b) Use the fact $\|y - p\| \geq 0$ to prove Cauchy-Schwarz inequality $\|x^T y\| \leq \|x\|\|y\|$.

4. Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that $\|U^T x\| \leq \|x\|$ for all $x \in \mathbb{R}^n$. When do we have $\|U^T x\| = \|x\|$?

5. A matrix $P \in \mathbb{R}^{n\times n}$ is called a *projection matrix* if $P = P^T$ and $P^2 = P$.
   (a) Show that if P is a projection matrix, so is $I - P$.
   (b) Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthogonal and each has unit length. Show that UU^T is a projection matrix.
   (c) Suppose $A \in \mathbb{R}^{n\times k}$ is full rank, with $k \leq n$. Show that $A(A^T A)^{-1} A^T$ is a projection matrix.
   (d) If S is a subspace of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, then the point in S closest to x is called the projection of x on S. Show that if $P \in \mathbb{R}^{n\times n}$ is a projection matrix, then $y = Px$ is the projection of x on $\mathcal{R}(P)$.
   Hint: Show that $\|x - Px\| \leq \|x - z\|$ for all $z \in \mathcal{R}(P)$.