Linear System Theory
Solution to Homework 3

1. We have

$$e^{\mathbf{A}t}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} e^{-3t} & e^{t} \\ -3e^{-3t} & e^{t} \end{bmatrix}$$

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{bmatrix} e^{-3t} & e^{t} \\ -3e^{-3t} & e^{t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-3t} & e^{t} \\ -3e^{-3t} & e^{t} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}e^{-3t}+\frac{3}{4}e^{t} & -\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t} \\ -\frac{3}{4}e^{-3t}+\frac{3}{4}e^{t} & \frac{3}{4}e^{-3t}+\frac{1}{4}e^{t} \end{bmatrix} \end{aligned}$$

To find $\mathbf{A}$, consider the relation $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$ for all t. Then

$$\left.\frac{d}{dt}e^{\mathbf{A}t}\right|_{t=0} = \mathbf{A}e^{\mathbf{0}} = \mathbf{A} = \begin{bmatrix} -\frac{3}{4}+\frac{3}{4} & \frac{3}{4}+\frac{1}{4} \\ \frac{9}{4}+\frac{3}{4} & -\frac{9}{4}+\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

2. (a) Use Matlab command expm(A) to find $e^{\mathbf{A}}$. The result is $e^{\mathbf{A}} = \begin{bmatrix} -1.4077 & 2.4164 \\ -2.4164 & 2.2169 \end{bmatrix}$

   (b) For simplicity, let $e^{\mathbf{A}} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where p, q, r, s have been found in part (a). Then

$$\begin{aligned} \mathbf{x}(1) = e^{\mathbf{A}}\mathbf{x}(0) \;&\Rightarrow\; \begin{bmatrix} x_1(1) \\ 2 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 1 \\ x_2(0) \end{bmatrix} \\ &\Rightarrow\; x_2(0) = \frac{2-r}{s} = 1.9922, \quad x_1(1) = p + qx_2(0) = 3.4062 \end{aligned}$$

   Then

$$\mathbf{x}(2) = e^{\mathbf{A}}\mathbf{x}(1) = \begin{bmatrix} -1.4077 & 2.4164 \\ -2.4164 & 2.2169 \end{bmatrix}\begin{bmatrix} 3.4062 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.0378 \\ -3.7970 \end{bmatrix}$$

3. Because $\mathbf{A}(\sigma\mathbf{I}) = (\sigma\mathbf{I})\mathbf{A}$, we have $e^{\mathbf{A}+\sigma\mathbf{I}} = e^{\mathbf{A}}e^{\sigma\mathbf{I}} = e^{\sigma}e^{\mathbf{A}}$. Hence

$$\mathbf{z}(t) = e^{(\mathbf{A}+\sigma\mathbf{I})t}\mathbf{z}(0) = e^{\sigma t}e^{\mathbf{A}t}\mathbf{x}(0) = e^{\sigma t}\mathbf{x}(t)$$

4. (a) The real modal form of $\mathbf{A}$ is $\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$, where $\mathbf{\Lambda} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$. Since $\mathbf{x}(t) = \mathbf{T}\mathbf{z}(t)$, we have $\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} = \mathbf{\Lambda}\mathbf{z}$. Note that

$$(s\mathbf{I}-\mathbf{\Lambda})^{-1} = \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix}^{-1} = \frac{1}{(s-\sigma)^2+\omega^2}\begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix}$$

Therefore

$$e^{\mathbf{\Lambda}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{\Lambda})^{-1}\right\} = e^{\sigma t}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

And

$$\mathbf{z}(t) = e^{\mathbf{\Lambda}t}\mathbf{z}(0) = e^{\sigma t}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\mathbf{z}(0) \tag{1}$$

(b) $\mathbf{T} = [\mathbf{u}, \mathbf{v}] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and then $\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Let $\mathbf{z}(t) = [z_1(t), z_2(t)]^T$ and define $\theta(t) = \tan^{-1}\frac{z_2(t)}{z_1(t)}$. From (1) we notice that $\|\mathbf{z}(t)\|_2 = e^{\sigma t}\|\mathbf{z}(0)\|_2$, and $\theta(t) = -\omega t + \theta(0)$.

$$\mathbf{z}(0) = \mathbf{T}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{z}(1) = \mathbf{T}^{-1}\mathbf{x}(1) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Since $\|\mathbf{z}(1)\| = \frac{1}{2}\|\mathbf{z}(0)\|$, we have $e^{\sigma} = \frac{1}{2}$ and therefore $\sigma = \log\frac{1}{2} = -0.6931$.

On the other hand,

$$\omega = \theta(0) - \theta(1) = -\frac{\pi}{2}$$

Therefore, the eigenvalues of $\mathbf{A}$ are $-0.6931 \pm j\frac{\pi}{2}$.

(c)

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{T}e^{\mathbf{\Lambda}}\mathbf{T}^{-1} = \mathbf{T}e^{\sigma}\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}\mathbf{T}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

5. The characteristic polynomial is

$$\chi(s) = (s - 0.5)^3 = s^3 - 1.5s^2 + 0.75s - 0.125$$

By Cayley-Hamilton theorem, we have

$$\chi(\mathbf{A}) = \mathbf{A}^3 - 1.5\mathbf{A}^2 + 0.75\mathbf{A} - 0.125\mathbf{I} = \mathbf{0} \Rightarrow \mathbf{I} = \frac{1}{0.125}\mathbf{A}(\mathbf{A}^2 - 1.5\mathbf{A} + 0.75\mathbf{I})$$

Hence

$$\mathbf{A}^{-1} = \frac{1}{0.125}(\mathbf{A}^2 - 1.5\mathbf{A} + 0.75\mathbf{I}) = 8\mathbf{A}^2 - 12\mathbf{A} + 6\mathbf{I}$$