Linear System Theory
Solution to Midterm

11/9/2023

1. (a) Suppose that $\mathbf{x} \in \mathcal{R}(\mathbf{P})$. Then $\exists \mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{x} = \mathbf{P}\mathbf{z} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{z} = \mathbf{A}\mathbf{w}$, where $\mathbf{w} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{z}$. Hence $\mathbf{x} \in \mathcal{R}(\mathbf{A})$, and therefore $\mathcal{R}(\mathbf{P}) \subset \mathcal{R}(\mathbf{A})$.

   On the other hand, if $\mathbf{x} \in \mathcal{R}(\mathbf{A})$, then $\exists \mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{x} = \mathbf{A}\mathbf{z}$. Define $\mathbf{y} = \mathbf{A}\mathbf{z}$; then $\mathbf{A}^T\mathbf{A}\mathbf{z} = \mathbf{A}^T\mathbf{y}$, or $\mathbf{z} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$. This means $\mathbf{x} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} = \mathbf{P}\mathbf{y}$. Thus $\mathbf{x} \in \mathcal{R}(\mathbf{P})$. and $\mathcal{R}(\mathbf{A}) \subset \mathcal{R}(\mathbf{P})$.

   According to the above arguments, we conclude that $\mathcal{R}(\mathbf{P}) = \mathcal{R}(\mathbf{A})$.

   (b) Suppose that $\mathbf{x} \in \mathcal{R}(\mathbf{I} - \mathbf{P})$. Then $\exists \mathbf{z} \in \mathbf{R}^m$ such that $\mathbf{x} = (\mathbf{I} - \mathbf{P})\mathbf{z}$. Hence $\mathbf{A}^T\mathbf{x} = (\mathbf{A}^T - \mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{z} = \mathbf{0}$, implying $\mathbf{x} \in \mathcal{N}(\mathbf{A}^T)$ and $\mathcal{R}(\mathbf{I} - \mathbf{P}) \subset \mathcal{N}(\mathbf{A}^T)$.

   On the other hand, if $\mathbf{x} \in \mathcal{N}(\mathbf{A}^T)$, i.e. $\mathbf{A}^T\mathbf{x} = \mathbf{0}$, then $\mathbf{P}\mathbf{x} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{x} = \mathbf{0}$. Thus $\mathbf{x} = \mathbf{x} - \mathbf{P}\mathbf{x} = (\mathbf{I} - \mathbf{P})\mathbf{x}$, implying that $\mathbf{x} \in \mathcal{R}(\mathbf{I} - \mathbf{P})$ and $\mathcal{N}(\mathbf{A}^T) \subset \mathcal{R}(\mathbf{I} - \mathbf{P})$.

   According to the above arguments, we conclude that $\mathcal{R}(\mathbf{I} - \mathbf{P}) = \mathcal{N}(\mathbf{A}^T)$.

2. (a) Since $\mathbf{v} = \mathbf{x}(t+1) - \mathbf{x}(t)$ for all $t \geq 0$, we have

$$\mathbf{v} = \mathbf{x}(2) - \mathbf{x}(1) = e^{\mathbf{A}}(\mathbf{x}(1) - \mathbf{x}(0)) = e^{\mathbf{A}}\mathbf{v} \tag{1}$$

   In other words, $e^{\mathbf{A}}$ has an eigenvalue 1 and the associated eigenvector is $\mathbf{v}$. By the spectral mapping theorem, $\lambda = \ln 1 = 0$ is an eigenvalue of $\mathbf{A}$, and the associated eigenvector is $\mathbf{v}$.

   (b)

$$\mathbf{x}(1) - \mathbf{x}(0) = \mathbf{v} \ \Rightarrow \ e^{\mathbf{A}}\mathbf{x}(0) = \mathbf{x}(0) + \mathbf{v} \tag{2}$$

   Thus (2) implies that $\mathbf{x}(0)$ is a generalized eigenvector of $e^{\mathbf{A}}$ associated with the eigenvalue 1. Combining (1) and (2) yields

$$e^{\mathbf{A}} \begin{bmatrix} \mathbf{v} & \mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{x}(0) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{3}$$

   Since $\mathbf{x}(0)$ and $\mathbf{v}$ are linearly independent, $e^{\mathbf{J}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the Jordan form of $e^{\mathbf{A}}$.

   (c) $\mathbf{v} = \mathbf{x}(1) - \mathbf{x}(0) = [1, 1]^T$. Let $\mathbf{T} = [\mathbf{v}, \mathbf{x}(0)] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Note that if $\mathbf{J} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $e^{\mathbf{J}t} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. For $t = 1$ and $\lambda = 0$, we have $e^{\mathbf{J}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then from (3) we have

$$\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

3. (a) The characteristic equation of $\mathbf{A}$ is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda+2 & -2 \\ 1 & \lambda \end{bmatrix} = \lambda(\lambda+2)+2 = \lambda^2+2\lambda+2 = 0$$

Hence the eigenvalues of $\mathbf{A}$ is $\lambda = \sigma \pm j\omega = -1 \pm j$, i.e. $\sigma = -1$ and $\omega = 1$. For $\lambda = -1+j$, the corresponding eigenvector is $\mathbf{v} = [v_1, v_2]^T \in \mathbb{C}^2$. Then

$$\big((-1+j)\mathbf{I} - \mathbf{A}\big)\mathbf{v} = \begin{bmatrix} 1+j & -2 \\ 1 & -1+j \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = (1-j)v_2$$

Hence the eigenvector is $\mathbf{v} = \begin{bmatrix} 1-j \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + j \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Choose $\mathbf{T} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Then

$$\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}, \text{ where } \mathbf{\Lambda} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

and

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\dot{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} = \mathbf{\Lambda}\mathbf{z}$$

(b)

$$\begin{aligned} \dot{r} &= \frac{1}{2}\frac{2z_1\dot{z}_1 + 2z_2\dot{z}_2}{\sqrt{z_1^2+z_2^2}} = \frac{z_1(\sigma z_1 + \omega z_2) + z_2(-\omega z_1 + \sigma z_2)}{r} = \frac{\sigma r^2}{r} = \sigma r = -r \\ \dot{\theta} &= \frac{\frac{\dot{z}_2 z_1 - z_2 \dot{z}_1}{z_1^2}}{1+(\frac{z_2}{z_1})^2} = \frac{z_1(-\omega z_1 + \sigma z_2) - z_2(\sigma z_1 + \omega z_2)}{z_1^2+z_2^2} = \frac{-\omega r^2}{r^2} = -\omega = -1 \end{aligned}$$

Hence the state equation is

$$\frac{d}{dt}\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{4}$$

(c) Given an initial state $[r(0), \theta(0)]^T$, the solution to (4) is

$$r(t) = e^{-t}r(0), \quad \theta(t) = -t + \theta(0).$$

4. (a) False

Suppose that $\mathbf{A} = \begin{bmatrix} 0 & \frac{3\pi}{4} \\ -\frac{3\pi}{4} & 0 \end{bmatrix}$. The eigenvalues of $\mathbf{A}$ are $\pm j\frac{3\pi}{4}$. Then by the spectral mapping theorem, the eigenvalues of $e^{\mathbf{A}}$ are $e^{\pm j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} \pm j \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}(-1 \pm j)$. Hence the eigenvalues of $e^{\mathbf{A}}$ have negative real parts.

(b) True

Let $\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1}$, where $\mathbf{J} = \mathrm{diag}(\mathbf{J}_1, \cdots, \mathbf{J}_r)$, and $\mathbf{J}_i \in \mathbf{R}^{n_i \times n_i}$ is the i-th Jordan block. $\mathbf{T} = [\mathbf{T}_1, \cdots, \mathbf{T}_r]$, where $\mathbf{T}_i \in \mathbf{R}^{n \times n_i}$. Without loss of generality, we can assume that $\mathbf{T}_1 = [\mathbf{v}, \cdots,]$. $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ. Note that $\mathbf{T}^{-1}\mathbf{T} = [\mathbf{T}^{-1}\mathbf{v}, \cdots] = \mathbf{I}$. Thus $\mathbf{T}^{-1}\mathbf{v} = \mathbf{e}_1 = [1, 0, \cdots, 0]^T$. In addition, $e^{\mathbf{J}t} = \mathrm{diag}(e^{\mathbf{J}_1 t}, \cdots, e^{\mathbf{J}_r t})$, and $e^{\mathbf{J}t}\mathbf{e}_1 = e^{\lambda t}\mathbf{e}_1$.

Suppose that $\mathbf{x}(0) = \mathbf{v}$. Then

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{v} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}\mathbf{v} = \mathbf{T}e^{\mathbf{J}t}\mathbf{e}_1 = e^{\lambda t}\mathbf{T}\mathbf{e}_1 = e^{\lambda t}\mathbf{v}$$

5. (a) The characteristic equation of $\mathbf{A}$ is

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\begin{bmatrix} \lambda-1 & 0 & 0 \\ -3 & \lambda-1 & 2 \\ -2 & 0 & \lambda+1 \end{bmatrix} = (\lambda-1)^2(\lambda+1) = 0$$

Hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ (with multiplicity 2 and $\lambda_2 = -1$. Let $\mathbf{t}_1 = [t_{11}, t_{21}, t_{31}]^T$ be the eigenvector associated with λ_1, i.e.

$$(\lambda_1\mathbf{I}-\mathbf{A})\mathbf{t}_1 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix} = \mathbf{0} \Rightarrow \begin{cases} -3t_{11}+2t_{31} &= 0 \\ -2t_{11}+2t_{31} &= 0 \end{cases} \Rightarrow \begin{cases} t_{11}=0 \\ t_{31}=0 \end{cases}$$

Therefore the eigenvalue λ_1 has only one linearly independent eigenvector $\mathbf{t}_1 = [0,1,0]^T$. The generalized eigenvector $\mathbf{t}_2 = [t_{12}, t_{22}, t_{32}]^T$ is

$$(\lambda_1\mathbf{I}-\mathbf{A})\mathbf{t}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \end{bmatrix} = -\mathbf{t}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{t}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

On the other hand, let $\mathbf{t}_3 = [t_{13}, t_{23}, t_{33}]^T$ be the eigenvector associated with λ_2, i.e.

$$(\lambda_2\mathbf{I}-\mathbf{A})\mathbf{t}_3 = \begin{bmatrix} -2 & 0 & 0 \\ -3 & -2 & 2 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} t_{13} \\ t_{23} \\ t_{33} \end{bmatrix} = \mathbf{0} \Rightarrow \begin{cases} t_{13} &= 0 \\ -t_{23}+t_{33} &= 0 \end{cases} \Rightarrow \mathbf{t}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $\mathbf{T} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3]$. Then, the Jordan form of $\mathbf{A}$ is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) Given $\mathbf{T}$ in part (a), we have $\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then $\mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$$\begin{aligned} H(s) &= \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} = \mathbf{c}\mathbf{T}(s\mathbf{I}-\mathbf{J})^{-1}\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{c}\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \end{bmatrix}\begin{bmatrix} s-1 & -1 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \mathbf{c}\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \end{bmatrix}\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \frac{\mathbf{c}\mathbf{t}_1+\mathbf{c}\mathbf{t}_2}{s-1} + \frac{\mathbf{c}\mathbf{t}_1}{(s-1)^2} + \frac{-\mathbf{c}\mathbf{t}_3}{s+1} \end{aligned}$$

(c) For the unstable eigenvalue $\lambda_1 = 1$ not shown in $H(s)$, the residues associated with λ_1 must vanish. In other words, $\mathbf{c}$ should be orthogonal to both $\mathbf{t}_1$ and $\mathbf{t}_2$. Thus we choose $\mathbf{c} = (\mathbf{t}_1 \times \mathbf{t}_2)^T = [1, 0, -1]$. In this case, we have $H(s) = -\frac{\mathbf{c}\mathbf{t}_3}{s+1} = \frac{1}{s+1}$.

(d) Let $\mathbf{X}(s)$ be the Laplace transform of $\mathbf{x}(t)$ and $\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix}$, where $\mathbf{w}_i \in \mathbb{R}^3$, $i = 1, 2, 3$. Following the same argument in part (b), we have

$$\begin{aligned}
\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix} \mathbf{x}(0) \\
&= \frac{(\mathbf{t}_1\mathbf{w}_1^T + \mathbf{t}_2\mathbf{w}_2^T)\mathbf{x}(0)}{s-1} + \frac{\mathbf{t}_1\mathbf{w}_1^T\mathbf{x}(0)}{(s-1)^2} + \frac{\mathbf{t}_3\mathbf{w}_3^T\mathbf{x}(0)}{s+1}
\end{aligned}$$

The first two terms must vanish for $\mathbf{x}(t)$ to converge to zero. In other words, $\mathbf{x}(0)$ must be orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_1$. Choose $\mathbf{x}(0) = \mathbf{w}_2 \times \mathbf{w}_1 = [0, 1, 1]^T$.

6. $\mathbf{A}$ has eigenvalue -1 with multiplicity 3. Since $\mathbf{A}$ is in the Jordan form and the maximum size of the Jordan block is 2, the minimal polynomial of $\mathbf{A}$ is $\varphi(s) = (s+1)^2$. This implies that $\varphi(\mathbf{A}) = (\mathbf{A} + \mathbf{I})^2 = \mathbf{0}$, or $\mathbf{A}^2 = -2\mathbf{A} - \mathbf{I}$.

(a) We show $\mathbf{A}^k = (-1)^{k-1}\big(k\mathbf{A} + (k-1)\mathbf{I}\big)$ by mathematical induction as follows.

i. For $k = 2$, we have just shown that the equality holds.

ii. Suppose that the equality holds for $k = n$, i.e. $\mathbf{A}^n = (-1)^{n-1}\big(n\mathbf{A} + (n-1)\mathbf{I}\big)$.

iii. Consider the case of $k = n + 1$.

$$\begin{aligned}
\mathbf{A}^{n+1} &= \mathbf{A}\mathbf{A}^n = (-1)^{n-1}\big(n\mathbf{A}^2 + (n-1)\mathbf{A}\big) \\
&= (-1)^{n-1}\big(n(-2\mathbf{A} - \mathbf{I}) + (n-1)\mathbf{A})\big) \\
&= (-1)^n\big((n+1)\mathbf{A} + n\mathbf{I}\big)
\end{aligned}$$

This implies that the equality holds for $k = n + 1$.

iv. By mathematical induction, the equality holds for all $k \geq 2$.

(b) Choose $p(s) = \alpha_0 + \alpha_1 s$. Then the interpolation conditions are

$$\begin{aligned}
p(-1) &= \alpha_0 - \alpha_1 = \sin(-1) = -\sin(1) \\
p'(-1) &= \alpha_1 = \cos(-1) = \cos(1)
\end{aligned}$$

Hence $\alpha_0 = -\sin(1) + \cos(1)$ and

$$\sin \mathbf{A} = p(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} = (\cos(1) - \sin(1))\mathbf{I} + \cos(1)\mathbf{A}$$