Linear System Theory
Solution to Final

1/4/2024

1. (a) Let $\mathbf{x} = [x_1, x_2]^T$. Then the state equation can be written as

$$\dot{x}_1 = \cos(t)x_1 + e^{\sin t}x_2 \tag{1}$$
$$\dot{x}_2 = -x_2 \tag{2}$$

(2) yields $x_2(t) = x_2(t_0)e^{-(t-t_0)}$. On the other hand, multiplying $e^{-\sin t}$ on both sides of (1) and rearranging the equation leads to

$$e^{-\sin t}\dot{x}_1 - \cos(t)e^{-\sin t}x_1(t) = \frac{d}{dt}\left(e^{-\sin t}x_1\right) = x_2(t_0)e^{-(t-t_0)} \tag{3}$$

Integrate both sides of (3) from t_0 to t and we have

$$e^{-\sin t}x_1(t) - e^{-\sin t_0}x_1(t_0) = x_2(t_0)\big(1 - e^{-(t-t_0)}\big)$$

Therefore

$$x_1(t) = e^{\sin t - \sin t_0}x_1(t_0) + x_2(t_0)e^{\sin t}\big(1 - e^{-(t-t_0)}\big)$$

Now, given the initial state $\mathbf{x}(t_0) = [1, 0]^T$, the solution of the state equation is

$$\mathbf{x}(t) = \begin{bmatrix} e^{\sin t - \sin t_0} \\ 0 \end{bmatrix}$$

On the other hand, given the initial state $\mathbf{x}(t_0) = [0, 1]^T$, the solution of the state equation is

$$\mathbf{x}(t) = \begin{bmatrix} e^{\sin t}(1 - e^{-(t-t_0)}) \\ e^{-(t-t_0)} \end{bmatrix}$$

Therefore, the state transition matrix is

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} e^{\sin t - \sin t_0} & e^{\sin t}(1 - e^{-(t-t_0)}) \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

(b) It is clear that $\mathbf{\Phi}(t, t_0)$ is a periodic matrix with period 2π, i.e. $\mathbf{\Phi}(t + 2\pi, t_0 + 2\pi) = \mathbf{\Phi}(t, t_0)$, and $\mathbf{\Phi}(2\pi, 0) = \begin{bmatrix} 1 & 1 - e^{-2\pi} \\ 0 & e^{-2\pi} \end{bmatrix}$. The characteristic polynomial of $\mathbf{\Phi}(2\pi, 0)$ is $p(s) = (s - 1)(s - e^{-2\pi})$ with eigenvalues 1 and $e^{-2\pi}$.
Therefore, we define $\log \mathbf{\Phi}(2\pi, 0) = \alpha_0\mathbf{I} + \alpha_1\mathbf{\Phi}(2\pi, 0)$, where α_0, α_1 satisfy

$$\alpha_0 + \alpha_1 = \log(1) = 0, \qquad \alpha_0 + \alpha_1 e^{-2\pi} = \log(e^{-2\pi}) = -2\pi$$

Thus $\alpha_0 = \frac{-2\pi}{1-e^{-2\pi}}$, and $\alpha_1 = \frac{2\pi}{1-e^{-2\pi}}$, and

$$\mathbf{B} = \frac{1}{2\pi}\log \mathbf{\Phi}(2\pi, 0) = \frac{1}{2\pi}\begin{bmatrix} \alpha_0+\alpha_1 & \alpha_1(1-e^{-2\pi}) \\ 0 & \alpha_0+\alpha_1 e^{-2\pi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Then

$$\left(s\mathbf{I}+\mathbf{B}\right)^{-1} = \begin{bmatrix} s & 1 \\ 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{-1}{s(s-1)} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s}-\frac{1}{s-1} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

We have $e^{-\mathbf{B}t} = \mathcal{L}^{-1}\left\{\left(s\mathbf{I}+\mathbf{B}\right)^{-1}\right\} = \begin{bmatrix} 1 & 1-e^t \\ 0 & e^t \end{bmatrix}$. Furthermore,

$$\mathbf{P}(t) = \mathbf{\Phi}(t,0)e^{-\mathbf{B}t} = \begin{bmatrix} e^{\sin t} & e^{\sin t}(1-e^{-t}) \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1-e^t \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{\sin t} & 0 \\ 0 & 1 \end{bmatrix}$$

(c) The eigenvalues of $\mathbf{B}$ are $0, -1$; hence the linear time-varying system is not stable.

2. Since only the velocity is included in the feedback control, define $v = x_2$ and treat it as the control input to the first state equation. Namely, the problem becomes

$$\begin{aligned} \min_v J \quad &= \int_0^\infty \left(x_1^2(t) + \rho v^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}_1 = v \end{aligned}$$

This is an LQR problem with $A = 0$, $B = 1$, $Q = 1$ and $R = \rho$. Hence the Riccati equation is

$$0 = Q + Ap + pA - \frac{1}{R}p^2B^2 = 1 - \frac{1}{\rho}p^2 \Rightarrow p = \sqrt{\rho}$$

The optimal control law for v is $v = -k_v x_1$, where $k_v = \frac{1}{\rho}p = \frac{1}{\sqrt{\rho}}$. Since $v = x_2 = -k_v x_1$, take time derivative on both sides and compare it with the second state equation. We have

$$\dot{x}_2 = -k_v\dot{x}_1 = -k_v x_2 = -2x_2 + u \Rightarrow u = -(k_v-2)x_2 = -\left(\frac{1}{\sqrt{\rho}} - 2\right)x_2$$

In other words, the velocity feedback gain is $k = \frac{1}{\sqrt{\rho}} - 2$.

3. (a) Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathcal{L}_2^n(\mathbb{R}_+)$. Then

$$\langle \mathcal{L}\mathbf{x}, \mathbf{y}\rangle = \int_0^\infty \mathbf{y}^T(t)e^{\mathbf{A}t}\mathbf{x}dt = \mathbf{x}^T\int_0^\infty e^{\mathbf{A}^Tt}\mathbf{y}(t)dt = \langle \mathbf{x}, \mathcal{L}^*\mathbf{y}\rangle$$

where $\mathcal{L}^* : \mathcal{L}_2^n(\mathbb{R}_+) \to \mathbb{R}^n$, $\mathcal{L}^*\mathbf{y} = \int_0^\infty e^{\mathbf{A}^Tt}\mathbf{y}(t)dt$ is the adjoint map of $\mathcal{L}$.

(b) $\mathcal{L}^*\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathcal{L}^*\mathcal{L}\mathbf{x} = \mathcal{L}^*e^{\mathbf{A}t}\mathbf{x} = \int_0^\infty e^{\mathbf{A}^Tt}e^{\mathbf{A}t}\mathbf{x}dt = \mathbf{M}\mathbf{x}$$

where $\mathbf{M} = \int_0^\infty e^{\mathbf{A}^Tt}e^{\mathbf{A}t}dt \in \mathbb{R}^{n\times n}$ is the representative matrix of $\mathcal{L}^*\mathcal{L}$.

(c) We first show that $\mathbf{M}$ is positive definite. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = \int_0^\infty \mathbf{x}^T e^{\mathbf{A}^T t} e^{\mathbf{A}t}\mathbf{x}dt = \int_0^\infty \|e^{\mathbf{A}t}\mathbf{x}\|^2 dt \geq 0$$

In other words, $\mathbf{M}$ is positive semidefinite. Furthermore, if $\mathbf{x}^T\mathbf{M}\mathbf{x} = 0$, then $e^{\mathbf{A}t}\mathbf{x} = \mathbf{0}$ for all $t \geq 0$. However, $e^{\mathbf{A}t}$ is nonsingular for all t. $e^{\mathbf{A}t}\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{M}$ is positive definite.

Then $\mathcal{N}(\mathcal{L}) = \mathcal{N}(\mathcal{L}^*\mathcal{L}) = \mathcal{N}(\mathbf{M}) = \{\mathbf{0}\}$. This is because $\mathbf{M}$ is nonsingular.

4. (a) First, we assume that $(\mathbf{A}, \mathbf{B}\mathbf{R}\mathbf{B}^T)$ is uncontrollable. By PBH test, there exists a left eigenvector of $\mathbf{A}$, denoted by $\mathbf{w}$, and its associated eigenvalue λ such that $\mathbf{w}^T\mathbf{A} = \lambda\mathbf{w}^T$ and $\mathbf{w}^T\mathbf{B}\mathbf{R}\mathbf{B}^T = \mathbf{0}$. Then $\mathbf{w}^T\mathbf{B}\mathbf{R}\mathbf{B}^T\mathbf{w} = 0$. Since $\mathbf{R}$ is positive definite, we must have $\mathbf{w}^T\mathbf{B} = \mathbf{0}$. This implies that $(\mathbf{A}, \mathbf{B})$ is uncontrollable.

   Conversely, assume that $(\mathbf{A}, \mathbf{B})$ is uncontrollable. Then $\exists\mathbf{w}$ and λ such that $\mathbf{w}^T\mathbf{A} = \lambda\mathbf{w}^T$ and $\mathbf{w}^T\mathbf{B} = \mathbf{0}$. Then $\mathbf{w}^T\mathbf{B}\mathbf{R}\mathbf{B}^T = \mathbf{0}$, implying that $(\mathbf{A}, \mathbf{B}\mathbf{R}\mathbf{B}^T)$ is uncontrollable.

   Combine both cases we conclude that $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $(\mathbf{A}, \mathbf{B}\mathbf{R}\mathbf{B}^T)$ is controllable.

   (b) Suppose that $(\mathbf{A}, \mathbf{B})$ is uncontrollable. By PBH test, there exists $\mathbf{w}_j$ and λ_j such that $\mathbf{w}_j^T\mathbf{A} = \lambda_j\mathbf{w}_j^T$ and $\mathbf{w}_j^T\mathbf{B} = \mathbf{0}$. Notice that $\mathbf{w}_j^T\mathbf{v}_i = 0$ for $i \neq j$ and $\mathbf{w}_j^T\mathbf{v}_j \neq 0$. Thus $\mathbf{w}_j^T(\mathbf{A} - \mathbf{X}) = \lambda_j\mathbf{w}_j^T - (\mathbf{w}_j^T\mathbf{v}_j)\mathbf{w}_j^T = (\lambda_j - (\mathbf{w}_j^T\mathbf{v}_j))\mathbf{w}_j^T$. In other words, $\mathbf{w}_j$ is a left eigenvector of $\mathbf{A} - \mathbf{X}$ associated with the eigenvalue $\lambda_j - (\mathbf{w}_j^T\mathbf{v}_j)$. In addition, $\mathbf{w}_j^T\mathbf{X}\mathbf{B} = (\mathbf{w}_j^T\mathbf{v}_j)\mathbf{w}_j^T\mathbf{B} = \mathbf{0}$. Therefore, $(\mathbf{A} - \mathbf{X}, \mathbf{X}\mathbf{B})$ is uncontrollable. This is absurd. As a result, $(\mathbf{A}, \mathbf{B})$ is controllable.

5. (a) The eigenvalues of $\mathbf{A}$ are $-2, 1, -1$. Use PBH rank test associated with the unstable eigenvalue $\lambda = 1$ to verify stabilizability w.r.t. $\mathbf{b}_1$ and $\mathbf{b}_2$.

$$\begin{aligned}
\text{rank}\begin{bmatrix} \mathbf{I} - \mathbf{A} & \mathbf{b}_1 \end{bmatrix} &= \begin{bmatrix} 2 & -2 & 0 & 1 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} = 3 \\
\text{rank}\begin{bmatrix} \mathbf{I} - \mathbf{A} & \mathbf{b}_2 \end{bmatrix} &= \begin{bmatrix} 2 & -2 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} = 2
\end{aligned}$$

   Therefore $(\mathbf{A}, \mathbf{b}_1)$ is stabilizable and $(\mathbf{A}, \mathbf{b}_2)$ is not stabilizable.

   (b)

$$\mathbf{A} - \mathbf{b}_1\mathbf{k}_1 = \begin{bmatrix} -1 - k_{11} & 2 - k_{12} & -k_{13} \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 - k_{11} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

   Hence the eigenvalues of $\mathbf{A} - \mathbf{b}_1\mathbf{k}_1$ are -1 and the solutions to

$$\lambda(\lambda + 1 + k_{11}) - 1 = \lambda^2 + (k_{11} + 1)\lambda - 1 = 0 \tag{4}$$

   Then $\lambda = \frac{1}{2}\big(-(k_{11} + 1) \pm \sqrt{(k_{11} + 1)^2 + 4}\big)$. Clearly, the eigenvalue $\lambda_+ = \frac{1}{2}\big(-(k_{11} + 1) + \sqrt{(k_{11} + 1)^2 + 4}\big)$ is positive, while the other eigenvalue is negative.

Hence we check stabilizability of λ_+ w.r.t. $\mathbf{b}_2$ by PBH rank test.

$$\begin{aligned}
\left[\begin{array}{cc} \lambda_+\mathbf{I}-\mathbf{A}+\mathbf{b}_1\mathbf{k}_1 & \mathbf{b}_2 \end{array}\right] &= \begin{bmatrix} \lambda_+ + k_{11}+1 & -1 & 0 & 0 \\ -1 & \lambda_+ & -1 & -1 \\ 0 & 0 & \lambda_+ + 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_+ + k_{11}+1 & -1 & 0 & 0 \\ 0 & \lambda_+ - \frac{1}{\lambda_+ + k_{11}+1} & -1 & -1 \\ 0 & 0 & \lambda_+ + 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_+ + k_{11}+1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & \lambda_+ + 1 & 2 \end{bmatrix} \qquad (5)
\end{aligned}$$

where the second equality comes from multiplying the first row by $\frac{1}{\lambda_+ + k_{11}+1}$ and adding the result to the second row. The third equality comes from (4). Notice that the first two columns of (5) are linearly dependent. Therefore, for $(\mathbf{A}-\mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ to be stabilizable, (5) must be full rank, which requires that the last two columns of (5) are linearly independent. Equivalently, $(\mathbf{A}-\mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is **NOT** stabilizable if

$$\begin{aligned}
0 &= \det\begin{bmatrix} -1 & -1 \\ \lambda_+ + 1 & 2 \end{bmatrix} = -2 + \lambda_+ + 1 \\
&\Rightarrow \lambda_+ = \frac{1}{2}\Big(-(k_{11}+1) + \sqrt{(k_{11}+1)^2+4}\Big) = 1 \\
&\Rightarrow \sqrt{(k_{11}+1)^2+4} = (k_{11}+1)+2 \\
&\Rightarrow (k_{11}+1)^2+4 = (k_{11}+1)^2 + 4(k_{11}+1)+4 \\
&\Rightarrow k_{11} = -1
\end{aligned}$$

Therefore, if $k_{11} \neq -1$, then $(\mathbf{A}-\mathbf{b}_1\mathbf{k}_1, \mathbf{b}_2)$ is stabilizable.

(c) Let $\mathbf{k}_1 = [k_{11}, k_{12}, k_{13}]$. Then

$$\begin{aligned}
\det\big(\lambda\mathbf{I}-\mathbf{A}+\mathbf{b}_1\mathbf{k}_1\big) &= \det\begin{bmatrix} \lambda + k_{11}+1 & k_{12}-2 & k_{13} \\ -1 & \lambda & -1 \\ 0 & 0 & \lambda+1 \end{bmatrix} \\
&= \big(\lambda^2 + (k_{11}+1)\lambda + k_{12} - 2\big)(\lambda+1) \qquad (6)
\end{aligned}$$

For $\mathbf{A}_1 = \mathbf{A}-\mathbf{b}_1\mathbf{k}_1$ to have two eigenvalues at -2, we should have $k_{11}=3$ and $k_{12}=6$ such that (6) becomes $(\lambda+2)^2(\lambda+1)$. k_{13} can be chosen arbitrarily and we choose $k_{13}=0$. Hence $\mathbf{k}_1 = [3, 6, 0]$.

Let $\mathbf{k}_2 = [k_{21}, k_{22}, k_{23}]$. Then

$$\begin{aligned}
\det\big(\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}\mathbf{K}\big) &= \det\big(\lambda\mathbf{I}-\mathbf{A}+\mathbf{b}_1\mathbf{k}_1+\mathbf{b}_2\mathbf{k}_2\big) = \det\big(\lambda\mathbf{I}-\mathbf{A}_1+\mathbf{b}_2\mathbf{k}_2\big) \\
&= \det\begin{bmatrix} \lambda+4 & 4 & 0 \\ -1-k_{21} & \lambda - k_{22} & -1-k_{23} \\ 2k_{21} & 2k_{22} & \lambda+1+2k_{23} \end{bmatrix}
\end{aligned}$$

We observe that if $k_{21}=k_{22}=0$, then

$$\det\big(\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}\mathbf{K}\big) = \det\begin{bmatrix} \lambda+4 & 4 & 0 \\ -1 & \lambda & -1-k_{23} \\ 0 & 0 & \lambda+1+2k_{23} \end{bmatrix} = (\lambda+2)^2(\lambda+1+2k_{23})$$

Choose $k_{23}=\frac{1}{2}$, then $\det\big(\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}\mathbf{K}\big) = (\lambda+2)^3$. Hence $\mathbf{k}_2 = [0, 0, \frac{1}{2}]$.

6. (a) **True**.

Suppose that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Then $\mathbf{C}\mathbf{A}\mathbf{v} = \lambda\mathbf{C}\mathbf{v} = \mathbf{0}$, $\mathbf{C}\mathbf{A}^2\mathbf{v} = \lambda^2\mathbf{C}\mathbf{v} = \mathbf{0}, \cdots, \mathbf{C}\mathbf{A}^k\mathbf{v} = \lambda^k\mathbf{C}\mathbf{v} = \mathbf{0}$. for $k = 1, 2, \cdots$. Therefore, $\mathcal{O}\mathbf{v} = \mathbf{0}$, i.e. $\mathbf{v} \in \mathcal{N}(\mathcal{O})$.

(b) **False**.

Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. Then $\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Note that $\mathbf{v} = [1, 0, 1]^T \in \mathcal{N}(\mathcal{O})$, but $\mathbf{v}$ is not an eigenvector of $\mathbf{A}$.

(c) **False**.

Let $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{C} = \mathbf{I}$. Then $\mathbf{H}(s)\frac{1}{s+1}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, i.e. -1 is a simple pole of every element of $\mathbf{H}(s)$, but -1 is an eigenvalue of $\mathbf{A}$ with multiplicity 2.