Linear System Theory
Homework 6

Due date: 12/7/2023

1. Consider the discrete-time system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$, where

$$\mathbf{A} = \frac{1}{16}\begin{bmatrix} 1 & 7 & 14 \\ -1 & 9 & 2 \\ -1 & 1 & 10 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   (a) Check whether the system is controllable or not.

   (b) Find $u(0)$ and $u(1)$ such that $\mathbf{x}(2) = [3,\ 5,\ 5]^T$.

   (c) Find $\mathbf{u} = [u(2),\ u(1),\ u(0]^T$ with minimum norm (i.e. the smallest $\|\mathbf{u}\|_2$) such that $\mathbf{x}(3) = [45,\ 11,\ 11]^T$.

2. Consider the system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^n$. Suppose that $(\mathbf{A}, \mathbf{B})$ is controllable. Find an input sequence $u(0), u(1), \cdots, u(K-1)$, $K > n$, that yields $\mathbf{x}(K) = \mathbf{0}$ and minimizes the following cost function:

$$J = \frac{1}{K}\left(u(0)^2 + \sum_{k=1}^{K-1}\big(u(k) - u(k-1)\big)^2\right)$$

3. Consider the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$. Suppose that $\mathbf{A}$ is *Hurwitz*, i.e. all eigenvalues of $\mathbf{A}$ have negative real parts, and $(\mathbf{A}, \mathbf{B})$ is controllable. Show that the minimum energy of the control input that is required to transfer the state from $\mathbf{x}(0) = \mathbf{0}$ to $\mathbf{x}(t_f) = \mathbf{x}_f$ is $\mathbf{x}_f^T \mathbf{W}_r^{-1}(t_f)\mathbf{x}_f$, where

$$\mathbf{W}_r(t_f) = \int_0^{t_f} e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt$$

and show that $\mathbf{W}_r(t_f)$ is the solution to the Lyapunov equation:

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T - e^{\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t_f} = \mathbf{0}$$

4. Consider the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$, where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   (a) Find an impulsive input that steers the state from $\mathbf{x}(0) = \mathbf{0}$ to $\mathbf{x}(0_+) = [1, -1]^T$.

   (b) Find a control input of the form $u(t) = a_0 + a_1 t$ which steers the state from $\mathbf{x}(0) = \mathbf{0}$ to $\mathbf{x}(1) = [1, -1]^T$. Compute the control energy $\int_0^1 \|u(t)\|^2 dt$.

5. Consider the discrete-time system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, m < n$, and $\mathbf{B}$ is full rank. The goal is to choose an input $\mathbf{u}(k)$ that causes $\mathbf{x}(k)$ to converge to zero as $k \to \infty$. A student proposes the following simple method: at time k, choose $\mathbf{u}(k)$ that minimizes $\|\mathbf{x}(k+1)\|$. He/She argues that this scheme will work well, since the norm of the state is made as small as possible at every step. In this problem you will analyze this scheme.

(a) Find an explicit expression for the proposed input $\mathbf{u}(k)$ in terms of $\mathbf{x}(k)$, $\mathbf{A}$, and $\mathbf{B}$.

(b) Now consider the linear dynamical system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)$ with $\mathbf{u}$ given by the proposed scheme. Show that $\mathbf{x}$ satisfies an autonomous linear dynamical system equation $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k)$. Express the matrix $\mathbf{F}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.

(c) Now for a specific case:

$$\mathbf{A} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Determine the system in part (c) is controllable or not. When applying the control scheme described in part (a) to the system in part (c), determine the closed-loop system is stable or not.