Linear System Theory
Homework 1

Due date: 9/28/2023

1. Let u and y be the input and output to a single-input single-output (SISO) discrete-time system which satisfies

$$y(k) = -a_1 y(k-1) - \cdots - a_n y(k-n) + b_0 u(k-d) + \cdots b_m u(k-n)$$

where $d = n - m > 0$, $a_1, \cdots, a_n$ and $b_0, \cdots, b_m$ are given real coefficients. Find an n-th order state space representation of this system.

2. Let P_n be the vector space of all real-coefficient polynomials with degree less than n. Consider the differentiation operation on P_n, i.e. $Dp(x) = \frac{dp(x)}{dx} \in P_n$ for $p(x) \in P_n$. It is well-known that differentiation is a linear operation. Find the representative matrix of the differentiation operation on P_n.

Hint: Firstly, find a one-to-one correspondence between P_n and $\mathbb{R}^n$. Let $p'(x) = Dp(x)$, and $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$ be the corresponding vectors of $p(x)$ and $p'(x)$, respectively. Then find the matrix that maps $\mathbf{p}$ to $\mathbf{p}'$.

3. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

   (a) Show that if f is affine, then

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i f(x_i)$$

   for all $x_1, \cdots, x_k \in \mathbb{R}^n$ and all $\alpha_1, \cdots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i = 1$.

   (b) Suppose $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that the function f defined by

$$f(x) = Ax + b \tag{1}$$

   is affine.

   (c) Conversely, every affine function must have the form (1). Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine, then f can be expressed as (1) for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Explicitly determine A and b.

4. Let V be a vector space. Let $+$ and $\cdot$ denote the vector addition and scalar multiplication defined on V. $0 \in V$ is the zero element of V, i.e. $x + 0 = x$ for all $x \in V$.

   (a) Show that the zero element of V is unique.

(b) For any $x \in V$, let $(-x) \in V$ denote the additive inverse of x, i.e. $x + (-x) = 0$. Show that $(-x)$ is unique.

(c) For any $x \in V$, show that $0 \cdot x = 0$, where the 0 on the left-hand side is a scalar, and the 0 on the right-hand side is the zero element of V.

5. Let S_{sn} be the set of all $n \times n$ skew symmetric matrices, i.e.

$$S_{sn} = \left\{ A \in \mathbb{R}^{n \times n} | \, A^T = -A \right\}$$

Show that S_{sn} is indeed a subspace of $\mathbb{R}^{n \times n}$ with ordinary matrix addition and scalar multiplication. What is the dimension of S_{sn}? Find a basis for S_{s3}.