Linear System Theory
Solution to Homework 6

1. (a) The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \frac{1}{16}\begin{bmatrix} -16 & 20 & 17 \\ 16 & 12 & 7 \\ 16 & 12 & 7 \end{bmatrix}$$

Since $\mathscr{C}$ has two identical rows, we have $\operatorname{rank}(\mathscr{C}) = 2$, and therefore $(\mathbf{A}, \mathbf{B})$ is uncontrollable.

(b)

$$\mathbf{x}(2) = \mathbf{AB}u(0) + \mathbf{B}u(1) = \frac{1}{4}\begin{bmatrix} -4 & 5 \\ 4 & 3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$$

Then $u(0) = 4$,and $u(1) = 2$.

(c)

$$\mathbf{x}(3) = \mathbf{A}^2\mathbf{B}u(0) + \mathbf{AB}u(1) + \mathbf{B}u(2) = \mathscr{C}\mathbf{u} \tag{1}$$

For $\mathbf{u}$ to be the least-norm solution, we should have $\mathbf{u} \in \mathcal{R}(\mathscr{C}^T)$. Notice that

$$\mathcal{R}(\mathscr{C}^T) = \operatorname{span}\left\{ \begin{bmatrix} -16 \\ 20 \\ 17 \end{bmatrix}, \begin{bmatrix} 16 \\ 12 \\ 7 \end{bmatrix} \right\}$$

Let

$$\mathbf{u} = \alpha_1 \begin{bmatrix} -16 \\ 20 \\ 17 \end{bmatrix} + \alpha_2 \begin{bmatrix} 16 \\ 12 \\ 7 \end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}$$

Then (1) becomes

$$\mathbf{x}(3) = \mathscr{C}\begin{bmatrix} -16 & 16 \\ 20 & 12 \\ 17 & 7 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 945 & 103 \\ 103 & 449 \\ 103 & 449 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0.7376 \\ 0.2228 \end{bmatrix}$$

Therefore, the least-norm contorl is

$$\mathbf{u} = \begin{bmatrix} -16 & 16 \\ 20 & 12 \\ 17 & 7 \end{bmatrix}\begin{bmatrix} 0.7376 \\ 0.2228 \end{bmatrix} = \begin{bmatrix} -8.2376 \\ 17.4257 \\ 14.099 \end{bmatrix}$$

2. We need to solve the underdetermined problem as follows:

$$
\begin{aligned}
\mathbf{x}(K) - \mathbf{A}^K\mathbf{x}(0) &= \mathbf{A}^{K-1}\mathbf{B}u(0) + \mathbf{A}^{K-2}\mathbf{B}u(1) + \cdots + \mathbf{B}u(K-1) \\
&= \begin{bmatrix} \mathbf{A}^{K-1}\mathbf{B} & \mathbf{A}^{K-1}\mathbf{B} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(K-1) \end{bmatrix} \qquad (2)
\end{aligned}
$$

The cost J can be expressed as the quadratic form of $\mathbf{u} = [u(0), \cdots, u(K-1)]^T$ as follows:

$$
\begin{aligned}
J &= \frac{1}{K}\left(u(0)^2 + \sum_{k=1}^{K-1}\big(u(k) - u(k-1)\big)^2\right) \\
&= \frac{1}{K}\left(\sum_{k=0}^{K-1} u(k)^2 - 2\sum_{k=1}^{K-1} u(k)u(k-1) + \sum_{k=0}^{K-2} u(k)^2\right) \\
&= \mathbf{u}^T\mathbf{Q}\mathbf{u}
\end{aligned}
$$

where $\mathbf{Q} = \frac{1}{K}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \in \mathbb{R}^{K\times K}$. Let $\mathbf{Q} = \mathbf{Q}^{\frac{T}{2}}\mathbf{Q}^{\frac{1}{2}}$ and $\mathbf{v} = \mathbf{Q}^{\frac{1}{2}}\mathbf{u}$. Then $J = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$ and (2) becomes

$$
-\mathbf{A}^K\mathbf{x}(0) = \mathbf{C}\mathbf{u} = \mathbf{C}\mathbf{Q}^{-\frac{1}{2}}\mathbf{v} \qquad (3)
$$

where $\mathbf{C} = \begin{bmatrix} \mathbf{A}^{K-1}\mathbf{B} & \mathbf{A}^{K-2}\mathbf{B} & \cdots & \mathbf{B} \end{bmatrix}$. Hence the solution of (2) that minimizes J is the least-norm solution of (3). The least-norm solution is

$$
\mathbf{v}^* = -\mathbf{Q}^{-\frac{T}{2}}\mathbf{C}^T\big(\mathbf{C}\mathbf{Q}^{-\frac{1}{2}}\mathbf{Q}^{-\frac{T}{2}}\mathbf{C}^T\big)^{-1}\mathbf{A}^K\mathbf{x}(0)
$$

and

$$
\mathbf{u}^* = \mathbf{Q}^{-\frac{1}{2}}\mathbf{v}^* = -\mathbf{Q}^{-1}\mathbf{C}^T\big(\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T\big)^{-1}\mathbf{A}^K\mathbf{x}(0)
$$

3. We have derived the minimum energy control for continuous-time time-varying systems. For time-invariant systems, the state transition matrix is $e^{\mathbf{A}t}$ and the initial time is fixed at $t_0 = 0$; hence the dependency of the reachability grammian on t_0 is removed. Then the minimum energy is $\mathbf{x}_f^T\mathbf{W}_r^{-1}(t_f)\mathbf{x}_f$, where

$$
\mathbf{W}_r(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-\tau)}d\tau = \int_0^t e^{\mathbf{A}s}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T s}ds \quad (\text{ Let } s = t-\tau)
$$

Consider the Lyapunov equation

$$
\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T - e^{\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t_f} = \mathbf{0} \qquad (4)
$$

The solution to (4) is

$$\begin{aligned}
\mathbf{X} &= \int_0^\infty e^{\mathbf{A}t}\left(\mathbf{B}\mathbf{B}^T - e^{\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t_f}\right)e^{\mathbf{A}^T t}dt \\
&= \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt - \int_0^\infty e^{\mathbf{A}(t+t_f)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T (t+t_f)}dt \\
&= \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt - \int_{t_f}^\infty e^{\mathbf{A}s}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T s}ds, \quad (s = t + t_f) \\
&= \int_0^{t_f} e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt \\
&= \mathbf{W}_r(t_f)
\end{aligned}$$

4. (a) Based on the derivation in class, we have

$$\mathbf{x}(0_+) = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} \end{bmatrix}\mathbf{f}$$

Then $\mathbf{f} = [2, 1]^T$, and $u(t) = 2\delta(t) + \delta'(t)$.

(b) Let $\mathbf{X}(s) = \mathcal{L}\{\mathbf{x}(t)\}$ and $U(s) = \mathcal{L}\{u(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2}$. Starting from $\mathbf{x}(0)$, the state trajectory is

$$\begin{aligned}
\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 \\ -1 \end{bmatrix}U(S) \\
&= \begin{bmatrix} \frac{a_0 - a_1}{s} + \frac{a_1}{s^2} + \frac{a_1 - a_0}{s+1} \\ \left(-\frac{a_0}{2} + \frac{3a_1}{4}\right)\frac{1}{s} - \frac{a_1}{2}\frac{1}{s^2} + \frac{a_0 - a_1}{s+1} + \left(-\frac{a_0}{2} + \frac{a_1}{4}\right)\frac{1}{s+2} \end{bmatrix} \\
\Rightarrow \mathbf{x}(t) &= \begin{bmatrix} \left(1 - e^{-t}\right)a_0 + \left(t + e^{-t} - 1\right)a_1 \\ \left(e^{-t} - \frac{1}{2}e^{-2t} - \frac{1}{2}\right)a_0 + \left(-\frac{t}{2} - e^{-t} + \frac{1}{4}e^{-2t} + \frac{3}{4}\right)a_1 \end{bmatrix}
\end{aligned}$$

Since $\mathbf{x}(1) = [1, -1]^T$, we have

$$\begin{bmatrix} 0.6321 & 0.3679 \\ -0.1998 & -0.084 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 13.9332 \\ -21.2229 \end{bmatrix}$$

Therefore, $u(t) = 13.9332 - 21.2229t$. The control energy is

$$\int_0^1 |u(t)|^2 dt = \int_0^1 \left(194.1348 - 591.4076t + 450.4126t^2\right)dt = 48.5685$$

5. (a) To minimize $\|\mathbf{x}(k+1)\|$, we compute the least squares solution of $\mathbf{B}\mathbf{u}(k) - \mathbf{A}\mathbf{x}(k)$ to get

$$\mathbf{u}(k) = -(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{A}\mathbf{x}(k)$$

(b) With the state feedback defined in part (a), the closed-loop system becomes

$$\mathbf{x}(k+1) = \left(\mathbf{A} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{A}\right)\mathbf{x}(k)$$

Hence $\mathbf{F} = \mathbf{A} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{A}$.

(c) The controllability matrix is $\mathcal{C} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$ and $\text{rank}(\mathcal{C}) = 2$. Hence the system is controllable. When applying the control scheme in part (a) to it, the $\mathbf{F}$ matrix is $\mathbf{F} = \begin{bmatrix} 0 & 1.5 \\ 0 & -1.5 \end{bmatrix}$. Since the eigenvalues are 0 and -1.5, the discrete-time closed-loop system is unstable.