Linear System Theory
Solution to Homework 5

1. Let $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$. The state equation can be expressed as

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) + \sin t x_2(t)\\ \dot{x}_2(t) &= -x_2(t)\end{aligned}$$

Thus $x_2(t) = x_{02}e^{-t}$ and suppose that $x_1(t) = c_1e^{-2t} + c_2e^{-t}\sin t + c_3e^{-t}\cos t$, where c_1, c_2, c_3 are coefficients to be determined. Then

$$\begin{aligned}\dot{x}_1(t) &= -2c_1e^{-2t} - c_2e^{-t}\sin t + c_2e^{-t}\cos t - c_3e^{-t}\cos t - c_3e^{-t}\sin t\\ &= -2\big(c_1e^{-2t} + c_2e^{-t}\sin t + c_3e^{-t}\cos t\big) + (c_2 - c_3)e^{-t}\sin t + (c_2 + c_3)e^{-t}\cos t\\ &= -2x_1(t) + x_{02}e^{-t}\sin t\end{aligned}$$

Therefore $c_2 + c_3 = 0$, and $c_2 - c_3 = x_{02} \Rightarrow c_2 = -c_3 = \frac{x_{02}}{2}$. In addition, $x_1(0) = x_{01} = c_1 + c_3 \Rightarrow c_1 = x_{01} + \frac{x_{02}}{2}$.

(a) Let $\mathbf{X}_0 = \mathbf{I}$,. Then the fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} e^{-2t} & \frac{1}{2}\big(e^{-2t} + e^{-t}\sin t - e^{-t}\cos t\big) \\ 0 & e^{-t}\end{bmatrix}$$

and the state transition matrix is $\mathbf{\Phi}(t,0) = \mathbf{X}(t)\mathbf{X}_0^{-1} = \mathbf{X}(t)$.

(b) As $t \to \infty$, $\mathbf{\Phi}(t,0) \to \mathbf{0}$. Hence the system is stable.

(c) Note that $\mathbf{A}(t)$ is periodic with period 2π, and $\mathbf{\Phi}(2\pi,0) = \begin{bmatrix} e^{-4\pi} & \frac{1}{2}(e^{-4\pi} - e^{-2\pi}) \\ 0 & e^{-2\pi}\end{bmatrix}$.
The minimal polynomial of $\mathbf{\Phi}(2\pi,0)$ is $\psi(s) = (s - e^{-4\pi})(s - e^{-2\pi})$. Thus $\mathbf{B} = \frac{1}{2\pi}\log\mathbf{\Phi}(2\pi,0) = \frac{1}{2\pi}\big(\alpha_0\mathbf{I} + \alpha_1\mathbf{\Phi}(2\pi,0)\big)$, and

$$\begin{aligned}\alpha_0 + \alpha_1e^{-4\pi} &= \log(e^{-4\pi}) = -4\pi\\ \alpha_0 + \alpha_1e^{-2\pi} &= \log(e^{-2\pi}) = -2\pi\end{aligned}$$

Then $\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} = \frac{2\pi}{e^{-2\pi} - e^{-4\pi}}\begin{bmatrix} -2e^{-2\pi} + e^{-4\pi} \\ 1\end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} -2 & -\frac{1}{2} \\ 0 & -1\end{bmatrix}$. The eigenvalues of $\mathbf{B}$ are -2 and -1.
Alternatively, we can find the eigenvalues of $\mathbf{B}$ from $\mathbf{\Phi}(2\pi,0)$. Since the eigenvalues of $\mathbf{\Phi}(2\pi,0)$ are $e^{-4\pi}$ and $e^{-2\pi}$, the eigenvalues of $\mathbf{B}$ are $\frac{1}{2\pi}\log e^{-4\pi} = -2$ and $\frac{1}{2\pi}\log e^{-2\pi} = -1$.

2. The state equation is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -g(x_1(t))x_2(t) - x_1(t)\end{aligned}$$

(a) The equilibrium point satisfies $x_2 = 0$ and $-g(x_1)x_2 - x_1 = -x_1 = 0$. Hence $(0,0)$ is the only equilibrium point of this system.

(b) Linearize the system around $(0,0)$ and we have

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The two eigenvalues of the linearized system are

$$\lambda_{1,2} = -\frac{g(0)}{2} \pm \sqrt{\frac{g^2(0)}{4} - 1}$$

Notice that if $\frac{g^2(0)}{4} - 1 \geq 0$, then $\lambda_{1,2} < 0$ because $\sqrt{\frac{g^2(0)}{4} - 1} < \frac{g(0)}{2}$. On the other hand, if $\frac{g^2(0)}{4} - 1 < 0$, then $\mathrm{Re}(\lambda_{1,2}) = -\frac{g(0)}{2} < 0$. Since all eigenvalues of the linearized system have negative real parts, $(0,0)$ is a stable equilibrium point of the original nonlinear system.

3. (a) Let $\mathbf{X}(t)$ be any fundamental matrix. By definition, $\mathbf{\Phi}(t,s) = \mathbf{X}(t)\mathbf{X}^{-1}(s)$. Then

$$\begin{aligned}\frac{\partial \mathbf{\Phi}(t,s)}{\partial s} &= \mathbf{X}(t)\frac{d}{ds}\left(\mathbf{X}^{-1}(s)\right) = -\mathbf{X}(t)\mathbf{X}^{-1}(s)\frac{d}{ds}\left(\mathbf{X}(s)\right)\mathbf{X}^{-1}(s) \\ &= -\mathbf{\Phi}(t,s)\mathbf{A}(s)\mathbf{X}(s)\mathbf{X}^{-1}(s) = -\mathbf{\Phi}(t,s)\mathbf{A}(s)\end{aligned}$$

Remark 1 *Since* $\mathbf{X}(s)\mathbf{X}(s)^{-1} = \mathbf{I}$, *we have*

$$\frac{d}{ds}\left(\mathbf{X}(s)\mathbf{X}^{-1}(s)\right) = \frac{d}{ds}\left(\mathbf{X}(s)\right)\mathbf{X}^{-1}(s) + \mathbf{X}(s)\frac{d}{ds}\left(\mathbf{X}^{-1}(s)\right) = \mathbf{0}$$

Therefore,

$$\frac{d}{ds}\left(\mathbf{X}^{-1}(s)\right) = -\mathbf{X}^{-1}(s)\frac{d}{ds}\left(\mathbf{X}(s)\right)\mathbf{X}^{-1}(s)$$

(b) First, $\mathbf{z}(t_0) = \mathbf{\Phi}^T(t_0,t_0)\mathbf{z}_0 = \mathbf{z}_0$. In addition,

$$\dot{\mathbf{z}}(t) = \frac{\partial \mathbf{\Phi}^T(t_0,t)}{\partial t}\mathbf{z}_0 = -\mathbf{A}^T(t)\mathbf{\Phi}^T(t_0,t)\mathbf{z}_0 = -\mathbf{A}^T(t)\mathbf{z}(t)$$

Hence $\mathbf{z}(t) = \mathbf{\Phi}^T(t_0,t)\mathbf{z}_0$ is the solution to $\dot{\mathbf{z}}(t) = -\mathbf{A}^T(t)\mathbf{z}(t)$, $\mathbf{z}(t_0) = \mathbf{z}_0$.

4. (a) Define $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$. Then

$$\dot{V} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} = \mathbf{x}^T(\mathbf{A}_{\sigma(t)}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{\sigma(t)})\mathbf{x} < -\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

Since $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \leq \lambda_{max}(\mathbf{P})\|\mathbf{x}\|^2$ and $\dot{V}(\mathbf{x}) < -\mathbf{x}^T\mathbf{Q}\mathbf{x} \leq -\lambda_{min}(\mathbf{Q})\|\mathbf{x}\|^2$, we have

$$\frac{\dot{V}}{V} < \frac{-\mathbf{x}^T\mathbf{Q}\mathbf{x}}{\mathbf{x}^T\mathbf{P}\mathbf{x}} \leq \frac{-\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} = -2\beta$$

where $\beta = \frac{1}{2}\frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})} > 0$. Integrate both sides over $[0,t]$, and we have

$$\log V(\mathbf{x}(t)) - \log V(\mathbf{x}(0)) \leq -2\beta t \Rightarrow V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))e^{-2\beta t}, \quad t \geq 0$$

This implies

$$\lambda_{min}(\mathbf{P})\|\mathbf{x}(t)\|^2 \leq \lambda_{max}(\mathbf{P})\|\mathbf{x}(0)\|^2 e^{-2\beta t}, \quad t \geq 0$$

Hence

$$\|\mathbf{x}(t)\| \leq \sqrt{\frac{\lambda_{max}(\mathbf{P})}{\lambda_{min}(\mathbf{P})}}\|\mathbf{x}(0)\|e^{-\beta t} \to 0, \text{ as } t \to \infty.$$

Hence the system is stable.

(b) Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are Hurwitz, i.e. all eigenvalues have negative real parts, $\mathbf{P}_1$ and $\mathbf{P}$ in steps i and ii are symmetric positive definite. Note that $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ implies $e^{\mathbf{A}_1}e^{\mathbf{A}_2} = e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_2}e^{\mathbf{A}_1}$. In addition,

$$\begin{aligned}
\mathbf{P} &= \int_0^\infty e^{\mathbf{A}_2^T t}\mathbf{P}_1 e^{\mathbf{A}_2 t}dt = \int_0^\infty e^{\mathbf{A}_2^T t}\left(\int_0^\infty e^{\mathbf{A}_1^T \tau}e^{\mathbf{A}_1 \tau}d\tau\right)e^{\mathbf{A}_2 t}dt \\
&= \int_0^\infty e^{\mathbf{A}_1^T \tau}\left(\int_0^\infty e^{\mathbf{A}_2^T t}e^{\mathbf{A}_2 t}dt\right)e^{\mathbf{A}_1 \tau}d\tau
\end{aligned}$$

Clearly $\mathbf{Q}_1 = \int_0^\infty e^{\mathbf{A}_2^T t}e^{\mathbf{A}_2 t}dt$ is positive definite. This implies that $\mathbf{P}$ is the solution to

$$\mathbf{A}_1^T\mathbf{P} + \mathbf{P}\mathbf{A}_1 = -\mathbf{Q}_1$$

From step ii, $\mathbf{P}$ is also the solution to $\mathbf{A}_2^T\mathbf{P}+\mathbf{P}\mathbf{A}_2 = -\mathbf{P}_1$. Given α defined in step iii, we have $\mathbf{Q} = \alpha\mathbf{I} < \mathbf{P}_1$ and $\mathbf{Q} = \alpha\mathbf{I} < \mathbf{Q}_1$. Therefore

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i < -\mathbf{Q}, \quad i = 1, 2$$