Linear System Theory
Solution to Homework 1

1. Define
$$z(k) = -a_1 z(k-1) - \cdots - a_n z(k-n) + u(k)$$
We claim that $y(k) = b_0 z(k-d) + b_1 z(k-d-1) + \cdots + b_m z(k-n)$. Indeed,
$$\begin{aligned}
y(k) &= b_0 z(k-d) + \cdots + b_m z(k-n) \\
&= b_0\big(-a_1 z(k-d-1) - \cdots - a_n z(k-d-n) + u(k-d)\big) \\
&+ b_1\big(-a_1 z(k-d-2) - \cdots - a_n z(k-d-n-1) + u(k-d-1)\big) \\
&\vdots \\
&+ b_m\big(-a_1 z(k-n-1) - \cdots - a_n z(k-2n) + u(k-n)\big) \\
&= -a_1\big(b_0 z(k-d-1) + b_1 z(k-d-2) + \cdots b_m z(k-n-1)\big) \\
&\vdots \\
&- a_n\big(b_0 z(k-d-n) + b_1(k-d-n-1) + \cdots + b_m z(k-2n)\big) \\
&+ b_0 u(k-d) + b_1 u(k-d-1) + \cdots b_m u(k-n) \\
&= -a_1 y(k-1) - \cdots - a_n y(k-n) + b_0 u(k-d) + \cdots b_m u(k-n)
\end{aligned}$$
Therefore, we choose $x(k) = [z(k-1),\, z(k-2),\, \cdots, z(k-n)]^T$ as the state, and the state space representation is
$$\begin{aligned}
x(k+1) &= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0 & \cdots & 0 & b_0 & \cdots & b_m \end{bmatrix} x(k)
\end{aligned}$$

2. For each $p(x) \in P_n$, $p(x) = a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, we can associate $\mathbf{p} = [a_1, \cdots, a_n]^T \in \mathbb{R}^n$ with $p(x)$. Since $\mathbf{p}$ is unique for each $p(x)$, the correspondence between $\mathbf{p}$ and $p(x)$ is one-to-one. Let $p'(x) = Dp(x) = (n-1)a_1 x^{n-2} + (n-2)a_2 x^{n-3} + \cdots + a_{n-1}$, and $\mathbf{p}, \mathbf{p}'$ be the corresponding vectors of $p(x)$ and $p'(x)$, respectively, i.e.
$$\begin{aligned}
\mathbf{p} &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T \\
\mathbf{p}' &= \begin{bmatrix} 0 & (n-1)a_1 & (n-2)a_2 & \cdots & a_{n-1} \end{bmatrix}^T
\end{aligned}$$

Hence the representative matrix that maps $\mathbf{p}$ to $\mathbf{p}'$ is

$$\mathbf{p}' = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{p}$$

3. (a) We prove this statement by induction. By the definition of affine functions, the statement is true for $k = 2$. Suppose that the statement is true for $k = l - 1$, i.e.

$$f\left(\sum_{i=1}^{l-1} \alpha_i x_i\right) = \sum_{i=1}^{l-1} \alpha_i f(x_i)$$

for $x_1, \cdots, x_{l-1} \in \mathbb{R}^n$ and all $\alpha_1, \cdots, \alpha_{l-1}$ such that $\sum_{i=1}^{l-1} \alpha_i = 1$. Now, let $k = l$ and consider all $x_1, \cdots, x_l \in \mathbb{R}^n$ and all $\alpha_1, \cdots, \alpha_l \in \mathbb{R}$ such that $\sum_{i=1}^{l} \alpha_i = 1$. Since the sum of all α_i, $i = 1, \cdots l$, is 1, there exists at lease one α_i such that $\alpha_i \neq 1$. Without loss of generality, we can assume that $\alpha_l \neq 1$. Then

$$\begin{aligned} f\left(\sum_{i=1}^{l} \alpha_i x_i\right) &= f\left(\sum_{i=1}^{l-1} \alpha_i x_i + \alpha_l x_l\right) \\ &= f\left((1-\alpha_l)\sum_{i=1}^{l-1} \frac{\alpha_i}{1-\alpha_l} x_i + \alpha_l x_l\right) \\ &= (1-\alpha_l) f\left(\sum_{i=1}^{l-1} \frac{\alpha_i}{1-\alpha_l} x_i\right) + \alpha_l f(x_l) \\ &= (1-\alpha_l)\sum_{i=1}^{l-1} \frac{\alpha_i}{1-\alpha_l} f(x_i) + \alpha_l f(x_l) \\ &= \sum_{i=1}^{l} \alpha_i f(x_i) \end{aligned}$$

Notice that the fourth equality holds because of the assumption of the induction and $\sum_{i=1}^{l-1} \frac{\alpha_i}{1-\alpha_l} = 1$. Therefore, the statement is true for all $k \in \mathbb{N}$.

(b) Let $x, y \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, and $\alpha + \beta = 1$. If $f(x) = Ax + b$, then

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b = \alpha(Ax + b) + \beta(Ay + b) = \alpha f(x) + \beta f(y)$$

Therefore $f(x)$ is affine.

(c) Let $e_i \in \mathbb{R}^n$ be the n-th column of the $n \times n$ identity matrix. For any $x \in \mathbb{R}^n$ and $x = [x_1, \cdots, x_n]^T$, we can express x as $x = \sum_{i=1}^{n} x_i e_i + (1 - \sum_{i=1}^{n} x_i)0$, where $0 \in \mathbb{R}^n$ is the zero vector. Then

$$\begin{aligned} f(x) &= f\left(\sum_{i=1}^n x_i e_i + \left(1 - \sum_{i=1}^n x_i\right) 0\right) \\ &= \sum_{i=1}^n x_i f(e_i) + \left(1 - \sum_{i=1}^n x_i\right) f(0) \\ &= \sum_{i=1}^n x_i \big(f(e_i) - f(0)\big) + f(0) \\ &= Ax + b \end{aligned}$$

where $A = [f(e_1) - f(0), f(e_2) - f(0), \cdots, f(e_n) - f(0)]$ and $b = f(0)$.

4. (a) Suppose $\hat{0} \in V$ is also a zero element. Since 0 is a zero element, $0 + \hat{0} = \hat{0}$; however, $\hat{0}$ is also a zero element; thus $0 + \hat{0} = 0$. This implies $0 + \hat{0} = 0 = \hat{0}$ and therefore the zero element is unique.

   (b) Let $(-\hat{x})$ is also an additive inverse of x, i.e. $x + (-\hat{x}) = 0$. Then

   $$(-x) = (-x) + 0 = (-x) + (x + (-\hat{x})) = ((-x) + x) + (-\hat{x}) = 0 + (-\hat{x}) = -\hat{x}$$

   Hence the additive inverse is unique.

   (c) For any $x \in V$,

   $$0 \cdot x + x = 0 \cdot x + 1 \cdot x = (0 + 1) \cdot x = 1 \cdot x = x$$

   Since the above equality is true for all $x \in V$, $0 \cdot x$ is the zero element of V. Because the zero element is unique, we have $0 \cdot x = 0$.

5. For any $A, B \in S_{sn}$ and any $\alpha \in \mathbb{R}$, we have $(A + B)^T = A^T + B^T = -(A + B)$ and $(\alpha A)^T = -(\alpha A)$. Therefore, S_{sn} is indeed a subspace of $\mathbb{R}^{n \times n}$.

   Notice that if $A \in S_{sn}$, then all diagonal elements of A must be zeros. Therefore A has $\frac{n(n-1)}{2}$ distinct elements, and $\dim S_{sn} = \frac{n(n-1)}{2}$.

   One basis of S_{s3} is

   $$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$