Nonlinear System Theory
Solution to Homework 9

1. (a)

$$\dot{y} = \dot{x}_3 = -x_1 + u$$

   Therefore, the relative degree is one.

   (b) To transform the nonlinear system into normal form, we should find $\phi_i(x) : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$, such that $\frac{\partial \phi_i}{\partial x} g(x) = 0$ for all $x \in D$ and $\phi_i(0) = 0$, where $g(x) = [0,\ 1,\ 1]^T$. In other words,

$$\frac{\partial \phi_i}{\partial x_2} + \frac{\partial \phi_i}{\partial x_3} = 0, \quad i = 1, 2$$

   Choose $\eta_1 = \phi_1(x) = x_1$ and $\eta_2 = \phi_2(x) = x_2 - x_3$. Let $\xi = y = x_3$. Then

$$\begin{aligned} \dot{\eta}_1 &= -x_1 + x_2 - x_3 = -\eta_1 + \eta_2 \\ \dot{\eta}_2 &= -x_1 x_3 - x_2 + u - (-x_1 + u) = -\eta_1 \xi - \eta_2 - \xi + \eta_1 \\ \dot{\xi} &= -x_1 + u = -\eta_1 + u \end{aligned}$$

   The transformation is valid for $\mathbb{R}^3$.

   (c) Let $\xi \equiv 0$. Then the zero dynamics is

$$\begin{aligned} \dot{\eta}_1 &= -\eta_1 + \eta_2 \\ \dot{\eta}_2 &= \eta_1 - \eta_2 \end{aligned}$$

   The eigenvalues of the zero dynamics are -2, and 0. The equilibrium point $(\eta_1, \eta_2) = (0, 0)$ is not asymptotically stable, hence the system is nonminimum phase.

2. (a)

$$\begin{aligned} \dot{y} &= \dot{x}_2 = x_2^2 + x_3 \\ \ddot{y} &= 2x_2 \dot{x}_2 + \dot{x}_3 = 2x_2^3 + 2x_2 x_3 - \sin x_3 - x_2 - x_1^3 + u \end{aligned} \tag{1}$$

   Hence the relative degree is 2.

   To transform to the normal form, we need to find a scalar function $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g = 0 \tag{2}$$

   where $g = [-2, 0, 1]^T$. Hence

$$\frac{\partial \phi}{\partial x} g = -2 \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_3} = 0$$

Choose $\phi(x) = x_1 + 2x_3$; then the conditions in (2) are satisfied. Define

$$\eta = \phi(x) = x_1 + 2x_3, \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2^2 + x_3 \end{bmatrix}$$

Conversely,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta - 2(\xi_2 - \xi_1^2) \\ \xi_1 \\ \xi_2 - \xi_1^2 \end{bmatrix}$$

Hence the normal form is

$$\begin{aligned}
\dot{\eta} &= \dot{x}_1 + 2\dot{x}_3 = x_2 + \tan x_3 - 2u + 2\big(-\sin x_3 - x_2 - x_1^3 + u\big) \\
&= -x_2 - 2x_1^3 + \tan x_3 - 2\sin x_3 \\
&= -\xi_1 - 2\big[\eta - 2(\xi_2 - \xi_1^2)\big]^3 + \tan\left(\xi_2 - \xi_1^2\right) - 2\sin\left(\xi_2 - \xi_1^2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= 2x_2^3 + 2x_2x_3 - \sin x_3 - x_2 - x_1^3 + u \\
&= -\xi_1 + 2\xi_1^3 + 2\xi_1\left(\xi_2 - \xi_1^2\right) - \sin\left(\xi_2 - \xi_1^2\right) - \big[\eta - 2(\xi_2 - \xi_1^2)\big]^3 + u
\end{aligned}$$

(b) Let

$$\dot{\eta} = f_0(\eta, \xi_1, \xi_2) = -\xi_1 - 2\big[\eta - 2(\xi_2 - \xi_1^2)\big]^3 + \tan\left(\xi_2 - \xi_1^2\right) - 2\sin\left(\xi_2 - \xi_1^2\right)$$

By definition, the zero dynamics is

$$\dot{\eta} = f_0(\eta, 0, 0) = -2\eta^3$$

Since the zero dynamics has an asymptotically stable equilibrium point at $\eta = 0$, the system is minimum phase.

(c) From (1) we choose

$$u = -\left(2x_2^3 + 2x_2x_3 - \sin x_3 - x_2 - x_1^3\right) + v \tag{3}$$

such that the input-output mapping from the auxiliary control input v to the output y is linearized as $\ddot{y} = v$. To achieve the desired transfer function, take

$$v = -15\dot{y} - 100y + 100r = -15\left(x_2^2 + x_3\right) - 100x_2 + 100r \tag{4}$$

Then the characteristic equation becomes

$$\ddot{y} + 15\dot{y} + 100y = 100r$$

Thus the transfer function from r to y has the desired form. Substitute (4) into (3) and the control law is

$$u = -\left(2x_2^3 + 2x_2x_3 - \sin x_3 - x_2 - x_1^3\right) - \left(15\left(x_2^2 + x_3\right) + 100x_2\right) + 100r$$