Nonlinear System Theory
Solution to Homework 9

1. (a) Since $G(s)$ is unstable, loop transformation is applied as shown in Figure 1, where $\alpha > 0$ is a constant. Then

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{s}{s^2 + (\alpha - 1)s + 1}$$

Choose $\alpha = 1.1$ and $\tilde{G}(s)$ is stable. The Popov plot of $\tilde{G}(s)$ is shown in Figure 2. To find the maximum sector, we choose the a vertical line (i.e. $\gamma = 0$) which intersects the horizontal axis at a point arbitrarily close to zero. In other words, $k > 0$ and k can be arbitrarily large. Note that for $\gamma = 0$, $1 + \lambda_i \gamma = 1 \neq 0$, where λ_i is any pole of $\tilde{G}(s)$. Moreover,

$$Z(j\omega) = \frac{1}{k} + \tilde{G}(j\omega) \to \frac{1}{k} > 0, \quad \text{as } \omega \to \infty$$

Consequently, $Z(s)$ is SPR if the Popov plot is to the right of the vertical line shown in Figure 2. This implies that if $\tilde{\psi} \in [0, k)$, where $k > 0$ can be arbitrarily large, or $\psi \in [\alpha, \infty) = [1.1, k)$, then the feedback system is absolutely stable.

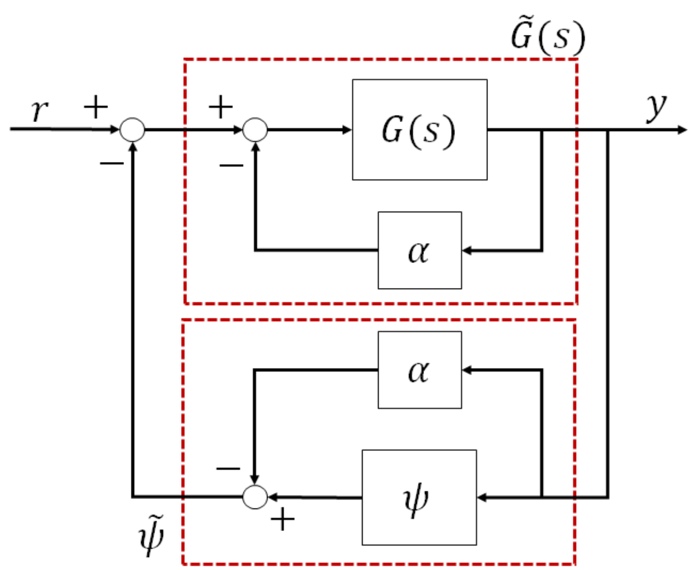

Figure 1: Loop Transformation for Problem 1(a)

(b) The Popov plot of $G(s)$ is shown in Figure 3. Choose $\gamma = 1$ and the intersection with the horizontal axis can be arbitrarily close to 0, implying

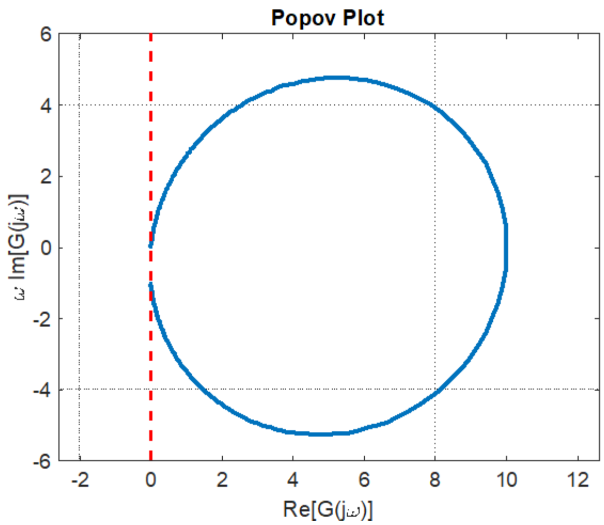

Figure 2: Popov plot for Problem 1(a)

that $k > 0$ can be arbitrarily large. Since the poles of $G(s)$ are $\lambda_{1,2} = -0.5 \pm 0.866j$, we have $1 + \lambda_i \gamma \neq 0$ for $i = 1, 2$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)G(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 3. Hence the feedback system is absolutely stable if $\psi \in [0,\, k)$, where $k > 0$ can be arbitrarily large.

However, we can further extend the lower bound of the sector to a negative value. Consider again the loop transformation in Figure 1. In this case, we have

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{1}{s^2 + s + (\alpha + 1)}$$

Choose $\alpha = -0.9$; then $\tilde{G}(s)$ is stable and its Popov plot is shown in Figure 4. Choose $\gamma = 1$ and the intersection with the horizontal axis can be arbitrarily close to 0, implying that $k > 0$ can be arbitrarily large. The poles of $\tilde{G}(s)$ are $\lambda_{1,2} = -0.887, -0.113$; therefore $1 + \lambda_i \gamma \neq 0$ for $i = 1, 2$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)\tilde{G}(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 4. Hence if $\tilde{\psi} \in [0,\, k)$, where $k > 0$ is arbitrarily large, or $\psi \in [\alpha,\, k) = [-0.9,\, k)$, the feedback system is absolutely stable.

(c) Because $G(s)$ is unstable, we need to do loop transformation as shown in Figure 1. Then

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{s + 1}{s^3 + 3s^2 + \alpha s + (\alpha - 4)}$$

By Routh's table, we can see that if $\alpha > 4$, then $\tilde{G}(s)$ is stable. Choose $\alpha = 4.1$ and the Popov plot is shown in Figure 5. Choose $\gamma = 0.5$ and

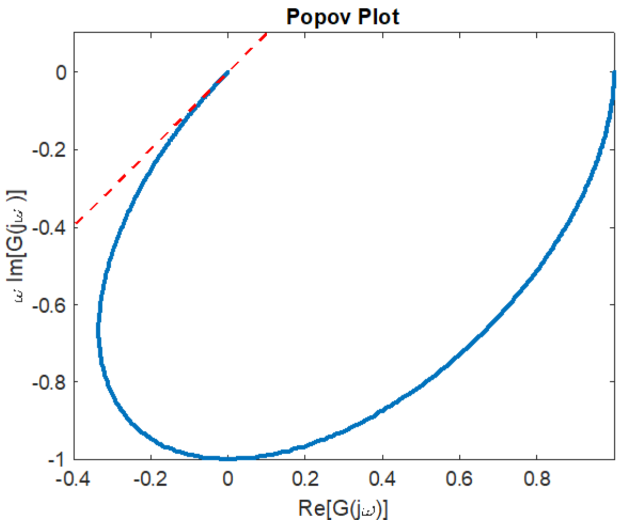

Figure 3: Popov plot for Problem 1(b)
– without Loop Transformation

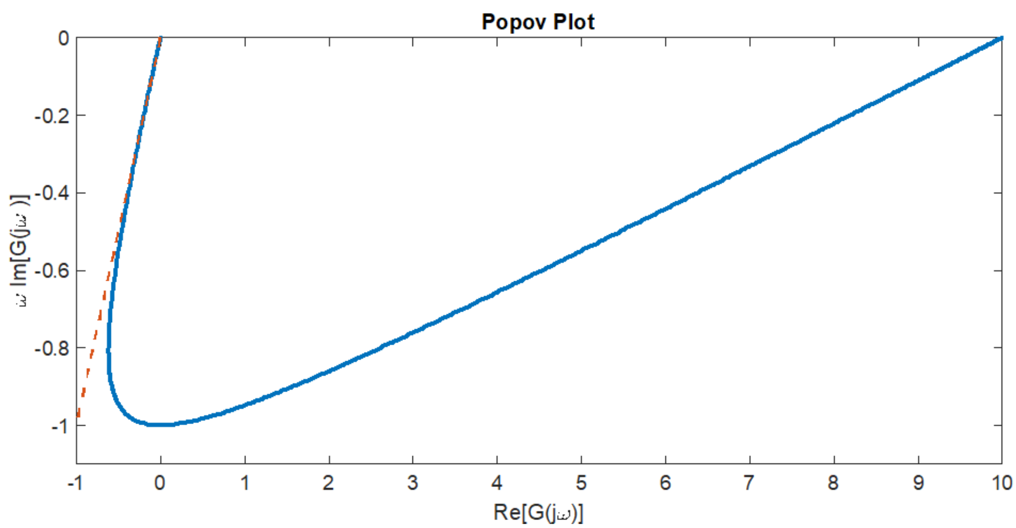

Figure 4: Popov plot for Problem 1(b) – with Loop Transformation

$k > 0$ arbitrarily large. Note that the poles of $\tilde{G}(s)$ are $\lambda_{1,2,3} = -1.488 \pm 1.347j, -0.024$; hence $1 + \lambda_i\gamma \neq 0$ for $i = 1, 2, 3$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)\tilde{G}(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 5. Hence if $\tilde{\psi} \in [0,\, k)$, where $k > 0$ is arbitrarily large, or $\psi \in [\alpha,\, k) = [4.1,\, k)$, the feedback system is absolutely stable.

2. (a) Let $V = \frac{1}{2}\big(k_1 e_1^2 + e_2^2 + \frac{1}{\gamma}\tilde{a}^2\big)$ be a Lyapunov function candidate. V is positive definite and radially unbounded. Then

$$\dot{V} = k_1 e_1 e_2 + e_2\big(-\tilde{a}\phi - k_1 e_1 - k_2 e_2\big) + \tilde{a} e_2 \phi = -k_2 e_2^2$$

Since $\dot{V}$ is negative semidefinite, the equilibrium point is globally stable.

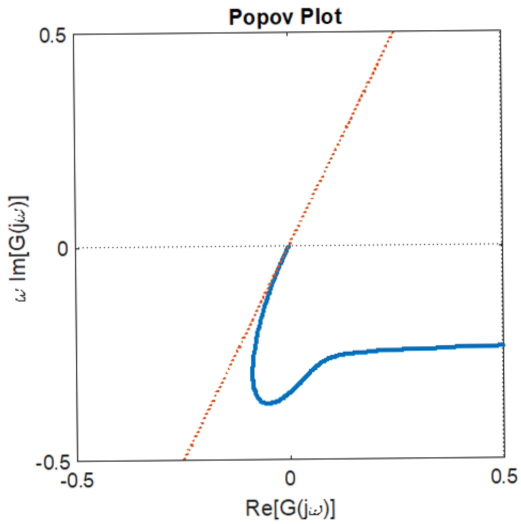

Figure 5: Popov plot for Problem 1(c)

(b) Since the equilibrium point is stable, the state variables, $e_1, e_2, \tilde{a}$ are bounded. Then V is bounded and from the state equation we see that $\dot{e}_2$ is also bounded. As a result, $\ddot{V} = -2k_2 e_2 \dot{e}_2$ is bounded. This implies that $\dot{V}$ is uniformly continuous. Hence by Barbalat's lemma, we conclude that $\dot{V} \to 0$ as $t \to \infty$, and $e_2 \to 0$ as $t \to \infty$.

3. (a) Rearrange the dynamic equation of the inverted pendulum system as follows:

$$\begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & M+m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ mL\dot{\theta}^2 \sin\theta - k\dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F \qquad (1)$$

Define

$$\Delta(\theta) = \det \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & M+m \end{bmatrix} = (I + mL^2)(M+m) - m^2L^2\cos^2\theta$$

Then

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} M+m & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \left(\begin{bmatrix} mgL\sin\theta \\ mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F \right)$$

Therefore the state equation is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{\Delta(x_1)}(M+m)mgL\sin x_1 - \frac{1}{\Delta(x_1)} mL\cos x_1 (mLx_2^2 \sin x_1 - kx_4 + F) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{1}{\Delta(x_1)} m^2L^2 g\cos x_1 \sin x_1 + \frac{1}{\Delta(x_1)} (I + mL^2)\big(mLx_2^2 \sin x_1 - kx_4 + F\big)
\end{aligned}$$

(b) Let $F = 0$. Then the equilibrium points are $(n\pi,\ 0,\ \bar{x}_3, 0)$, where $n \in \mathbb{Z}$ and $\bar{x}_3$ can be any real number.

(c) To facilitate the derivation, we linearize (1) about $\theta = \dot{\theta} = 0$. Note that $\cos\theta \approx 1$, $\sin\theta \approx \theta$, $\dot{\theta}^2 \approx 0$ as $\theta \approx 0$ and $\dot{\theta} \approx 0$. Hence

$$\begin{bmatrix} I + mL^2 & mL \\ mL & M + m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\theta \\ -k\dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

Then the linearzied state equation is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 0 & \Delta & 0 & 0 \\ (M+m)mgL & 0 & 0 & mLk \\ 0 & 0 & 0 & \Delta \\ -m^2L^2g & 0 & 0 & -(I+mL^2)k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} 0 \\ -mL \\ 0 \\ I + mL^2 \end{bmatrix} F$$

where $\Delta = (M+m)(I+mL^2) - m^2L^2$. The eigenvalues of the linearized system are the solutions to the following characteristic equation

$$\begin{aligned} 0 &= \det\begin{bmatrix} s & -\Delta & 0 & 0 \\ -(M+m)mgL & s & 0 & -mLk \\ 0 & 0 & s & -\Delta \\ m^2L^2g & 0 & 0 & s + (I+mL^2)k \end{bmatrix} \\ &= s^3(s + (I+mL^2)k) + \Delta\Big(-(M+m)mgLs(s + (I+mL^2)k) + m^3L^3kgs \Big) \\ &= s\left\{ s^3 + (I+mL^2)ks^2 - \Delta(M+m)mgLs - \Delta mgLk\big((M+m)(I+mL^2) - m^2L^2\big) \right\} \\ &= s\left\{ s^3 + (I+mL^2)ks^2 - \Delta(M+m)mgLs - \Delta^2 mgLk \right\} \end{aligned}$$

Apparently, there is an eigenvalue at $s = 0$. To determine stability of the other eigenvalues, we establish the Routh table for the 3rd order polynomial as follows:

$$\begin{array}{lcc} s^3: & 1 & -\Delta(M+m)mgL \\ s^2: & (I+mL^2)k & -\Delta^2 mgLk \\ s^1: & -\frac{\Delta m^3L^3g}{(I+mL^2)} & 0 \\ s^0: & -\Delta^2 mgLk & 0 \end{array}$$

Since there is one sign change in the first column of the Routh table, the system has one pole with positive real part. Hence the system is unstable.

(d) Use LQR to design the state feedback controller. The linearized model is given is part (c). Choose $Q = \mathrm{diag}(10, 1, 500, 1)$ and $R = 200$. Let $u = F$, The LQR controller is

$$u^* = \arg\min_u \int_0^\infty \big(x^T(t)Qx(t) + u^T(t)Ru(t)\big)dt = -Kx$$

where $K = [-32.8033, -8.3136, -1.5811, -2.7666]$. The simulation results are shown in Figure 6. From Figure 6 we see that the initial pendulum angle is 30°. It returns to the upright position $\theta = 0$. The control force F is saturated at 10 N, and the cart position y is limited within 2 m.

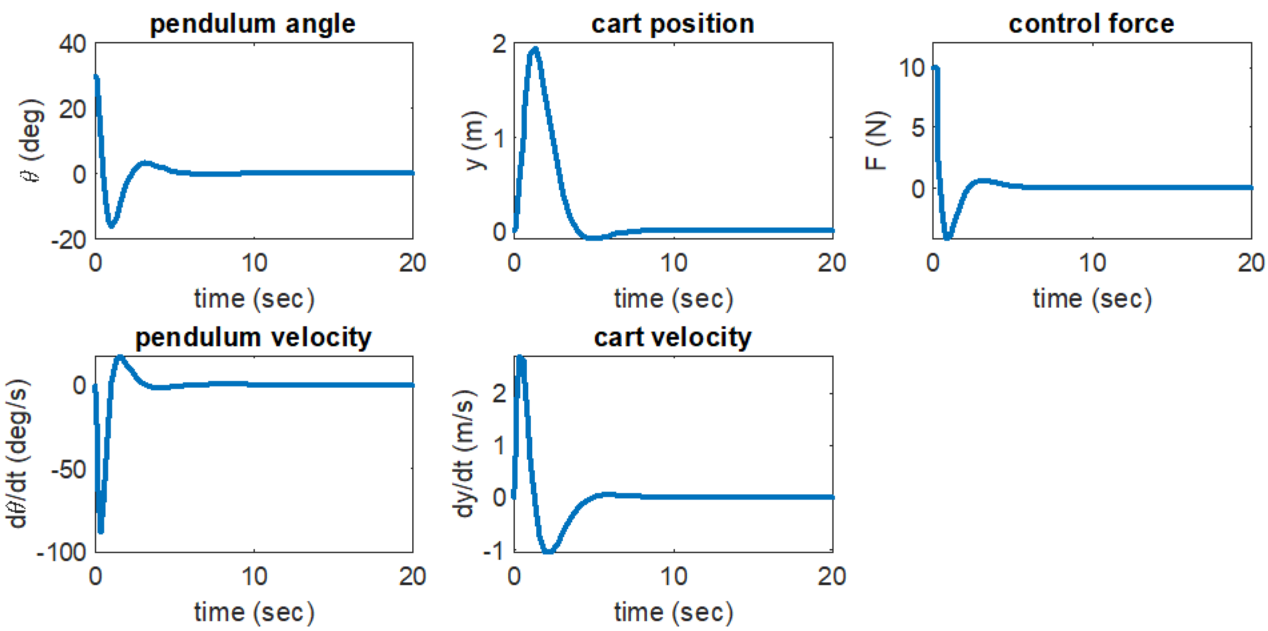

Figure 6: Simulation results for Problem 3