

Nonlinear System Theory

Solution to Homework 1

1. Note that $\bar{\alpha}_f = \frac{\bar{F}_{yf}}{C_{\alpha f}} = 0.0436$ and $\bar{\alpha}_r = \frac{\bar{F}_{yr}}{C_{\alpha r}} = 0.0404$. We consider different cases to find the equilibrium points.

- Case I: $|\alpha_f| \leq \bar{\alpha}_f$ and $|\alpha_r| \leq \bar{\alpha}_r$

The dynamic equations are

$$\begin{aligned}\dot{\beta} &= \frac{2}{Mv_x} \left[C_{\alpha f} \left(\delta_f - \beta - \frac{rl_f}{v_x} \right) + C_{\alpha r} \left(-\beta + \frac{rl_r}{v_x} \right) \right] - r \\ \dot{r} &= \frac{2}{I_z} \left[C_{\alpha f} \left(\delta_f - \beta - \frac{rl_f}{v_x} \right) l_f - C_{\alpha r} \left(-\beta + \frac{rl_r}{v_x} \right) l_r \right]\end{aligned}$$

Rearrange the equation and we have

$$\begin{aligned}\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{-2(C_{\alpha f} + C_{\alpha r})}{Mv_x} & \frac{-2(C_{\alpha f}l_f - C_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{-2(C_{\alpha f}l_f - C_{\alpha r}l_r)}{I_z} & \frac{-2(C_{\alpha f}l_f^2 + C_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{2C_{\alpha f}}{Mv_x} \\ \frac{2C_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\ &= \underbrace{\begin{bmatrix} -17.7183 & -0.9247 \\ 5.1756 & -15.8140 \end{bmatrix}}_{\mathbf{A}_1} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} 0.9275 \\ 5.6595 \end{bmatrix}}_{\mathbf{B}_1} \delta_f\end{aligned}$$

The equilibrium point is

$$\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = -\mathbf{A}_1^{-1} \mathbf{B}_1 = \begin{bmatrix} 0.0331 \\ 0.3687 \end{bmatrix}$$

Then the corresponding tire slip angles are

$$\alpha_f^* = \delta_f - \beta^* - \frac{r^* l_f}{v_x} = 0.0214, \quad \alpha_r^* = -\beta^* + \frac{r^* l_r}{v_x} = 0.0199$$

Note that the assumptions of this case ($|\alpha_f| \leq \bar{\alpha}_f$ and $|\alpha_r| \leq \bar{\alpha}_r$) hold.

The eigenvalues of \mathbf{A}_1 are $-16.7661 \pm 1.9696j$; hence the equilibrium point is a stable focus.

- Case II: $|\alpha_f| > \bar{\alpha}_f$ and $|\alpha_r| \leq \bar{\alpha}_r$

Follow the same argument and the dynamic equation is

$$\begin{aligned}\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{2(C'_{\alpha f} - C_{\alpha r})}{Mv_x} & \frac{2(C'_{\alpha f}l_f + C_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{2(C'_{\alpha f}l_f + C_{\alpha r}l_r)}{I_z} & \frac{2(C'_{\alpha f}l_f^2 - C_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} -\frac{2C'_{\alpha f}}{Mv_x} \\ \frac{-2C'_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\ &\quad + 2 \begin{bmatrix} \frac{1}{Mv_x} \\ \frac{l_f}{I_z} \end{bmatrix} (\bar{F}_{yf} + C'_{\alpha f} \bar{\alpha}_f) \text{sgn}(\alpha_f) \\ &= \underbrace{\begin{bmatrix} -6.8775 & 0.0373 \\ 71.3256 & -9.9438 \end{bmatrix}}_{\mathbf{A}_2} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0185 \\ -0.1132 \end{bmatrix}}_{\mathbf{B}_2} \pm \begin{bmatrix} 0.4732 \\ 2.8872 \end{bmatrix}\end{aligned}$$

The equilibrium points are

$$\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = \begin{bmatrix} 0.0703 \\ 0.7836 \end{bmatrix} \text{ for } \alpha_f > 0, \text{ or } \begin{bmatrix} -0.0761 \\ -0.8475 \end{bmatrix} \text{ for } \alpha_f < 0$$

For $(\beta^*, r^*) = (0.0703, 0.7836)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-0.0526, 0.0422)$, which violate the assumption that $\alpha_f^* > 0$.

For $(\beta^*, r^*) = (-0.0761, -0.8475)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (0.2386, -0.0456)$, which violate the assumption that $\alpha_f^* < 0$.

Therefore, no equilibrium point is found for this case.

- Case III: $|\alpha_f| \leq \bar{\alpha}_f$ and $|\alpha_r| > \bar{\alpha}_r$

Follow the same argument and the dynamic equation is

$$\begin{aligned} \begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{-2(C_{\alpha f} - C'_{\alpha r})}{Mv_x} & \frac{-2(C_{\alpha f}l_f + C'_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{-2(C_{\alpha f}l_f + C'_{\alpha r}l_r)}{I_z} & \frac{-2(C_{\alpha f}l_f^2 - C'_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{2C_{\alpha f}}{Mv_x} \\ \frac{2C_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\ &\quad + 2 \begin{bmatrix} \frac{1}{\frac{Mv_x}{l_r}} \\ \frac{1}{I_z} \end{bmatrix} (\bar{F}_{yr} + C'_{\alpha r} \bar{\alpha}_r) \text{sgn}(\alpha_r) \\ &= \underbrace{\begin{bmatrix} -10.4864 & -1.9635 \\ -66.2535 & -5.5539 \end{bmatrix}}_{\mathbf{A}_3} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} 0.9275 \\ 5.6595 \end{bmatrix} \pm \begin{bmatrix} 0.2923 \\ -2.8872 \end{bmatrix}}_{\mathbf{B}_3} \end{aligned}$$

The equilibrium points are

$$\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = \begin{bmatrix} -0.0185 \\ 0.7202 \end{bmatrix} \text{ for } \alpha_r > 0, \text{ or } \begin{bmatrix} 0.1845 \\ -0.6617 \end{bmatrix} \text{ for } \alpha_r < 0$$

For $(\beta^*, r^*) = (-0.0185, 0.7202)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (0.0419, 0.1220)$, which satisfy the assumptions that $|\alpha_f| \leq \bar{\alpha}_f$ and $\alpha_r > \bar{\alpha}_r > 0$.

For $(\beta^*, r^*) = (0.1845, -0.6617)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-0.0385, -0.2795)$, which satisfy the assumptions that $|\alpha_f| \leq \bar{\alpha}_f$ and $\alpha_r < -\bar{\alpha}_r < 0$.

Hence both $(\beta^*, r^*) = (-0.0185, 0.7202)$ and $(\beta^*, r^*) = (0.1845, -0.6617)$ are equilibrium points. They correspond to the same Jacobian matrix \mathbf{A}_3 , whose eigenvalues are -19.6895 and 3.6491 . In other words, the equilibrium points in Case III are saddle points and they are unstable.

- Case IV: $|\alpha_f| > \bar{\alpha}_f$ and $|\alpha_r| > \bar{\alpha}_r$

Follow the same argument and the dynamic equation is

$$\begin{aligned} \begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{2(C'_{\alpha f} + C'_{\alpha r})}{Mv_x} & \frac{2(C'_{\alpha f}l_f - C'_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{2(C'_{\alpha f}l_f - C'_{\alpha r}l_r)}{I_z} & \frac{2(C'_{\alpha f}l_f^2 + C'_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{-2C'_{\alpha f}}{Mv_x} \\ \frac{-2C'_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\ &\quad \pm 2 \begin{bmatrix} \frac{1}{\frac{Mv_x}{l_f}} \\ \frac{1}{I_z} \end{bmatrix} (\bar{F}_{yf} + C'_{\alpha f} \bar{\alpha}_f) \text{sgn}(\alpha_f) \pm 2 \begin{bmatrix} \frac{1}{\frac{Mv_x}{l_r}} \\ \frac{1}{I_z} \end{bmatrix} (\bar{F}_{yr} + C'_{\alpha r} \bar{\alpha}_r) \text{sgn}(\alpha_r) \\ &= \underbrace{\begin{bmatrix} 0.3544 & -1.0015 \\ -0.1035 & 0.3163 \end{bmatrix}}_{\mathbf{A}_4} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0185 \\ -0.1132 \end{bmatrix} \pm \begin{bmatrix} 0.4732 \\ 2.8872 \end{bmatrix} \pm \begin{bmatrix} 0.2923 \\ -2.8872 \end{bmatrix}}_{\mathbf{B}_4} \end{aligned}$$

For $\alpha_f > 0$ and $\alpha_r > 0$, the equilibrium point is $(\beta^*, r^*) = (-14.6091, -4.4234)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (15.0889, 13.9737)$, which satisfy the assumption that $\alpha_f > \bar{\alpha}_f > 0$ and $\alpha_r > \bar{\alpha}_r > 0$.

For $\alpha_f < 0$ and $\alpha_r < 0$, the equilibrium point is $(\beta^*, r^*) = (42.9594, 14.4176)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-44.1516, -40.8885)$, which satisfy the assumption that $\alpha_f < -\bar{\alpha}_f$ and $\alpha_r < -\bar{\alpha}_r$.

For $\alpha_f > 0$ and $\alpha_r < 0$, the equilibrium point is $(\beta^*, r^*) = (-680.1881, -240.5115)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (701.6183, 645.6410)$, which violate the assumption that $\alpha_f > \bar{\alpha}_f > 0$ and $\alpha_r < -\bar{\alpha}_r$.

For $\alpha_f < 0$ and $\alpha_r > 0$, the equilibrium point is $(\beta^*, r^*) = (708.5384, 250.5057)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-730.6811, -672.5558)$, which violate the assumption that $\alpha_f < -\bar{\alpha}_f$ and $\alpha_r > \bar{\alpha}_r > 0$.

Therefore, there are two equilibrium points for case IV: $(\beta^*, r^*) = (-14.6091, -4.4234)$ and $(\beta^*, r^*) = (42.9594, 14.4176)$. Since the eigenvalues of \mathbf{A}_4 are 0.6579, and 0.0128. These two equilibrium points are unstable nodes.

2. (a) The loop transfer function is

$$L(s) = \frac{K}{s}G(s) = \frac{500K}{s(s^2 + 60s + 500)}$$

The root locus is shown in Figure 1.

- (b) For $K = 60$, the characteristic equation of the closed-loop system is

$$0 = s^3 + 60s^2 + 500s + 30000 = (s + 60)(s^2 + 500)$$

Hence the closed-loop poles are $p = -60$, and $\pm 10\sqrt{5}j$. The step response and the state trajectory are shown in Figure 2 and Figure 3, respectively.

- (c) The step response and the state trajectory are shown in Figure 4 and Figure 5, respectively. The state trajectory exhibits the property of limit cycle.

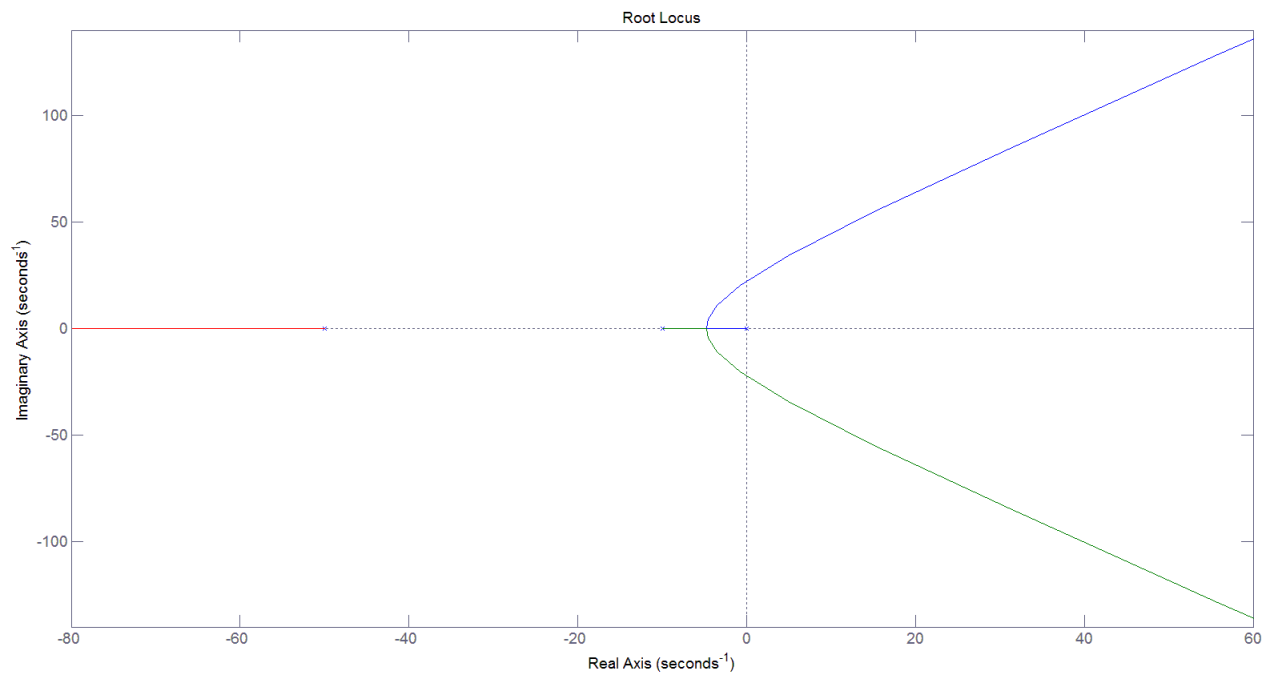


Figure 1: Root locus of the system in Problem 2

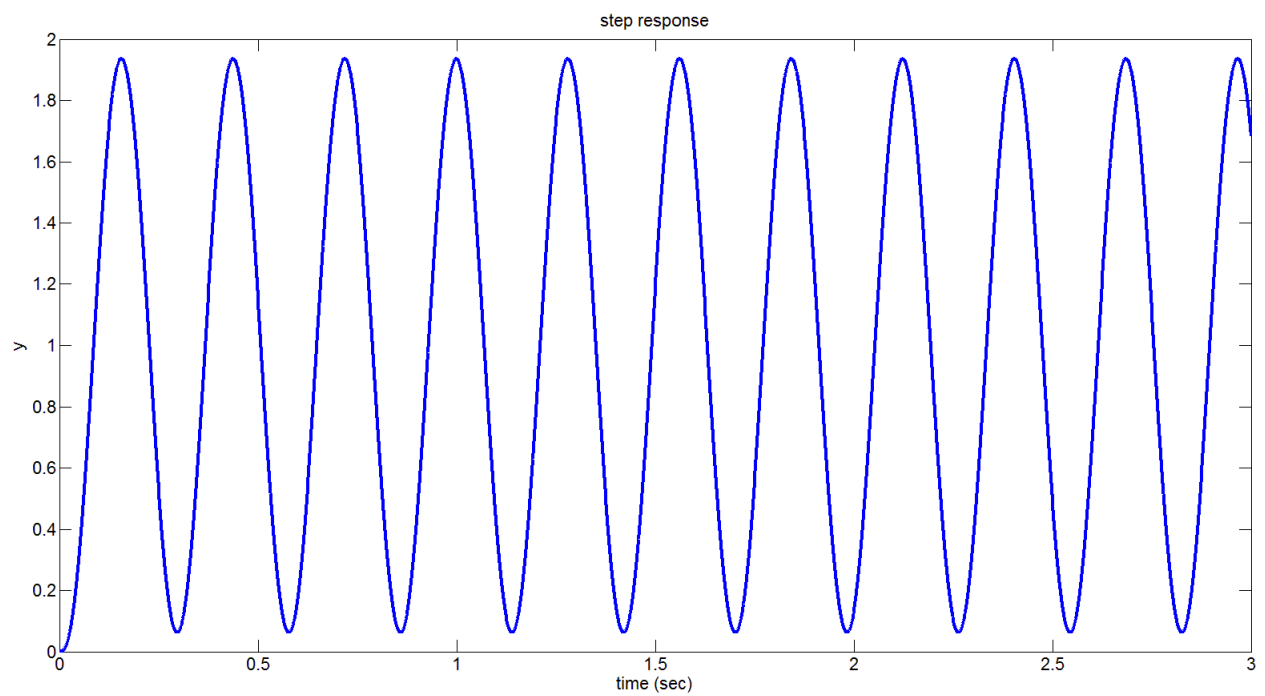


Figure 2: Step response of the system in Problem 2

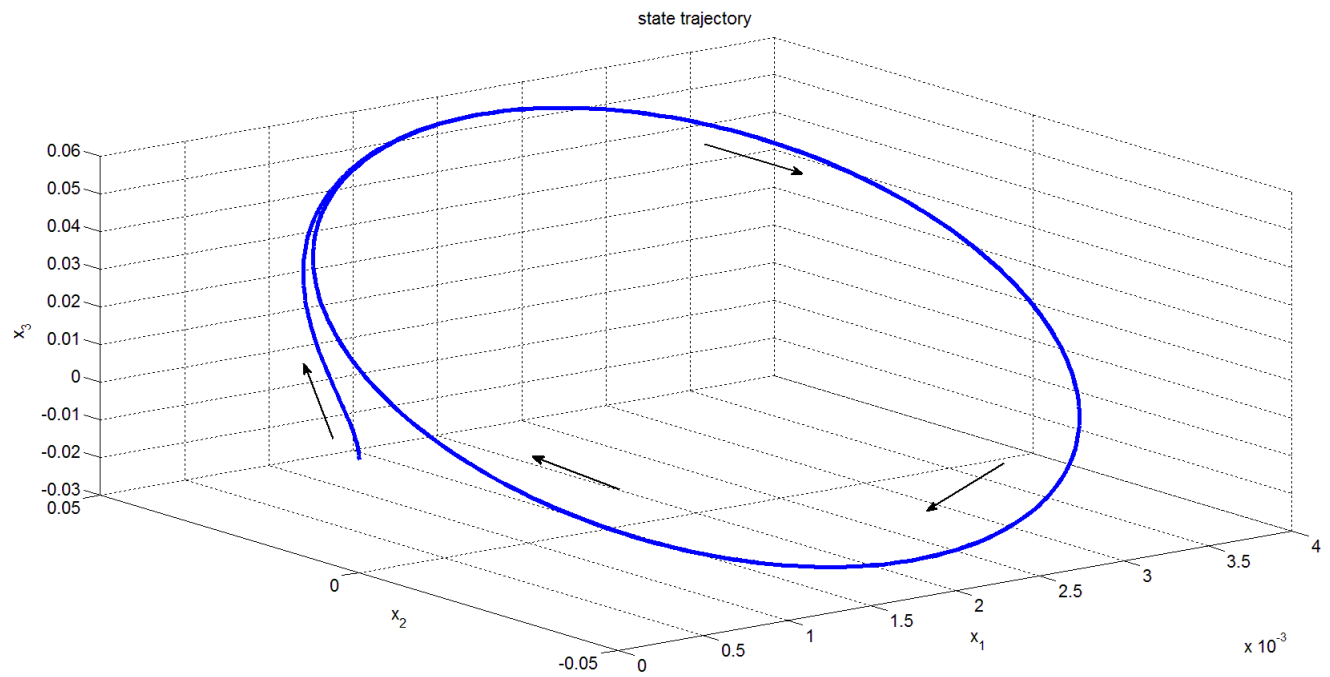


Figure 3: state trajectory of the system in Problem 2

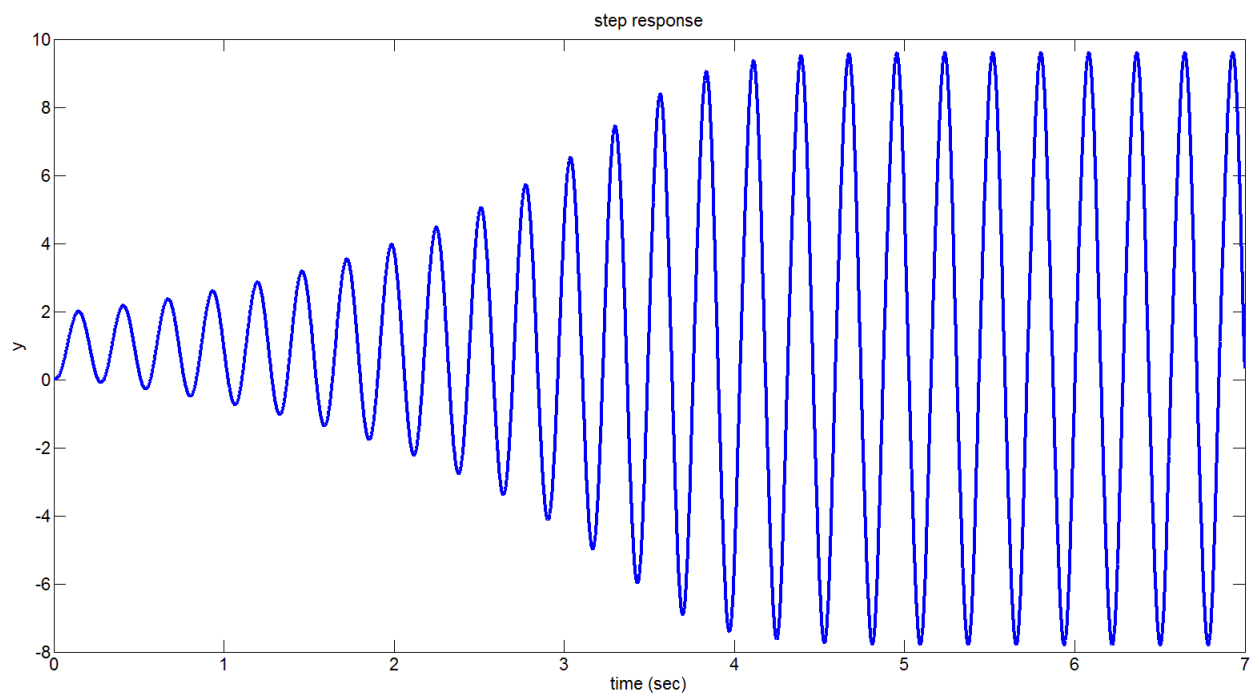


Figure 4: Step response of the system in Problem 2

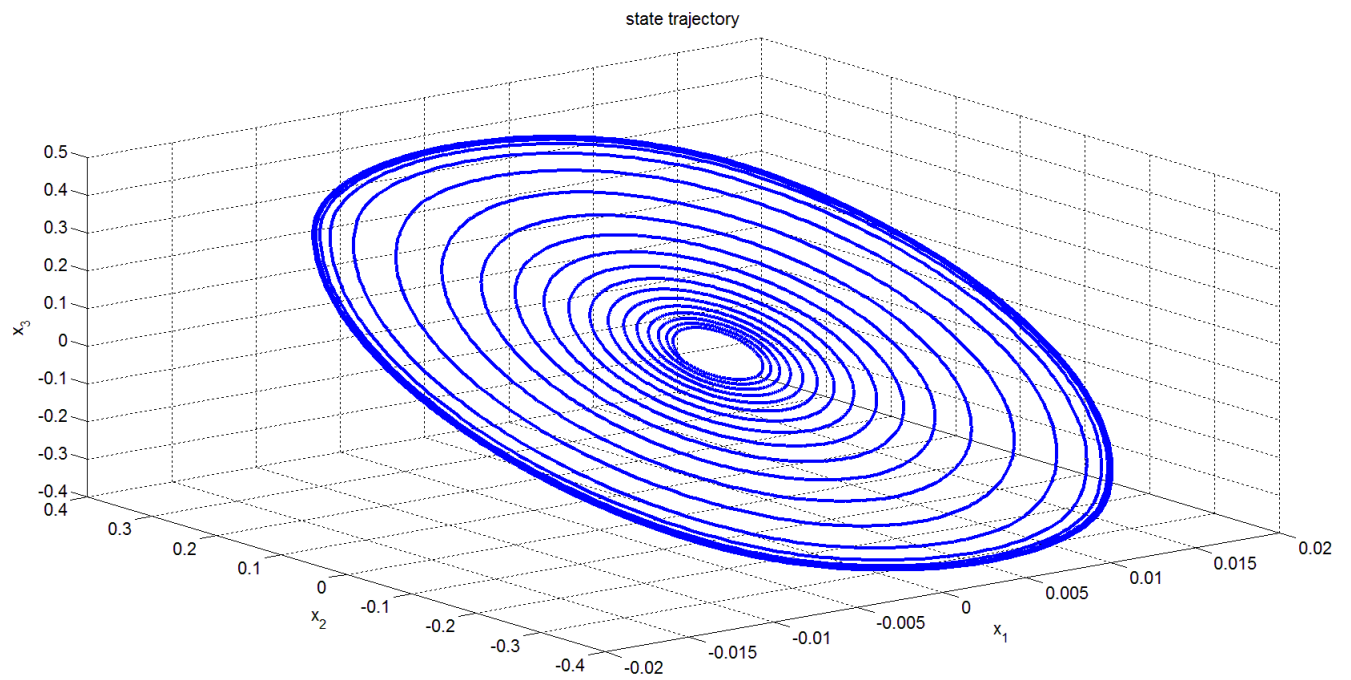


Figure 5: State trajectory of the system in Problem 2