

# Nonlinear System Theory

## Solution to Homework 3

1. (a) Choose  $\tilde{q}$  and  $\dot{\tilde{q}}$  as the state variables. Since  $q_d$  is constant,  $\dot{\tilde{q}} = -\dot{q}$ . Then the state equation is

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix} &= \begin{bmatrix} \dot{\tilde{q}} \\ -\ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}} \\ -M^{-1}(q)(\tau - C(q, \dot{q})\dot{q} - G(q) - B\dot{q}) \end{bmatrix} \\ &= \begin{bmatrix} \dot{\tilde{q}} \\ -M^{-1}(q)(K_P\tilde{q} - K_D\dot{q} - C(q, \dot{q})\dot{q} - B\dot{q}) \end{bmatrix} \end{aligned}$$

The equilibrium point satisfies  $\dot{\tilde{q}} = 0$  and  $\ddot{q} = 0$ . Then  $-M^{-1}(q)K_P\tilde{q} = 0$ . Since both  $M(q)$  and  $K_P$  are positive definite, so is  $M^{-1}(q)K_P$ . Therefore  $\tilde{q} = 0$  is the unique solution. In other words,  $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$  is the unique equilibrium point of the closed-loop system.

- (b) Since  $M(q)$  and  $K_P$  are positive definite matrices,  $V = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_P\tilde{q}$  is a positive definite function. The time derivative of  $V$  along the state trajectory is

$$\begin{aligned} \dot{V} &= \dot{\tilde{q}}^T M(q)\ddot{q} + \frac{1}{2}\dot{\tilde{q}}^T \dot{M}(q)\dot{\tilde{q}} - \tilde{q}^T K_P\dot{\tilde{q}} \\ &= \dot{\tilde{q}}^T \left[ K_P\tilde{q} - K_D\dot{q} - C(q, \dot{q})\dot{q} - B\dot{q} + \frac{1}{2}\dot{M}(q)\dot{\tilde{q}} - K_P\tilde{q} \right] \\ &= -\dot{\tilde{q}}^T (K_D + B)\dot{q} + \dot{\tilde{q}}^T \left( \frac{1}{2}\dot{M}(q) - C(q, \dot{q}) \right) \dot{\tilde{q}} \\ &= -\dot{\tilde{q}}^T (K_D + B)\dot{q} \end{aligned}$$

Because both  $K_D$  and  $B$  are positive definite,  $\dot{V} = -\dot{\tilde{q}}^T (K_D + B)\dot{\tilde{q}}$  is negative semidefinite. Hence the origin is a stable equilibrium point.

- (c) If  $\dot{V} = 0$ , then  $\dot{\tilde{q}} = 0$ . Suppose that the state stays identically in the set  $S = \{(\tilde{q}, \dot{\tilde{q}}) \in \mathbb{R}^{2n} \mid \dot{V} = 0\}$ . Then

$$\dot{\tilde{q}} \equiv 0 \Rightarrow \ddot{q} \equiv 0 \Rightarrow \tilde{q} = 0$$

Therefore  $(0, 0)$  is the only solution in  $S$ . By LaSalle Theorem,  $(0, 0)$  is an asymptotically stable equilibrium point.

2. Note that the origin is the unique equilibrium point of the system. Consider the Lyapunov function candidate  $V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ . Clearly  $V$  is positive definite and radially unbounded. Then the time derivative of  $V$  along the state trajectory is

$$\dot{V} = x_1^3\dot{x}_1 + x_2\dot{x}_2 = x_1^3x_2 - x_1^3x_2 - x_2^4 = -x_2^4$$

Since  $\dot{V}$  is negative semidefinite, the origin is globally stable. Consider the state trajectories that stay identically in the set  $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid \dot{V} = 0\}$ . Then

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$$

Since  $(0, 0)$  is the only trajectory in  $S$ , by LaSalle Theorem, the origin is globally asymptotically stable.

3. (a) Note that  $\dot{x} < 0$  for  $x > 0$ , implying that  $x(t)$  is decreasing towards 0 if  $x > 0$ . On the other hand,  $\dot{x} > 0$  for  $x < 0$ , implying that  $x(t)$  is increasing towards 0 if  $x < 0$ . Hence  $x(t)$  eventually converges to zero. In other words,  $x = 0$  is an asymptotically stable equilibrium point.

Alternatively, we can linearize the system at  $x = 0$ . Since this is a scalar system, the Jacobian matrix is actually a scalar, which is

$$A = -2x \tan^{-1} x - 1 = -1, \quad \text{for } x = 0$$

Hence  $x = 0$  is asymptotically stable.

- (b) Let  $v = \tan^{-1} x$ . Then

$$\dot{v} = \frac{\dot{x}}{1+x^2} = -\tan^{-1} x + \frac{e^{-2t}}{1+x^2} \leq -v + e^{-2t}$$

Consider the differential equation  $\dot{u} = -u + e^{-2t}$ ,  $u(0) = v(0) = \tan^{-1} x_0$ . The solution is

$$u(t) = C_1 e^{-t} + C_2 e^{-2t} \Rightarrow \dot{u} = -C_1 e^{-t} - 2C_2 e^{-2t} = -(C_1 e^{-t} + C_2 e^{-2t}) + e^{-2t} \Rightarrow C_2 = -1$$

Then

$$u(0) = C_1 + C_2 = C_1 - 1 = \tan^{-1} x_0 \Rightarrow C_1 = 1 + \tan^{-1} x_0$$

By comparison lemma, we have

$$v(t) = \tan^{-1} x(t) \leq u(t) = (1 + \tan^{-1} x_0) e^{-t} - e^{-2t}$$

Therefore,

$$x(t) \leq y(t) = \tan u(t) = \tan \left( (1 + \tan^{-1} x_0) e^{-t} - e^{-2t} \right), \quad \forall t \geq 0$$

4. The Jacobian matrix is

$$A = \begin{bmatrix} \frac{\pi}{2} - 2x_1 + \frac{\sin x_2}{10}, & \frac{x_1 \cos x_2}{10} \\ \cos x_1 - x_1 \sin x_1, & -\frac{3}{10} x_2^2 \end{bmatrix}$$

- (a) It is easy to check that  $(x_1, x_2) = (\frac{\pi}{2}, 0)$  is indeed an equilibrium point. Then the Jacobian matrix about  $(\frac{\pi}{2}, 0)$  is

$$A_1 = \begin{bmatrix} -\frac{\pi}{2}, & \frac{\pi}{20} \\ -\frac{\pi}{2}, & 0 \end{bmatrix}$$

The characteristic polynomial of  $A_1$  is  $\lambda^2 + \frac{\pi}{2}\lambda + \frac{\pi^2}{40}$  and the eigenvalues of  $A_1$  are  $-1.3938$  and  $-0.177$ . Hence  $(\frac{\pi}{2}, 0)$  is an asymptotically stable equilibrium point.

(b) The Jacobian matrix about  $(x_1, x_2) = (0, 0)$  is

$$A_2 = \begin{bmatrix} \frac{\pi}{2} & 0 \\ 1 & 0 \end{bmatrix}$$

The corresponding eigenvalues are  $\frac{\pi}{2}$  and 0. Since one eigenvalue has positive real part,  $(x_1, x_2) = (0, 0)$  is an unstable equilibrium point.

(c) Choose  $r = 1$  and let  $B_r = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq r\}$ , where  $x = [x_1, x_2]^T$ . Define  $V(x) = x_1 x_2$ . Then there is always an  $x_0$  in any neighborhood of  $x = 0$  such that  $V(x_0) > 0$ . Let  $U = \{x \in B_r \mid V(x) > 0\}$ . Notice that for  $x \in U$ , we have  $x_1 x_2 > 0$ ,  $\|x\|_2 \leq 1$ , and therefore  $\cos x_1 > 0$ . Thus the time derivative of  $V$  for  $x \in U$  is

$$\begin{aligned} \dot{V} &= \dot{x}_1 x_2 + x_1 \dot{x}_2 \\ &= x_1 x_2 \left( \frac{\pi}{2} - x_1 \right) + \frac{x_1 x_2 \sin x_2}{10} + x_1^2 \cos x_1 - \frac{x_1 x_2^3}{10} \\ &= x_1 x_2 \left( \frac{\pi}{2} - x_1 + \frac{\sin x_2}{10} - \frac{x_2^2}{10} \right) + x_1^2 \cos x_1 \\ &\geq x_1 x_2 \left( \frac{\pi}{2} - 1 - \frac{1}{10} - \frac{1}{10} \right) + x_1^2 \cos x_1 \\ &> 0.3 x_1 x_2 + x_1^2 \cos x_1 > 0 \end{aligned}$$

By Chetaev's Theorem,  $x = 0$  is an unstable equilibrium point.