Nonlinear System Theory
Solution to Homework 3

1. (a) Choose $\tilde{q}$ and $\dot{\tilde{q}}$ as the state variables. Since q_d is constant, $\dot{\tilde{q}} = -\dot{q}$. Then the state equation is

$$\frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}} \\ -\ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}} \\ -M^{-1}(q)\big(\tau - C(q,\dot{q})\dot{q} - G(q) - B\dot{q}\big) \end{bmatrix}$$
$$= \begin{bmatrix} \dot{\tilde{q}} \\ -M^{-1}(q)\big(K_P\tilde{q} - K_D\dot{q} - C(q,\dot{q})\dot{q} - B\dot{q}\big) \end{bmatrix}$$

The equilibrium point satisfies $\dot{\tilde{q}} = 0$ and $\ddot{\tilde{q}} = 0$. Then $-M^{-1}(q)K_P\tilde{q} = 0$. Since both $M(q)$ and K_P are positive definite, so is $M^{-1}(q)K_P$. Therefore $\tilde{q} = 0$ is the unique solution. In other words, $(\tilde{q}, \dot{\tilde{q}}) = (0,0)$ is the unique equilibrium point of the closed-loop system.

(b) Since $M(q)$ and K_P are positive definite matrices, $V = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_P \tilde{q}$ is a positive definite function. The time derivative of V along the state trajectory is

$$\begin{aligned}
\dot{V} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} - \tilde{q}^T K_P \dot{q} \\
&= \dot{q}^T\left[K_P\tilde{q} - K_D\dot{q} - C(q,\dot{q})\dot{q} - B\dot{q} + \frac{1}{2}\dot{M}(q)\dot{q} - K_P\tilde{q}\right] \\
&= -\dot{q}^T\big(K_D + B\big)\dot{q} + \dot{q}^T\left(\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right)\dot{q} \\
&= -\dot{q}^T\big(K_D + B\big)\dot{q}
\end{aligned}$$

Because both K_D and B are positive definite, $\dot{V} = -\dot{q}^T\big(K_D + B\big)\dot{q}$ is negative semidefinite. Hence the origin is a stable equilibrium point.

(c) If $\dot{V} = 0$, then $\dot{q} = 0$. Suppose that the state stays identically in the set $S = \{(\tilde{q}, \dot{q}) \in \mathbb{R}^{2n} \mid \dot{V} = 0\}$. Then

$$\dot{q} \equiv 0 \Rightarrow \ddot{q} \equiv 0 \Rightarrow \tilde{q} = 0$$

Therefore $(0,0)$ is the only solution in S. By LaSalle Theorem, $(0,0)$ is an asymptotically stable equilibrium point.

2. Note that the origin is the unique equilibrium point of the system. Consider the Lyapunov function candidate $V(x_1,x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. Clearly V is positive definite and radially unbounded. Then the time derivative of V along the state trajectory is

$$\dot{V} = x_1^3\dot{x}_1 + x_2\dot{x}_2 = x_1^3x_2 - x_1^3x_2 - x_2^4 = -x_2^4$$

Since $\dot{V}$ is negative semidefinite, the origin is globally stable. Consider the state trajectories that stay identically in the set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid \dot{V} = 0\}$. Then

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$$

Since $(0,0)$ is the only trajectory in S, by LaSalle Theorem, the origin is globally asymptotically stable.

3. (a) Note that $\dot{x} < 0$ for $x > 0$, implying that $x(t)$ is decreasing towards 0 if $x > 0$. On the other hand, $\dot{x} > 0$ for $x < 0$, implying that $x(t)$ is increasing towards 0 if $x < 0$. Hence $x(t)$ eventually converges to zero. In other words, $x = 0$ is an asymptotically stable equilibrium point.

   Alternatively, we can linearize the system at $x = 0$. Since this is a scalar system, the Jacobian matrix is actually a scalar, which is

$$A = -2x \tan^{-1} x - 1 = -1, \quad \text{for } x = 0$$

   Hence $x = 0$ is asymptotically stable.

   (b) Let $v = \tan^{-1} x$. Then

$$\dot{v} = \frac{\dot{x}}{1+x^2} = -\tan^{-1} x + \frac{e^{-2t}}{1+x^2} \leq -v + e^{-2t}$$

   Consider the differential equation $\dot{u} = -u + e^{-2t}$, $u(0) = v(0) = \tan^{-1} x_0$. The solution is

$$u(t) = C_1 e^{-t} + C_2 e^{-2t} \Rightarrow \dot{u} = -C_1 e^{-t} - 2C_2 e^{-2t} = -(C_1 e^{-t} + C_2 e^{-2t}) + e^{-2t} \Rightarrow C_2 = -1$$

   Then

$$u(0) = C_1 + C_2 = C_1 - 1 = \tan^{-1} x_0 \Rightarrow C_1 = 1 + \tan^{-1} x_0$$

   By comparison lemma, we have

$$v(t) = \tan^{-1} x(t) \leq u(t) = (1 + \tan^{-1} x_0) e^{-t} - e^{-2t}$$

   Therefore,

$$x(t) \leq y(t) = \tan u(t) = \tan\left((1 + \tan^{-1} x_0) e^{-t} - e^{-2t}\right), \quad \forall t \geq 0$$

4. The Jacobian matrix is

$$A = \begin{bmatrix} \frac{\pi}{2} - 2x_1 + \frac{\sin x_2}{10}, & \frac{x_1 \cos x_2}{10} \\ \cos x_1 - x_1 \sin x_1, & -\frac{3}{10} x_2^2 \end{bmatrix}$$

   (a) It is easy to check that $(x_1, x_2) = (\frac{\pi}{2}, 0)$ is indeed an equilibrium point. Then the Jacobian matrix about $(\frac{\pi}{2}, 0)$ is

$$A_1 = \begin{bmatrix} -\frac{\pi}{2}, & \frac{\pi}{20} \\ -\frac{\pi}{2} & 0 \end{bmatrix}$$

   The characteristic polynomial of A_1 is $\lambda^2 + \frac{\pi}{2}\lambda + \frac{\pi^2}{40}$ and the eigenvalues of A_1 are -1.3938 and -0.177. Hence $(\frac{\pi}{2}, 0)$ is an asymptotically stable equilibrium point.

(b) Choose $r = 1$ and let $B_r = \{x \in \mathbb{R}^2 \big| \|x\|_2 \leq r\}$, where $x = [x_1, x_2]^T$. Define $V(x) = x_1 x_2$. Then there is always an x_0 in any neighborhood of $x = 0$ such that $V(x_0) > 0$. Let $U = \{x \in B_r \big| V(x) > 0\}$. Notice that for $x \in U$, we have $x_1 x_2 > 0$, $\|x\|_2 \leq 1$, and therefore $\cos x_1 > 0$. Thus the time derivative of V for $x \in U$ is

$$\begin{aligned}
\dot{V} &= \dot{x}_1 x_2 + x_1 \dot{x}_2 \\
&= x_1 x_2 \big(\frac{\pi}{2} - x_1\big) + \frac{x_1 x_2 \sin x_2}{10} + x_1^2 \cos x_1 - \frac{x_1 x_2^3}{10} \\
&= x_1 x_2 \left(\frac{\pi}{2} - x_1 + \frac{\sin x_2}{10} - \frac{x_2^2}{10}\right) + x_1^2 \cos x_1 \\
&\geq x_1 x_2 \left(\frac{\pi}{2} - 1 - \frac{1}{10} - \frac{1}{10}\right) + x_1^2 \cos x_1 \\
&> 0.3 x_1 x_2 + x_1^2 \cos x_1 > 0
\end{aligned}$$

By Chetaev's Theorem, $x = 0$ is an unstable equilibrium point.