Nonlinear System Theory
Homework 5

Due date: 4/7/2022

1. Consider the following system

$$\begin{aligned} \dot{x}_1 &= k_1(\sin t)x_2 - k_2 x_1^3 \\ \dot{x}_2 &= -k_1(\sin t)x_1 - k_2 x_2^3 \end{aligned}$$

where $k_1, k_2 > 0$.

   (a) Show that the linearized system around $x = 0$ is NOT exponentially stable.

   (b) Show that $x = 0$ is a uniformly asymptotically stable equilibrium point.

2. Consider the following system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 - a x_2^3 \end{aligned}$$

where $a > 0$.

   (a) Show that the solution $x(t)$ is ultimately bounded.

   (b) Find an estimate of the ultimate bound.

3. Consider the following non-autonomous nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2^2 \\ \dot{x}_2 &= \delta(t)x_1 - 2x_2 + x_1 x_2 + u(t) \end{aligned}$$

where $\delta(t)$ is continuous and $|\delta(t)| \leq k$ for some $k > 0$ and $\forall t \geq t_0$.

   (a) Let $u(t) \equiv 0$. Linearize the nonlinear system about $x = [x_1, x_2]^T = 0$, and show that the linearized system is exponentially stable by using a quadratic Lyapunov function.

   (b) Let $k = 1$ and $u(t) = \cos(t)$. Show that the nonlinear system is uniformly ultimately bounded.

   (c) Continued from part (b). Find the ultimate bound of the nonlinear system.