Nonlinear System Theory
Solution to Homework 2

1. Let $v = \frac{1}{2}x^2$. Then

$$\dot{v} = x\dot{x} = -x^2 + \frac{x^2}{1+x^2} \leq -2v + 1$$

Let $u(t)$ be the solution to the following differential equation

$$\dot{u} = -2u + 1, \qquad u(0) = 1$$

Then

$$u(t) = e^{-2t} + \int_0^t e^{-2(t-\tau)} d\tau = e^{-2t} + \frac{1}{2}\left(1 - e^{-2t}\right) = \frac{1}{2}\left(1 + e^{-2t}\right)$$

Therefore

$$|x(t)| = \sqrt{2v(t)} \leq \sqrt{2u(t)} \leq \sqrt{1 + e^{-2t}}$$

2. (a) Clearly $x = 0$ is an equilibrium point of the differential equation in consideration. Define the Lyapunov function candidate as $V(x) = \frac{1}{2}x^2$. $V(x)$ is positive definite, and

$$\dot{V} = x\dot{x} = -\alpha x^{\frac{r+1}{r}}$$

Since r is an odd integer, $x^{\frac{r+1}{r}} > 0$ for $x \neq 0$. Hence $\dot{V}$ is negative definite and $x = 0$ is an asymptotically stable equilibrium point, i.e. $x(t) \to 0$ as $t \to \infty$.

(b) By separating the variables and integrating from t_0 to t, we have

$$\begin{aligned}
& x^{-\frac{1}{r}} dx = -\alpha dt \\
\Rightarrow \quad & \int_{x_0}^{x(t)} z^{-\frac{1}{r}} dz = \int_{t_0}^{t} -\alpha d\tau \\
\Rightarrow \quad & \frac{r}{r-1}\left(x^{\frac{r-1}{r}}(t) - x_0^{\frac{r-1}{r}}\right) = -\alpha(t - t_0) \\
\Rightarrow \quad & x^{\frac{r-1}{r}}(t) = -\alpha\frac{r-1}{r}(t - t_0) + x_0^{\frac{r-1}{r}}
\end{aligned}$$

Since $r > 1$ is an odd integer, $x^{\frac{r-1}{r}}(t) \geq 0$, and decrease towards zero. For $t \geq t_0 + \frac{r}{\alpha(r-1)} x_0^{\frac{r-1}{r}}$, $x^{\frac{r-1}{r}}(t) = 0$, and so is $x(t)$.

3. Firstly, we can easily check that $(x_1, x_2) = (0, 0)$ is indeed an equilibrium point of the system. Let $V(x_1, x_2) = \frac{1}{2}x_1^2 + (1 - \cos x_2)$ be the Lyapunov function candidate. Clearly, $V(x_1, x_2)$ is positive definite. Then

$$\begin{aligned} \dot{V} &= -x_1^4 - x_1(\sin x_1 \cos x_2 - \sin x_2 \cos x_1) + \sin x_2(-x_1 \cos x_1 - \sin x_2) \\ &= -x_1^4 - x_1 \sin x_1 \cos x_2 - \sin^2 x_2 \end{aligned}$$

For $-\frac{\pi}{2} < x_1, x_2 < \frac{\pi}{2}$, we have $\cos x_2 > 0$ and $x_1 \sin x_1 \geq 0$. Therefore $x_1 \sin x_1 \cos x_2 \geq 0$ and $\dot{V} \leq -x_1^4 - \sin^2 x_2$. This shows that $\dot{V}$ is negative definite and thus $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

4. Since (A, B) is controllable and A is Hurwitz, the controllability grammian W is well-defined and positive definite. Therefore W^{-1} is also positive definite. Note that the controllability grammian satisfies the Lyapunov equation $AW + WA^T + BB^T = 0$.

   Consider the state-feedback system $\dot{x} = (A - BK)x = (A - BB^TW^{-1})x$ and given $V(x) = x^TW^{-1}x$. Then the time derivative of V is

$$\begin{aligned} \dot{V}(x) &= x^TW^{-1}(A - BB^TW^{-1})x + x^T(A - BB^TW^{-1})^TW^{-1}x \\ &= x^TW^{-1}\big(AW + WA^T - 2BB^T\big)W^{-1}x \\ &= -3x^TBB^Tx \leq 0 \end{aligned}$$

   Hence, by Lyapunov theorem, we conclude that $x = 0$ is a stable equilibrium point.

   Suppose that $x = 0$ is NOT asymptotically stable. This implies that at least one eigenvalue of $A - BK$ lies on the imaginary axis. Let λ be such an eigenvalue and $v \in \mathbb{C}^n$ be the corresponding left eigenvector, i.e. $v^*(A - BK) = \lambda v^*$ and $Re(\lambda) = 0$, where v^* denotes the conjugate transpose of v. Recall that (A, B) is controllable, so is $(A - BK, B)$ for any state-feedback gain K. In addition,

$$(A - BK)W + W(A - BK)^T = AW + WA^T - 2BB^T = -3BB^T$$

   Multiply both sides by v^* from the left and by v from the right, and we have

$$v^*(A - BK)Wv + v^*W(A - BK)^Tv = \lambda v^*Wv + \lambda^*v^*Wv = -3v^*BB^Tv$$

   Since $Re(\lambda) = 0$, we obtain

$$\lambda v^*Wv + \lambda^*v^*Wv = 2Re(\lambda)v^*Wv = 0 = -3v^*BB^Tv \Rightarrow v^*B = 0$$

   By PBH test, we conclude that $(A - BK, B)$ is uncontrollable, which is a contradiction. Therefore, $A - BK$ cannot have any eigenvalue on the imaginary axis. Consequently, $x = 0$ is asymptotically stable.