Nonlinear System Theory
Solution to Homework 6

1. (a) Let $V(x) = \frac{\alpha}{2}(x_1^2 + x_2^2)$, where $\alpha > 0$. Then the Hamilton-Jacobin inequality is

$$\begin{aligned}
\mathcal{H} &= \alpha \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -g_1 + x_2 \\ -x_1 - g_2 \end{bmatrix} + \frac{\alpha^2}{2\gamma^2} x_2^2 + \frac{1}{2} x_2^2 \\
&= -\alpha x_1 g_1 - \alpha x_2 g_2 + \frac{\alpha^2}{2\gamma^2} x_2^2 + \frac{1}{2} x_2^2 \\
&\le -\alpha k_1 x_1^2 - \left(\alpha k_2 - \frac{\alpha^2}{2\gamma^2} - \frac{1}{2} \right) x_2^2
\end{aligned}$$

Then the system is finite-gain $\mathcal{L}_2$ stable if $\alpha k_2 - \frac{\alpha^2}{2\gamma^2} - \frac{1}{2} \ge 0$. In other words, the $\mathcal{L}_2$ gain of the system is less than or equal to γ, and

$$\gamma \ge \frac{\alpha}{\sqrt{2\alpha k_2 - 1}} = A(\alpha)$$

If can be shown that $A(\alpha)$ has the minimum at $\alpha = \frac{1}{k_2}$ and $\min_{\alpha > 0} A(\alpha) = \frac{1}{k_2}$. Therefore, we choose $\alpha = \frac{1}{k_2}$, and then $\gamma \ge \frac{1}{k_2}$. Hence the $\mathcal{L}_2$ gain of the system is less than or equal to $\frac{1}{k_2}$.

(b) Let $V(x) = \frac{\alpha}{2}(x_1^2 + x_2^2)$, where $\alpha > 0$. Then the Hamilton-Jacobin inequality is

$$\begin{aligned}
\mathcal{H} &= \alpha \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -g_1 + x_2 \\ -x_1 - g_2 \end{bmatrix} + \frac{\alpha^2}{2\gamma^2} x_2^2 + \frac{1}{2} x_1^2 \\
&= -\alpha x_1 g_1 - \alpha x_2 g_2 + \frac{\alpha^2}{2\gamma^2} x_2^2 + \frac{1}{2} x_1^2 \\
&\le -\left(\alpha k_1 - \frac{1}{2} \right) x_1^2 - \alpha \left(k_2 - \frac{\alpha}{2\gamma^2} \right) x_2^2
\end{aligned}$$

Hence if $\alpha > \frac{1}{2k_1}$ and $k_2 > \frac{\alpha}{2\gamma^2}$, then $\mathcal{H} \le 0$, implying that the system is finite-gain $\mathcal{L}_2$ stable. The $\mathcal{L}_2$ gain is less than or equal to γ and

$$\gamma > \frac{\sqrt{\alpha}}{\sqrt{2k_2}} > \frac{1}{2\sqrt{k_1 k_2}}$$

Hence the $\mathcal{L}_2$ gain of the system is less than or equal to $\frac{1}{2\sqrt{k_1 k_2}}$.

2. Choose $x_1 = q$ and $x_2 = \dot{q}$ as the state of the robotic system. Then the state equation is

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}(x_1)\big(C(x_1, x_2)x_2 + G(x_1) + Bx_2\big) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} \tau$$

Choose the storage function as

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q) = \frac{1}{2}x_2^T M(x_1)x_2 + U(x_1)$$

Then V is positive semidefinite.

(a)

$$\begin{aligned}
\frac{\partial V}{\partial x_1} &= \frac{1}{2}\left[x_2^T\frac{\partial M(x_1)}{\partial x_{11}}x_2,\ x_2^T\frac{\partial M(x_1)}{\partial x_{12}}x_2\ \cdots x_2^T\frac{\partial M(x_1)}{\partial x_{1n}}x_2\right] + G^T(x_1)\\
\frac{\partial V}{\partial x_2} &= x_2^T M(x_1)
\end{aligned}$$

where $x_1 = [x_{11}, x_{12}, \cdots x_{1n}]^T$. Note that

$$\frac{\partial V}{\partial x_1}x_2 = x_2^T\sum_{i=1}^{n}\left[\frac{\partial M}{\partial x_{1i}}x_{2i}\right]x_2 + G^T(x_1)x_2 = \frac{1}{2}x_2^T\dot{M}(x_1)x_2 + G^T(x_1)x_2$$

Let γ be a positive constant. Substitute $\frac{\partial V}{\partial x} = [\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}]^T$ and the state equation into the Hamilton-Jacobi inequality. Then we have

$$\begin{aligned}
&\frac{1}{2}x_2^T\dot{M}(x_1)x_2 + G^T(x_1)x_2 - x_2^T\big(C(x_1,x_2)x_2 + G(x_1) + Bx_2\big) + \frac{1}{2\gamma^2}x_2^Tx_2 + \frac{1}{2}x_2^Tx_2\\
=\ & x_2^T\left[\frac{1}{2}\dot{M}(x_1) - C(x_1,x_2)\right]x_2 - x_2^TBx_2 + \frac{1}{2}\left(\frac{1}{\gamma^2}+1\right)x_2^Tx_2\\
\le\ & -\left[\lambda_{min}(B) - \frac{1}{2}\left(\frac{1}{\gamma^2}+1\right)\right]x_2^Tx_2
\end{aligned}$$

If $\lambda_{min}(B) - \frac{1}{2}\left(\frac{1}{\gamma^2}+1\right) \ge 0$, or equivalently, $\gamma^2 \ge \frac{1}{2\lambda_{min}(B)-1}$, then the Hamilton-Jacobi inequality holds. In other words, the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain is less than or equal to $\gamma \ge \frac{1}{\sqrt{2\lambda_{min}(B)-1}}$.

(b) Suppose that $B = 0$. Then

$$\begin{aligned}
\dot{V} &= \dot{q}^TM(q)\ddot{q} + \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^T\frac{\partial U(q)}{\partial q}\\
&= \dot{q}^T\big[\tau - C(q,\dot{q})\dot{q} - G(q)\big] + \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + \dot{q}^TG(q)\\
&= \dot{q}^T\tau + \dot{q}^T\left[\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right]\dot{q}\\
&= u^Ty
\end{aligned}$$

Hence the system is lossless.

(c) Suppose that B is positive definite. Then

$$\dot{V} = \tau^T\dot{q} - \dot{q}^TB\dot{q} \le \tau^T\dot{q} - \lambda_{min}(B)\dot{q}^T\dot{q}$$

where $\lambda_{min}(B)$ is the smallest eigenvalue of B, which is positive. Hence the system is output strictly passive.

3. We can redraw the feedback control system in Figure 1 below, where

$$H_1(s) = -C(s)G(s) = -\frac{2K}{(s+1)(s+2)(s+3)} = -K\left[\frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3}\right]$$

and

$$h_1(t) = \mathcal{L}^{-1}\{H(s)\} = -K\left(e^{-t} - 2e^{-2t} + e^{-3t}\right)$$

where $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transform and $h_1(t)$ is the impulse response of $H_1(s)$. Let e and u_1 denote the input and output of H_1, respectively. Then

$$\|u_{1\tau}\|_{\mathcal{L}_\infty} \le \|h_1\|_{\mathcal{L}_1}\|e_\tau\|_{\mathcal{L}_\infty}, \quad \forall e \text{ and } \forall \tau \in [0, \infty)$$

Namely, the $\mathcal{L}_\infty$ gain of H_1 is

$$\begin{aligned} \gamma_1 &= \|h_1\|_{\mathcal{L}_1} = K\int_0^\infty \left|e^{-t} - 2e^{-2t} + e^{-3t}\right| dt \\ &\le K\int_0^\infty \left(e^{-t} + 2e^{-2t} + e^{-3t}\right) dt \\ &= K\left(1 + 1 + \frac{1}{3}\right) = \frac{7}{3}K \end{aligned}$$

On the other hand, H_2 is finite-gain $\mathcal{L}_\infty$ stable since $|u(t)| \le |u_1(t)|$ for all $t \ge 0$. Hence the $\mathcal{L}_\infty$ gain of H_2 is $\gamma_2 = 1$.

By small gain theorem, the feedback system is finite-gain $\mathcal{L}_\infty$ stable if $\gamma_1\gamma_2 < 1$, or $K < \frac{3}{7}$.

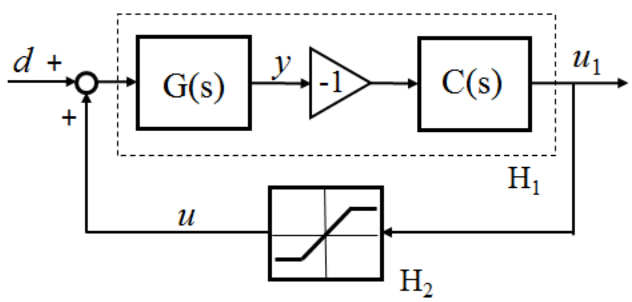

Figure 1: Feedback control system of Problem 3

4. Since the system is input strictly passive with $\psi(u) = \epsilon u$, we have

$$u^T y \ge \dot{V} + \epsilon u^T u$$

On the other hand, the system is finite-gain $\mathcal{L}_2$ stable with zero bias, we have

$$\int_{\tau_1}^{\tau_2} y^T(t)y(t)dt \le \gamma \int_{\tau_1}^{\tau_2} u^T(t)u(t)dt, \quad \forall \tau_2 > \tau_1 > 0$$

Then

$$\begin{aligned} V(x(\tau_2)) - V(x(\tau_1)) &\le \int_{\tau_1}^{\tau_2} \left[u^T(t)y(t) - \epsilon u^T(t)u(t)\right] dt \\ &= \int_{\tau_1}^{\tau_2} u^T(t)y(t)dt - \frac{\epsilon}{2}\int_{\tau_1}^{\tau_2} u^T(t)u(t)dt - \frac{\epsilon}{2}\int_{\tau_1}^{\tau_2} u^T(t)u(t)dt \\ &\le \int_{\tau_1}^{\tau_2} u^T(t)y(t)dt - \frac{\epsilon}{2}\int_{\tau_1}^{\tau_2} u^T(t)u(t)dt - \frac{\epsilon}{2\gamma}\int_{\tau_1}^{\tau_2} y^T(t)y(t)dt \end{aligned}$$

Since the foregoing inequality is valid for all $\tau_2 > \tau_1 \geq 0$, we have

$$\dot{V} \leq u^T y - \frac{\epsilon}{2} u^T u - \frac{\epsilon}{2\gamma} y^T y \;\Rightarrow\; u^T y \geq \dot{V} + \frac{\epsilon}{2} u^T u + \frac{\epsilon}{2\gamma} y^T y$$

Namely, $\epsilon_1 = \frac{\epsilon}{2}$ and $\delta_1 = \frac{\epsilon}{2\gamma}$.