Nonlinear System Theory
Homework 2

Due date: 3/21/2023

1. Find an upper bound on the solution of the following scalar equation
$$\dot{x} = -x + \frac{x}{1+x^2}, \qquad x(0) = 1$$

2. Consider the following scalar differential equation
$$\dot{x} = -\alpha \cdot \operatorname{sgn}(x)|x|^{\frac{1}{r}}, \quad x(t_0) = x_0$$
where $\alpha > 0$, and $r > 1$ is an odd positive integer. $\operatorname{sgn}(\cdot)$ is the signum function (i.e. the relay) defined in the lecture notes. For simplicity, $\operatorname{sgn}(x)|x|^{\frac{1}{r}}$ is denoted by $x^{\frac{1}{r}}$.
   (a) Show that $x(t) \to 0$ as $t \to \infty$.
   (b) Show that $x(t) = 0$ for all $t \geq t_r = t_0 + \frac{r}{\alpha(r-1)} x_0^{\frac{r-1}{r}}$, i.e. $x(t)$ vanishes in finite time.

3. Consider the following second-order system
$$\begin{aligned} \dot{x}_1 &= -x_1^3 - \sin(x_1 - x_2) \\ \dot{x}_2 &= -x_1 \cos x_1 - \sin x_2 \end{aligned}$$
Suppose that $-\frac{\pi}{2} < x_1, x_2 < \frac{\pi}{2}$. Show that $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

4. Consider the following system:
$$\begin{aligned} \dot{x}_1 &= x_2 - \frac{1}{2} x_2 \sin x_1 \\ \dot{x}_2 &= -x_1 - x_2 + \frac{1}{2} x_1 \sin x_1 \end{aligned}$$
   (a) Use the Lyapunov function candidate $V = \frac{1}{2}(x_1^2 + x_2^2)$ to show that $(x_1, x_2) = (0, 0)$ is a stable equilibrium point.
   (b) Use LaSalle theorem to show that $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

5. Consider the linear time-invariant system $\dot{x} = Ax + Bu$, where (A, B) is controllable and A is Hurwitz. Let $W = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ be the controllability grammian. Suppose that the state-feedback control $u = -Kx$ is applied, where $K = B^T W^{-1}$. Use $V(x) = x^T W^{-1} x$ as a Lyapunov function candidate to show that $x = 0$ is an asymptotically stable equilibrium point of the closed-loop system.

   Hint: Use $V(x)$ to show that $x = 0$ is stable. Then use the controllability condition to show that $x = 0$ is indeed asymptotically stable.