Nonlinear System Theory
Homework 3

Due date: 3/28/2023

1. Consider an n-joint robot which satisfies the following dynamic equation:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + B\dot{q} = \tau \tag{1}$$

where $q, \dot{q}, \ddot{q}, \tau \in \mathbb{R}^n$ denote the joint angle, joint velocity, joint acceleration, and joint torque, respectively. $M(q), C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ are the inertia matrix and the Coriolis and centrifugal matrix, respectively. $G(q) \in \mathbb{R}^n$ is the gravitational vector. $B = \mathrm{diag}\big(b_1, b_2, \cdots, b_n\big) \in \mathbb{R}^{n\times n}$, where $b_i > 0$, for $i = 1, 2, \cdots, n$, is the viscous friction coefficient. It is well-known in robotics that the following properties hold for all q and $\dot{q}$.

- $M(q)$ is symmetric positive definite for all $q \in \mathbb{R}^n$.
- $\frac{1}{2}\dot{M}(q) - C(q,\dot{q})$ is skew-symmetric, i.e. $x^T\left[\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right]x = 0$ for all $x \in \mathbb{R}^n$.

Suppose that the desired joint angle $q_d \in \mathbb{R}^n$ is a constant vector, i.e. we are considering the set-point control problem, and the following *PD control law with gravity compensation* is implemented:

$$\tau = K_P\tilde{q} + K_D\dot{\tilde{q}} + G(q) \tag{2}$$

where $\tilde{q} = q_d - q$; $K_P, K_D \in \mathbb{R}^{n\times n}$ are symmetric positive definite.

(a) Show that $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$ is the unique equilibrium point of the closed-loop system consisting of (1) and (2).

(b) Use $V(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_P\tilde{q}$ as the Lyapunov function candidate to show that the equilibrium point is stable.

(c) Use LaSalle Theorem to show that the equilibrium point is asymptotically stable.

2. Show that the origin of

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^3 - x_2^3\end{aligned}$$

is globally asymptotically stable.

3. Consider the following scalar differential equation

$$\dot{x} = -(1 + x^2)\tan^{-1} x + r(t), \quad x(0) = x_0$$

(a) Let $r(t) \equiv 0$. Check whether the equilibrium point $x = 0$ is (asymptotically) stable or unstable.

(b) Let $r(t) = e^{-2t}$. Find an upper bound $y(t)$ such that $x(t) \leq y(t)$ for all $t \geq 0$.

4. Consider the following nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= x_1\left(\frac{\pi}{2} - x_1\right) + \frac{x_1 \sin x_2}{10} \\
\dot{x}_2 &= x_1 \cos x_1 - \frac{x_2^3}{10}
\end{aligned}$$

(a) Show that $(x_1, x_2) = (\frac{\pi}{2}, 0)$ is an asymptotically stable equilibrium point.

(b) Can you conclude stability of the equilibrium point $(x_1, x_2) = (0, 0)$ based on the linearzied system? Why?

(c) Use Chetaev's theorem to show that $(x_1, x_2) = (0, 0)$ is an unstable equilibrium point.