Nonlinear System Theory
Homework 6

Due date: 5/9/2023

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 + u \\ y &= kx_2 + u \end{aligned}$$

   where $a > 0$, $k > 0$, $h \in [\alpha_1, \infty]$, and $\alpha_1 > 0$. Let $V(x) = k\int_0^{x_1} h(s)ds + x^T P x$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, $p_{11} = ap_{12}$, $p_{22} = \frac{k}{2}$, and $0 < p_{12} < \min\{2\alpha_1, \frac{ak}{2}\}$. Using $V(x)$ as a storage function, show that the system is strictly passive.

2. Show that the transfer function $G(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$ is strictly positive real if and only if all coefficients are positive and $b_1 < a_1 b_0$.

3. Show that the parallel connection of two passive (respectively, input strictly passive, output strictly passive, strictly passive) dynamical systems is passive (respectively, input strictly passive, output strictly passive, strictly passive). Note that for the case of output strictly passive parallel systems, we assume that there exist $\delta_i > 0$ such that $u_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i$ for $i = 1, 2$, where u_i, y_i, and V_i are the input, output, and storage function of the i-th system in the parallel connection.

4. Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - f(x_2) - h(x_3) + u \\ \dot{x}_3 &= -h(x_3) + x_2 \end{aligned}$$

   where $f, h : \mathbb{R} \to \mathbb{R}$, $x_2 f(x_2) \geq kx_2^2$, for some $k > 0$, $x_3 h(x_3) > 0$ for all $x_3 \neq 0$, and $f(0) = h(0) = 0$.

   (a) Let $u = 0$. Show that $(x_1, x_2, x_3) = (0, 0, 0)$ is asymptotically stable.

   (b) Let $y = x_2$ be the output of the system. Show that the system from input u to output y is finite-gain $\mathcal{L}_2$ stable with the $\mathcal{L}_2$-gain less than or equal to $\frac{1}{k}$.

   (c) Suppose that the 3rd state equation is replaced by

$$\dot{x}_3 = -(1 + \Delta(x_3))h(x_3) + x_2$$

   where $\Delta : \mathbb{R} \to \mathbb{R}$. Find a range $S \subset \mathbb{R}$ such that $(x_1, x_2, x_3) = (0, 0, 0)$ is asymptotically stable for $u = 0$ and for all $\Delta(x_3) \in S$.

   Hint: One way, but not the only way, to solve the problem is to decompose the system into two feedback connected (dynamic) subsystems. Then use passivity theorems to show asymptotic stability of the equilibrium point.