Nonlinear System Theory
Solution to Homework 8

1. (a) $G(s)$ has two unstable poles and its Nyquist plot is shown in Figure 1, which is a circle centered at $(-\frac{1}{2}, 0)$ with radius $\frac{1}{2}$. Note that $G(j\omega) \to 0$ as $\omega \to 0$ and $\omega \to \infty$. Hence for positive frequency $0 < \omega < \infty$, $G(j\omega)$ is a complete circle, starting from and ending at $(0, 0)$. Consider the whole frequency $\omega \in (-\infty, \infty)$, $G(j\omega)$ in Figure 1 actually consists of two circles. Because $G(s)$ is unstable, α must be positive, and $G(j\omega)$ is outside $D(\alpha, \beta)$ and encircles $D(\alpha, \beta)$ twice in the counterclockwise direction. As a result, $D(\alpha, \beta)$ must be contained in the circle of $G(j\omega)$. Choose $-\frac{1}{\alpha} = -\frac{1}{1+\varepsilon_1}$ and $-\frac{1}{\beta} = -\varepsilon_2$ for $0 < \varepsilon_1, \varepsilon_2 \ll 1$. Hence the system is absolutely stable for the sector $[1 + \varepsilon_1, \frac{1}{\varepsilon_2}]$.

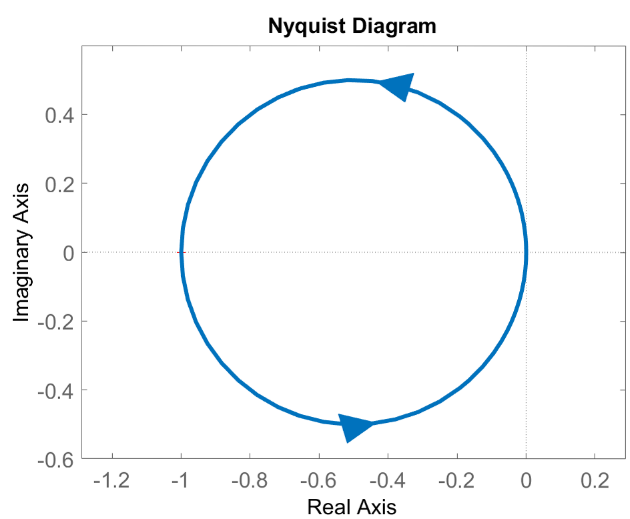

Figure 1: Nyquist Plot for Problem 1(a)

(b) $G(s)$ is stable and its Nyquist plot is shown in Figure 2. In this case, α can be positive, negative or zero. If $\alpha > 0$, the disk $D(\alpha, \beta)$ must be on the left-hand side of $G(j\omega)$ since $G(j\omega)$ cannot encircle $D(\alpha, \beta)$. Thus β is a finite value. To extend the sector, we can consider $\alpha = 0$, and $G(j\omega)$ is on the right-hand side of the vertical line passing through $(-0.35, 0)$ (see Figure 2); hence $\beta = \frac{1}{0.35} = 2.857$ and the sector is $[0,\ 2.857]$.
On the other hand, if $\alpha < 0$, then $G(j\omega)$ should be contained inside $D(\alpha, \beta)$. Consider that $D(\alpha, \beta)$ is centered at $(0.35, 0)$ with radius 1.07 (see Figure 2); then $\alpha = \frac{-1}{0.35+1.07} = -0.704$ and $\beta = \frac{-1}{0.35-1.07} = 1.389$. Hence the sector is $[-0.704,\ 1.389]$.

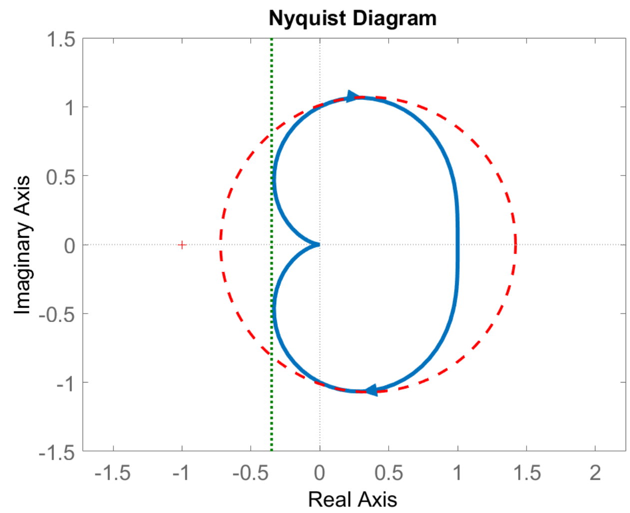

Figure 2: Nyquist Plot for Problem 1(b)

(c) $G(s)$ has an unstable pole at $s = 1$ and its Nyquist plot is shown in Figure 3. Therefore, α must be positive, and $G(j\omega)$ should be outside the disk $D(\alpha, \beta)$ and encircles $D(\alpha, \beta)$ once in the counterclockwise direction. As a result, $D(\alpha, \beta)$ should be inside the closed contour of $G(j\omega)$. Let $D(\alpha, \beta)$ be centered at $(-0.16, 0)$ with radius 0.09 (see Figure 3). So, $\alpha = \frac{-1}{-0.16-0.09} = 4$ and $\beta = \frac{-1}{-0.16+0.09} = 14.29$. In other words, the sector is $[4,\ 14.29]$.

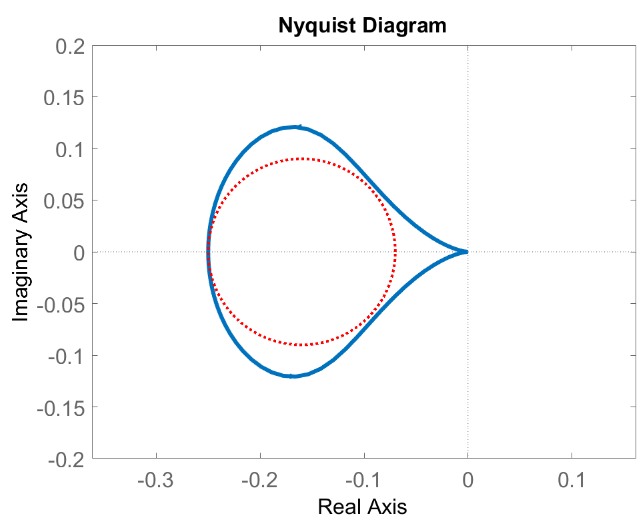

Figure 3: Nyquist Plot for Problem 1(c)

2. (a) Since $G(s)$ is unstable, loop transformation is applied as shown in Figure 4, where $\alpha > 0$ is a constant. Then

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{s}{s^2 + (\alpha - 1)s + 1}$$

Choose $\alpha = 1.1$ and $\tilde{G}(s)$ is stable. The Popov plot of $\tilde{G}(s)$ is shown in Figure 5. To find the maximum sector, we choose the a vertical line (i.e.

$\gamma = 0$) which intersects the horizontal axis at a point arbitrarily close to zero. In other words, $k > 0$ and k can be arbitrarily large. Note that for $\gamma = 0$, $1 + \lambda_i \gamma = 1 \neq 0$, where λ_i is any pole of $\tilde{G}(s)$. Moreover,

$$Z(j\omega) = \frac{1}{k} + \tilde{G}(j\omega) \to \frac{1}{k} > 0, \quad \text{as } \omega \to \infty$$

Consequently, $Z(s)$ is SPR if the Popov plot is to the right of the vertical line shown in Figure 5. This implies that if $\tilde{\psi} \in [0, k)$, where $k > 0$ can be arbitrarily large, or $\psi \in [\alpha, \infty) = [1.1, k)$, then the feedback system is absolutely stable.

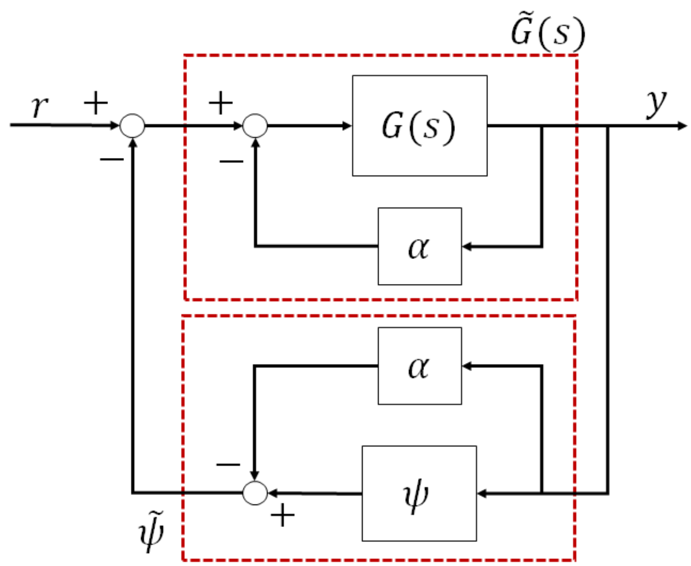

Figure 4: Loop Transformation for Problem 2(a)

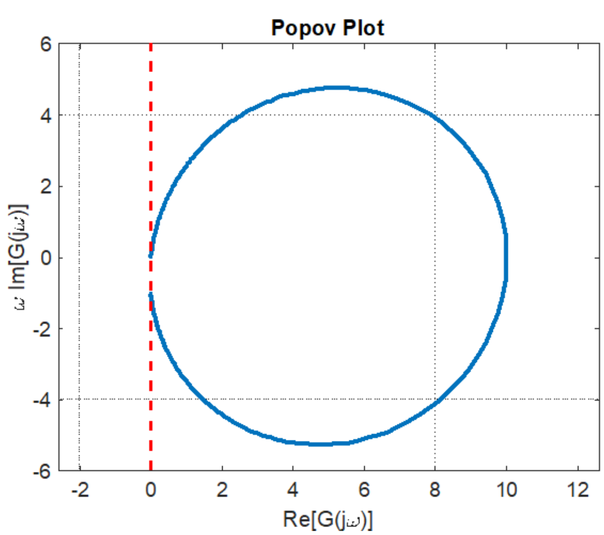

Figure 5: Popov plot for Problem 2(a)

(b) The Popov plot of $G(s)$ is shown in Figure 6. Choose $\gamma = 1$ and the intersection with the horizontal axis can be arbitrarily close to 0, implying that $k > 0$ can be arbitrarily large. Since the poles of $G(s)$ are $\lambda_{1,2} = -0.5 \pm 0.866j$, we have $1 + \lambda_i \gamma \neq 0$ for $i = 1, 2$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)G(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 6. Hence the feedback system is absolutely stable if $\psi \in [0, k)$, where $k > 0$ can be arbitrarily large.

However, we can further extend the lower bound of the sector to a negative value. Consider again the loop transformation in Figure 4. In this case, we have

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{1}{s^2 + s + (\alpha + 1)}$$

Choose $\alpha = -0.9$; then $\tilde{G}(s)$ is stable and its Popov plot is shown in Figure 7. Choose $\gamma = 1$ and the intersection with the horizontal axis can be arbitrarily close to 0, implying that $k > 0$ can be arbitrarily large. The poles of $\tilde{G}(s)$ are $\lambda_{1,2} = -0.887, -0.113$; therefore $1 + \lambda_i \gamma \neq 0$ for $i = 1, 2$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)\tilde{G}(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 7. Hence if $\tilde{\psi} \in [0, k)$, where $k > 0$ is arbitrarily large, or $\psi \in [\alpha, k) = [-0.9, k)$, the feedback system is absolutely stable.

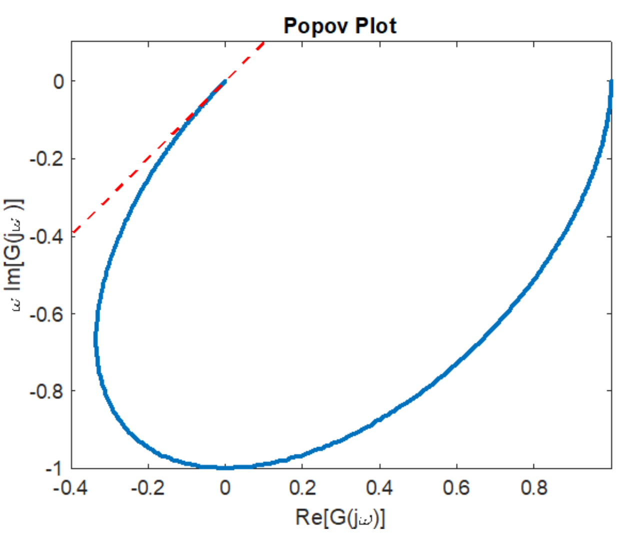

Figure 6: Popov plot for Problem 2(b) – without Loop Transformation

(c) Because $G(s)$ is unstable, we need to do loop transformation as shown in Figure 4. Then

$$\tilde{G}(s) = \frac{G(s)}{1 + \alpha G(s)} = \frac{s + 1}{s^3 + 3s^2 + \alpha s + (\alpha - 4)}$$

By Routh's table, we can see that if $\alpha > 4$, then $\tilde{G}(s)$ is stable. Choose $\alpha = 4.1$ and the Popov plot is shown in Figure 8. Choose $\gamma = 0.5$ and

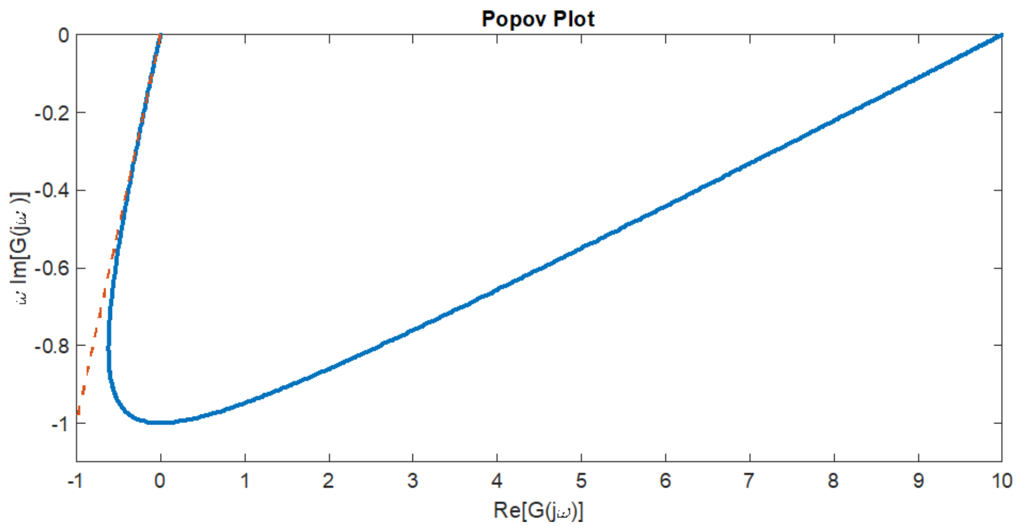

Figure 7: Popov plot for Problem 2(b) – with Loop Transformation

$k > 0$ arbitrarily large. Note that the poles of $\tilde{G}(s)$ are $\lambda_{1,2,3} = -1.488 \pm 1.347j, -0.024$; hence $1 + \lambda_i \gamma \neq 0$ for $i = 1, 2, 3$. Moreover,

$$Z(j\omega) = \frac{1}{k} + (1 + j\omega\gamma)\tilde{G}(j\omega) \to \frac{1}{k}, \quad \text{as } \omega \to \infty$$

Thus $Z(s)$ is SPR if the Popov plot is to the right of the line shown in Figure 8. Hence if $\tilde{\psi} \in [0, k)$, where $k > 0$ is arbitrarily large, or $\psi \in [\alpha, k) = [4.1, k)$, the feedback system is absolutely stable.

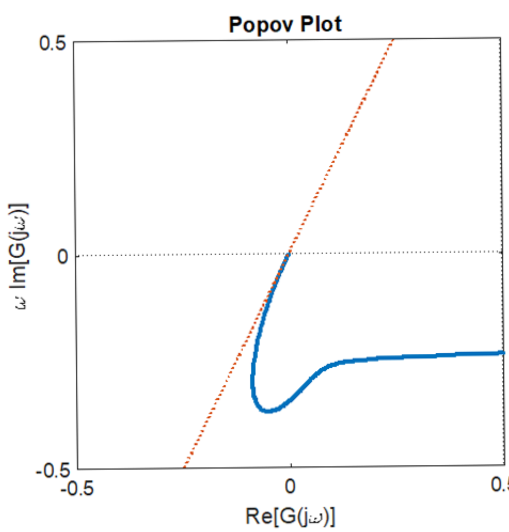

Figure 8: Popov plot for Problem 2(c)

3. (a) Let $V = \frac{1}{2}\left(k_1 e_1^2 + e_2^2 + \frac{1}{\gamma}\tilde{a}^2\right)$ be a Lyapunov function candidate. V is positive definite and radially unbounded. Then

$$\dot{V} = k_1 e_1 e_2 + e_2\left(-\tilde{a}\phi - k_1 e_1 - k_2 e_2\right) + \tilde{a} e_2 \phi = -k_2 e_2^2$$

Since $\dot{V}$ is negative semidefinite, the equilibrium point is globally stable.

(b) Since the equilibrium point is stable, the state variables, $e_1, e_2, \tilde{a}$ are bounded. Then V is bounded and from the state equation we see that $\dot{e}_2$ is also bounded. As a result, $\ddot{V} = -2k_2 e_2 \dot{e}_2$ is bounded. This implies that $\dot{V}$ is uniformly continuous. Hence by Barbalat's lemma, we conclude that $\dot{V} \to 0$ as $t \to \infty$, and $e_2 \to 0$ as $t \to \infty$.