Nonlinear System Theory
Final

6/6/2023

1. (40%)Consider the following system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + h(x_2) - u \\ \dot{x}_2 &= -x_2 - h(x_2) + u\end{aligned}$$

where h is a continuously differentiable function, $h(0) = 0$, and $x_2 h(x_2) > 0$, $\forall x_2 \neq 0$.

   (a) (8%)Let $u \equiv 0$. Show that $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

   Hint: Notice the sign of $h'(0)$, where h' is the derivative of h with respect to x_2.

   (b) (8%)Define $y = -x_1 + h(x_2)$ as the output of the system. Show that the system is strictly passive from input u to output y.

   (c) (8%)Continued from part (b). Show that the system is finite-gain $\mathcal{L}_2$ stable from input u to output y. Furthermore, the $\mathcal{L}_2$ gain is less than or equal to 2.

   (d) (8%)Redefine the output as $\tilde{y} = x_1$. Let $V(x_1, x_2) = \frac{\alpha}{2}x_1^2 + \beta\int_0^{x_2} h(\tau)d\tau$, where α and β can be any arbitrary positive constants. Use $V(x_1, x_2)$ and the Hamilton-Jacobi inequality to show that the system is finite-gain $\mathcal{L}_2$ stable from input u to output $\tilde{y}$. Furthermore, the $\mathcal{L}_2$ gain is less than or equal to $\frac{1}{2}$.

   (e) (8%)Continued from part (d). Consider the feedback control law $u = r - w$, where r is an external reference input, and $W(s) = C(s)\tilde{Y}(s)$. $W(s)$ and $\tilde{Y}(s)$ are the Laplace transforms of $w(t)$ and $\tilde{y}(t)$, respectively. $C(s)$ is a lead controller of the form $C(s) = k\frac{1+aTs}{1+Ts}$, where $a > 1$, $k > 0$, and $T > 0$. Show that if $ka < 2$, then the feedback system from r to $\tilde{y}$ is finite-gain $\mathcal{L}_2$ stable.

2. (14%)Consider the feedback connected system in Figure 1(a), where

$$G(s) = \frac{10(s^2 - 6s + 10)}{(s - 0.5)(s + 4)(s + 8)}$$

   (a) (8%)The Nyquist plot of $G(s)$ is shown in Figure 1(b). Suppose that ψ belongs to the sector (α, β), where $\beta > \alpha$. Find the maximum possible α and β such that the feedback system is absolutely stable.

   Note: Roughly estimating the values of α and β from the Nyquist plot will be fine.

   (b) (6%)Let $\psi(y) = ky$, where $k = \frac{\alpha+\beta}{2}$ for α and β in part (a). Denote the feedback connected system in Figure 1(a) as $\tilde{G}(s)$. Suppose that $\tilde{G}(s)$ and a nonlinear static function ϕ are connected in a feedback loop as shown in Figure 1(c). Find the sector to which ϕ belongs (in terms of α and β) such that the feedback system in Figure 1(c) is absolutely stable.

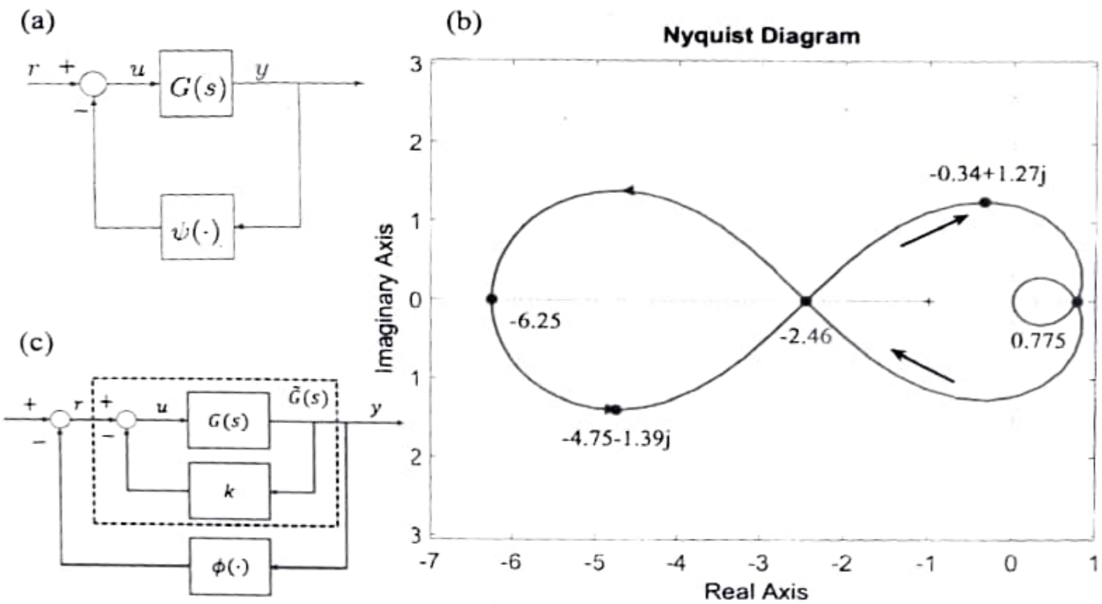

Figure 1: (a)Feedback connected system in Problem 2(a) (b)Nyquist plot of $G(s)$ (c)Feedback connected system in Problem 2(b)

3. (16%)Consider the following system

$$
\begin{aligned}
\dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e}(t) + \mathbf{b}\phi^T(t)\theta(t) \\
\dot{\theta}(t) &= -\mathbf{\Gamma}\phi(t)\mathbf{b}^T\mathbf{P}\mathbf{e}(t)
\end{aligned}
$$

where $\mathbf{e}(t) \in \mathbb{R}^n$ and $\theta(t) \in \mathbb{R}^m$ are states of the system. $\phi(t) \in \mathbb{R}^m$ is a known vector of bounded functions. $\mathbf{A}, \mathbf{P} \in \mathbb{R}^{n \times n}$, $\mathbf{\Gamma} \in \mathbb{R}^{m \times m}$, and $\mathbf{b} \in \mathbb{R}^{n \times 1}$ are constant matrices and vectors, where $\mathbf{A}$ is Hurwitz, $\mathbf{P}, \mathbf{\Gamma}$ are symmetric positive definite and $\mathbf{P}$ satisfies the Lyapunov equation: $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$ for some positive definite matrix $\mathbf{Q}$.

(a) (8%)Show that $(\mathbf{e}, \theta) = (\mathbf{0}, \mathbf{0})$ is a stable equilibrium point.

(b) (8%)Show that $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$.

4. (30%)Consider the following system

$$
\begin{aligned}
\dot{x}_1 &= x_2 + u \\
\dot{x}_2 &= -x_2 - \sin x_1 + x_3 \\
\dot{x}_3 &= x_2^2 - x_3 - 2u \\
y &= x_2
\end{aligned}
$$

Let $x = [x_1, x_2, x_3]^T$. The above state equation can be expressed as $\dot{x} = f(x) + g(x)u$ for some vector-valued functions $f(x)$ and $g(x)$.

(a) (8%)What is the relative degree of this system?

(b) (8%)Is the system minimum phase?

(c) (6%)Find state variables of the normal form in terms of x. Note: expression of the normal form is NOT required.

(d) (8%)Is span$\{g, ad_f g(x)\}$ involutive for all $0 < |x_1| < \pi$?