Nonlinear System Theory
Solution to Midterm

1. Define $v(t) = \ln(1 + x(t))$ and $v(0) = \ln(1 + x(0)) = 0$. Then

$$\dot{v} = \frac{\dot{x}}{1+x} = \cos^2\big(\ln(1+x)\big) - \frac{x^2}{(1+x)^2} \le \cos^2 v, \quad v(0) = 0$$

   Let us solve the differential equation:

$$\frac{du}{dt} = \cos^2 u, \quad u(0) = 0 \Rightarrow \int_0^u \sec^2 \tau d\tau = \int_0^t dt \Rightarrow \tan u = t \Rightarrow u(t) = \tan^{-1} t, \quad t \ge 0$$

   By comparison lemma,

$$v(t) \le u(t) = \tan^{-1} t, \quad t \ge 0$$

   and

$$x(t) = e^{v(t)} - 1 \le e^{\tan^{-1} t} - 1, \quad t \ge 0$$

2. (a) Consider the case that $|x_2| > \epsilon$. At the equilibrium point, we have $x_2 = \frac{1}{2}x_1 - g(t)$, and

$$0 = -x_2^2 - 2x_1x_2 + 1 - 2g(t)x_1 = -(\frac{1}{4}x_1^2 - g(t)x_1 + g^2(t)) - (x_1^2 - 2g(t)x_1) + 1 - 2g(t)x_1$$

   Hence the equilibrium point should satisfy

$$\frac{5}{4}x_1^2 - g(t)x_1 + (g^2(t) - 1) = 0 \tag{1}$$

   The discriminant of the parabolic equation (1) is

$$\Delta = g^2(t) - 5(g^2(t) - 1) = -4g^2(t) + 5 = -16\sin^2(10t) + 5$$

   Clearly $\Delta < 0$ for some t, implying that (1) has no real solution for some time. Therefore, we cannot find an equilibrium point that satisfies (1) for all $t \ge 0$. Similar argument applies to the case that $|x_2| \le \epsilon$. Hence we conclude that the system has no equilibrium point.

   (b) Let $L = x_1^2 + x_2^2 - 1$. Then for either case ($|x_2| > \epsilon$ or $|x_2| \le \epsilon$), we have

$$\dot{L} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -(x_1^2 + x_2^2 - 1) = -L$$

   Therefore $L \to 0$ as $t \to \infty$. This implies that the state trajectory converges to the circle $L = 0$ from any initial state; Therefore, $L = 0$ is the stable limit cycle of the system.

3. (a) The time derivative of the Lyapunov function candidate is

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 - \frac{1}{2}x_1x_2\sin x_1 - x_1x_2 - x_2^2 + \frac{1}{2}x_1x_2\sin x_1 = -x_2^2$$

Since V is positive definite and $\dot{V}$ is negative semidefinite, $(x_1, x_2) = (0, 0)$ is a stable equilibrium point.

(b) Let $S = \{x \in \mathbb{R}^2 | \dot{V} = 0\} = \{x \in \mathbb{R}^2 | x_2 = 0\}$. If $x_2 \equiv 0$, then $\dot{x}_2 \equiv 0$, and

$$0 = -x_1 - x_2 + \frac{1}{2}x_1\sin x_1 = -x_1\big(1 - \frac{1}{2}\sin x_1\big) \Rightarrow x_1 = 0$$

Hence $(x_1, x_2) \equiv (0, 0)$ is the only trajectory that can stay in S. By LaSalle theorem, $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

4. (a) Let $f(x_1) = ax_1^2 - (1 - \cos x_1)$ for $|x_1| \le \frac{\pi}{4}$, where $a > 0$. Note that $f(x_1)$ is an even function, i.e. $f(-x_1) = f(x_1)$, and $f(0) = 0$. We want to find the coefficient a such that either $f(x_1) \ge 0$ or $f(x_1) \le 0$ for all $|x_1| \le \frac{\pi}{4}$.
Notice that $f'(x_1) = 2ax_1 - \sin x_1$. If $f'(x_1) \ge 0$ for all $0 \le x_1 \le \frac{\pi}{4}$, then $f(x_1) \ge 0$ for all $|x_1| \le \frac{\pi}{4}$. Similarly, if $f'(x_1) \le 0$ for all $0 \le x_1 \le \frac{\pi}{4}$, then $f(x_1) \le 0$ for all $|x_1| \le \frac{\pi}{4}$.
For $0 \le x_1 \le \frac{\pi}{4}$, $\sin x_1$ is bounded by two straight lines, i.e. $mx_1 \le \sin x_1 \le x_1$, where $m = \frac{1}{\sqrt{2}\frac{\pi}{4}} \approx 0.9$. Hence if $a = \frac{1}{2}$, then $f(x_1) \ge 0$ for all $|x_1| \le \frac{\pi}{4}$, and if $a = 0.45$, then $f(x_1) \le 0$ for all $|x_1| \le \frac{\pi}{4}$.
Based on the previous observation, we have

$$0.45x_1^2 \le 1 - \cos x_1 \le 0.5x_1^2, \quad \forall |x_1| < \frac{\pi}{4}$$

Hence

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \le \frac{1}{2}\|x\|^2 = \alpha_2(\|x\|)$$

On the other hand,

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \ge 0.45x_1^2 + \frac{1}{2}x_2^2 \ge 0.45\|x\|^2 = \alpha_1(\|x\|)$$

We also note that $\alpha_2^{-1}(y) = \sqrt{2y}$ and $\alpha_1^{-1}(y) = \sqrt{\frac{y}{0.45}} \approx 1.49\sqrt{y}$.

(b) The time derivative of V is

$$\begin{aligned}\dot{V} &= \sin x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1\sin x_1 + x_2\sin x_1 - x_2\sin x_1 - x_2^2 + g(t)x_2 \\ &\le -x_1\sin x_1 - x_2^2 + k|x_2|\end{aligned}$$

From part (a) we see that $mx_1 \le \sin x_1 \le x_1$ for $0 \le x_1 \le \frac{\pi}{4}$ and $m \approx 0.9$. Hence $mx_1^2 \le x_1\sin x_1 \le x_1^2$ for $|x_1| < \frac{\pi}{4}$. Thus

$$\dot{V} \le -mx_1^2 - x_2^2 + k|x_2| \le -m\|x\|^2 + k\|x\| = -m(1-\theta)\|x\|^2 - (m\theta\|x\| - k)\|x\|$$

where $0 < \theta < 1$. If $\|x\| > \frac{k}{m\theta} = \mu$, then $\dot{V} \le -m(1-\theta)\|x\|^2$, i.e. $\dot{V}$ is negative definite for $\|x\| > \mu$. Since μ should satisfy $\mu < \alpha_2^{-1}(\alpha_1(\frac{\pi}{4})) = \sqrt{0.9}\frac{\pi}{4} = 0.745$, we choose $\theta = 0.9$, and then $\mu = \frac{\frac{\pi}{6}}{m\theta} = 0.6464$.
Thus, the ultimate bound is

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = 1.49\sqrt{0.5}\mu = 0.681$$

5. (a) The equilibrium point satisfies

$$\begin{aligned} x_2 &= \sin x_1 \\ x_2^2 &= \frac{3}{2}(1-\cos x_1) \Rightarrow \sin^2 x_1 = 1-\cos^2 x_1 = \frac{3}{2} - \frac{3}{2}\cos x_1 \end{aligned}$$

Thus

$$0 = \cos^2 x_1 - \frac{3}{2}\cos x_1 + \frac{1}{2} = \big(\cos x_1 - \frac{1}{2}\big)\big(\cos x_1 - 1\big) \Rightarrow x_1 = \pm\frac{\pi}{3} \text{ or } 0$$

The equilibrium points are

$$(x_1, x_2) = (0,0),\ (\frac{\pi}{3}, \frac{\sqrt{3}}{2}),\ (-\frac{\pi}{3}, -\frac{\sqrt{3}}{2})$$

(b) Let $S_1 = x_2 - \sin x_1$ and $S_2 = x_2 - \sqrt{\frac{3}{2}(1-\cos x_1)}$. For $(x_1, x_2) \in U_1$, we have $S_1 > 0$ and $S_2 > 0$, or equivalently, $x_2^2 > \frac{3}{2}(1-\cos x_1)$. Then

$$\dot{S}_1 = (\sin x_1 - x_2) - \cos x_1 \Big(-\frac{3}{2}(1-\cos x_1) + x_2^2 \Big) < 0$$

This implies that the trajectory will moves towards the boundary between U_1 and U_3, i.e. the solid curve $S_1 = 0$.

On the other hand, if $(x_1, x_2) \in U_2$, then $S_1 < 0$ and $S_2 < 0$. Hence

$$\dot{S}_2 = (\sin x_1 - x_2) - \frac{1}{2} \frac{\frac{3}{2}\sin x_1(-\frac{3}{2}(1-\cos x_1) + x_2^2)}{\sqrt{\frac{3}{2}(1-\cos x_1)}} > 0$$

This implies that the trajectory will moves towards the boundary between U_2 and U_3, i.e. the dotted curve $S_2 = 0$.

If $(x_1, x_2) \in U_3$, then $S_1 < 0$ and $S_2 > 0$, implying that $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$. In other words, x_1 and x_2 increase simultaneously and therefore the trajectory moves away from the origin.

(c) The Jacobain matrix of the system is

$$A = \begin{bmatrix} -\frac{3}{2}\sin x_1 & 2x_2 \\ \cos x_1 & -1 \end{bmatrix}$$

For $(x_1, x_2) = (0,0)$, $A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, and the eigenvalues of A are 0 and -1. There is no conclusion about stability of the equilibrium point based on the linearized system. However, from part (b) we see that any trajectory starting from the first quadrant will enter U_3 and leave away from the origin; hence $(0,0)$ is an unstable equilibrium point.

For $(x_1, x_2) = (\frac{\pi}{3}, \frac{\sqrt{3}}{2})$, $A = \begin{bmatrix} -1.299 & 1.7321 \\ 0.5 & -1 \end{bmatrix}$, and the eigenvalues of A are -2.0921 and -0.207. Hence the equilibrium point is asymptotically stable.

For $(x_1, x_2) = (-\frac{\pi}{3}, -\frac{\sqrt{3}}{2})$, $A = \begin{bmatrix} 1.299 & -1.7321 \\ 0.5 & -1 \end{bmatrix}$, and the eigenvalues of A are 0.8243 and -0.5253. Hence the equilibrium point is unstable.