Nonlinear System Theory
Solution to Homework 8

1. (a) $G(s)$ has two unstable poles and its Nyquist plot is shown in Figure 1, which is a circle centered at $(-\frac{1}{2}, 0)$ with radius $\frac{1}{2}$. Note that $G(j\omega) \to 0$ as $\omega \to 0$ and $\omega \to \infty$. Hence for positive frequency $0 < \omega < \infty$, $G(j\omega)$ is a complete circle, starting from and ending at $(0, 0)$. Consider the whole frequency $\omega \in (-\infty, \infty)$, $G(j\omega)$ in Figure 1 actually consists of two circles.

   Because $G(s)$ is unstable, α must be positive, and $G(j\omega)$ is outside $D(\alpha, \beta)$ and encircles $D(\alpha, \beta)$ twice in the counterclockwise direction. As a result, $D(\alpha, \beta)$ must be contained in the circle of $G(j\omega)$. Choose $-\frac{1}{\alpha} = -\frac{1}{1+\varepsilon_1}$ and $-\frac{1}{\beta} = -\varepsilon_2$ for $0 < \varepsilon_1, \varepsilon_2 \ll 1$. Hence the system is absolutely stable for the sector $[1 + \varepsilon_1, \frac{1}{\varepsilon_2}]$.

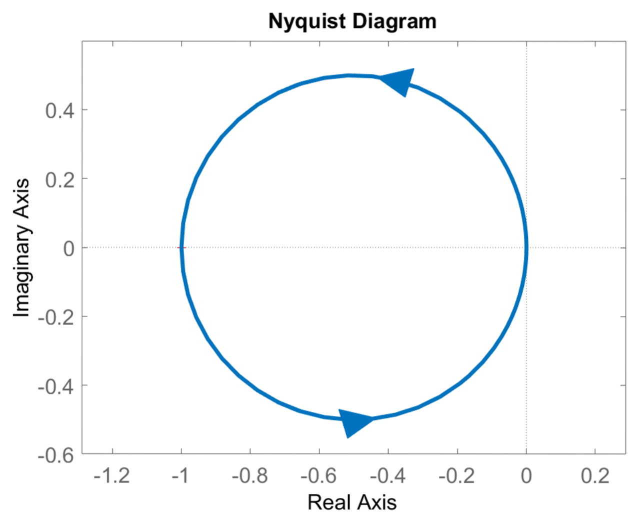

Figure 1: Nyquist Plot for Problem 1(a)

   (b) $G(s)$ is stable and its Nyquist plot is shown in Figure 2. In this case, α can be positive, negative or zero. If $\alpha > 0$, the disk $D(\alpha, \beta)$ must be on the left-hand side of $G(j\omega)$ since $G(j\omega)$ cannot encircle $D(\alpha, \beta)$. Thus β is a finite value. To extend the sector, we can consider $\alpha = 0$, and $G(j\omega)$ is on the right-hand side of the vertical line passing through $(-0.35, 0)$ (see Figure 2); hence $\beta = \frac{1}{0.35} = 2.857$ and the sector is $[0,\ 2.857]$.

   On the other hand, if $\alpha < 0$, then $G(j\omega)$ should be contained inside $D(\alpha, \beta)$. Consider that $D(\alpha, \beta)$ is centered at $(0.35, 0)$ with radius 1.07 (see Figure 2); then $\alpha = \frac{-1}{0.35+1.07} = -0.704$ and $\beta = \frac{-1}{0.35-1.07} = 1.389$. Hence the sector is $[-0.704,\ 1.389]$.

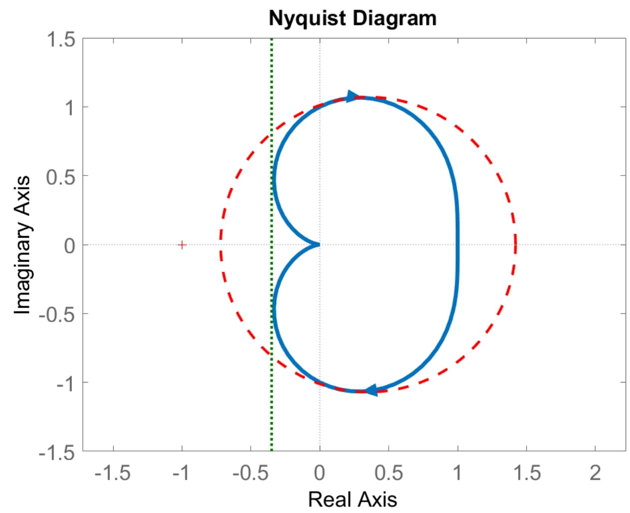

Figure 2: Nyquist Plot for Problem 1(b)

(c) $G(s)$ has an unstable pole at $s = 1$ and its Nyquist plot is shown in Figure 3. Therefore, α must be positive, and $G(j\omega)$ should be outside the disk $D(\alpha, \beta)$ and encircles $D(\alpha, \beta)$ once in the counterclockwise direction. As a result, $D(\alpha, \beta)$ should be inside the closed contour of $G(j\omega)$. Let $D(\alpha, \beta)$ be centered at $(-0.16, 0)$ with radius 0.09 (see Figure 3). So, $\alpha = \frac{-1}{-0.16-0.09} = 4$ and $\beta = \frac{-1}{-0.16+0.09} = 14.29$. In other words, the sector is $[4,\ 14.29]$.

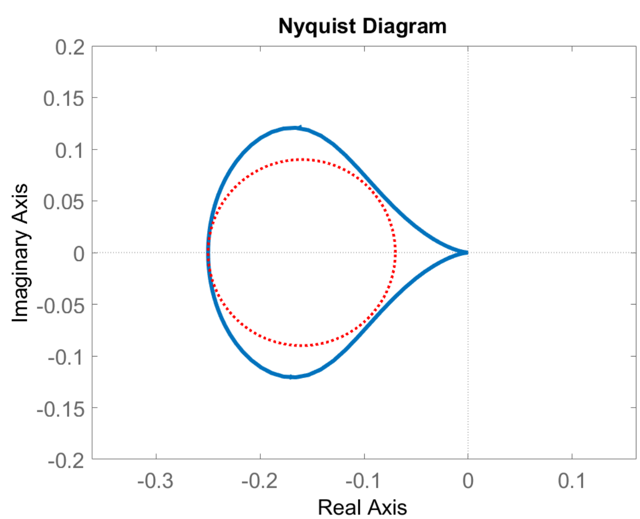

Figure 3: Nyquist Plot for Problem 1(c)

2. (a) $G(s)$ is stable and its Nyquist plot is shown in Figure 4. Since $G(j\omega)$ is on the right-hand side of the vertical line passing through $(-1, 0)$, the feedback system is absolutely stable for the sector $[0,\ 1]$.

   (b) Note that $\psi(y) = \text{sat}(y) \in [0,\ 1]$. Because the feedback system is absolutely stable for the sector $[0,\ 1]$, $x = 0$ is globally asymptotically stable. In other words, starting from any initial state $x(0) \in \mathbb{R}^2$, the state trajectory converges to 0 as $t \to \infty$. Therefore, there cannot exist periodic solutions, and cannot have limit cycles.

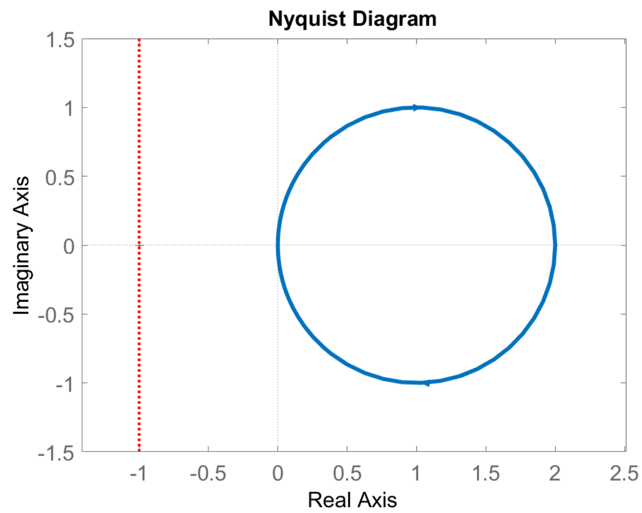

Figure 4: Nyquist Plot for Problem 2

3. (a) Since $G(s)$ is Hurwitz, the lower bound of the sector of ψ can be positive, negative, or zero. The Nyquist plot of $G(s)$ is shown in Figure 5. Let α and β be the lower and upper bounds of the sector of ψ, respectively. To find the maximum possible range of the sector, we first consider $\alpha = 0$. By the circle criterion, we choose the vertical line passing through the origin. Hence $\frac{1}{\beta} = 0$, i.e. $\psi \in [0, \infty)$.

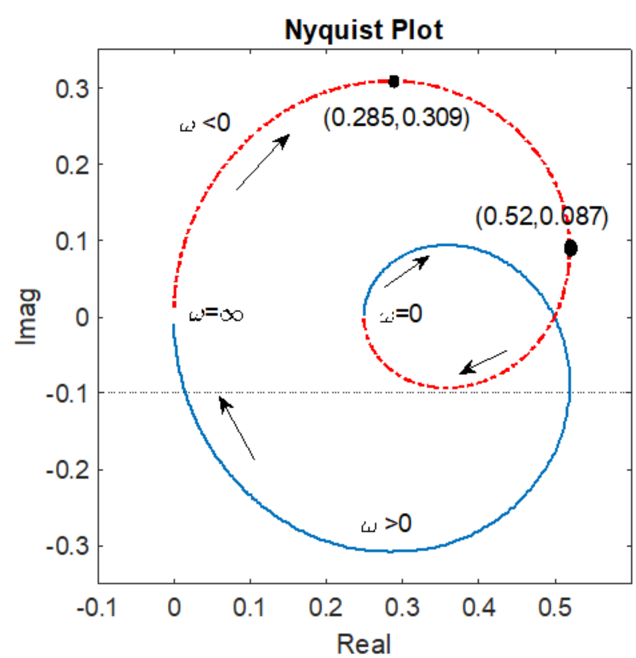

Figure 5: Nyquist Plot of $G(s)$ in Problem 3

On the other hand, let us consider the case that $\alpha < 0$. Then the Nyquist plot of $G(s)$ should be contained in the disk $D(\alpha, \beta)$. Let $D(\alpha, \beta)$ be centered at $(0.25, 0)$ with radius $r = 0.325$. The result is shown in Figure 6. The intersections of $D(\alpha, \beta)$ with the real axis are -0.075 and 0.575. Hence $\alpha = \frac{-1}{0.575} = -1.739$ and $\beta = \frac{1}{0.075} = 13.33$. Namely, $\psi \in [-1.739, 13.33]$.

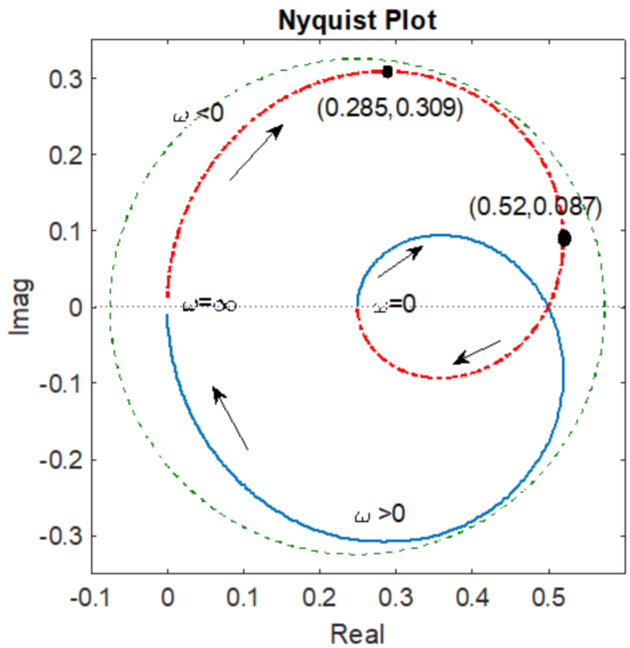

Figure 6: Nyquist Plot of $G(s)$ in Problem 3

(b)

$$\frac{d\psi}{dy} = \omega_1 \cos(\omega_1 y)\cos(\omega_2 y) - \omega_2 \sin(\omega_1 y)\sin(\omega_2 y)$$

Then $\frac{d\psi(0)}{dy} = \omega_1$. Hence $\psi \in [-\omega_1, \omega_1] = [-0.5, 0.5]$. The graph of $\psi(y)$ is shown in Figure 7. From the result of part (a), we conclude that the origin of the feedback system is asymptotically stable.

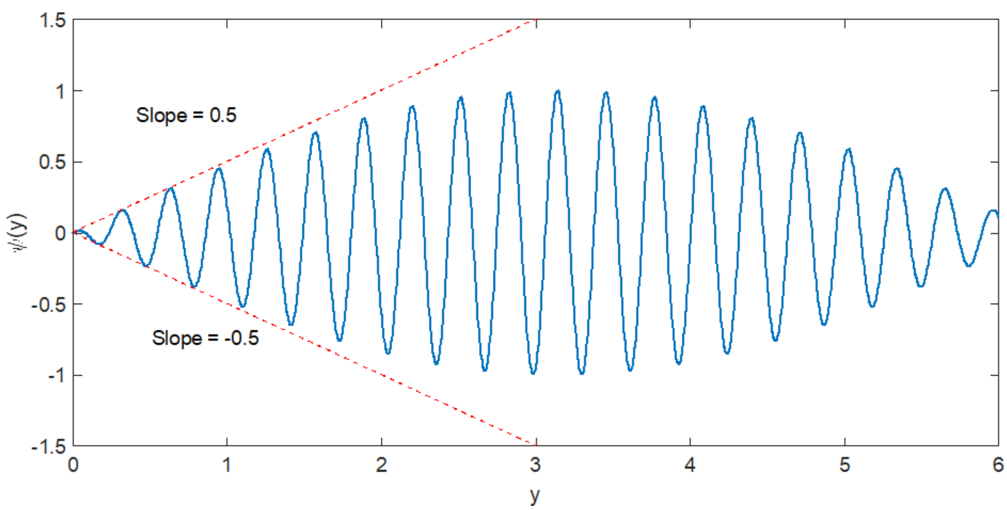

Figure 7: $\psi(y)$ in Problem 3(b)