Nonlinear System Theory
Solution to Homework 4

1. (a) The equilibrium point is

$$x_1 = 0, \quad -x_2^3 - x_2 = -x_2(x_1^2 + 1) = 0 \Rightarrow x_2 = 0$$

Hence $x = 0$ is the unique equilibrium point.

(b) The Jacobian matrix around $x = 0$ is

$$\left. \begin{bmatrix} -1 & 0 \\ x_2^4 + x_2^2 + 2x_1x_2 & 4x_1x_2^3 - 3x_2^2 + 2x_1x_2 - 1 + x_1^2 \end{bmatrix} \right|_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since both eigenvalues of the Jacobian matrix have negative real parts, $x = 0$ is asymptotically stable.

(c) Define $V = x_1x_2$. Then

$$\dot{V} = -x_1x_2 + (x_1x_2 - 1)x_1x_2^3 + (x_1x_2 - 1 + x_1^2)x_1x_2$$

Note that $\dot{V}|_{x_1x_2=2} = 2x_1^2 + 2x_2^2 > 0$. This implies that V is increasing at the boundary of Γ, pushing the state trajectory staying in Γ. Hence Γ is a positively invariant set.

(d) $x = 0$ is not globally asymptotically stable since trajectories starting in Γ do not converge to $x = 0$.

2. If $r_1 \geq r_2$, then $r_1 + r_2 \leq 2r_1$. Hence

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) \leq \alpha(2r_1) + \alpha(2r_2)$$

On the other hand, if $r_2 \geq r_1$, then $r_1 + r_2 \leq 2r_2$. Hence

$$\alpha(r_1 + r_2) \leq \alpha(2r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

Combine both cases and we have

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

3. Consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^Tx$. Then $V(x)$ is positive definite and raidally unbounded. Furthermore,

$$\dot{V} = -ax^T\left(I + S(x) + xx^T\right)x = -ax^Tx - ax^TS(x)x - a(x^Tx)x^Tx = -a(1 + \|x\|^2)\|x\|^2$$

Since $\dot{V}$ is negative definite, we conclude that $x = 0$ is globally asymptotically stable.

4. (a) The linearized system is $\dot{x} = A(t)x$, where

$$A(t) = \left[\begin{array}{cc} -3k_2x_1^2 & k_1 \sin t \\ -k_1 \sin t & -3k_2x_2^2 \end{array} \right]_{(x_1=0,x_2=0)} = \left[\begin{array}{cc} 0 & k_1 \sin t \\ -k_1 \sin t & 0 \end{array} \right]$$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be a Lyapunov function candidate for the linearized system. Then

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = k_1(\sin t)x_1x_2 - k_1(\sin t)x_1x_2 = 0$$

This shows that solutions starting on the surface $V(x) = c$ remain on that surface for all t. Hence $x = 0$ is not an exponentially stable equilibrium point of the linearized system.

Alternatively, we can solve the linearized system directly. The solution to $\dot{x} = A(t)x$ is

$$x(t) = \exp\left[\int_0^t A(\tau)d\tau\right]x(0)$$

and

$$\int_0^t A(\tau)d\tau = \left[\begin{array}{cc} 0 & \int_0^t k_1 \sin \tau d\tau \\ -\int_0^t k_1 \sin \tau d\tau & 0 \end{array} \right] = \left[\begin{array}{cc} 0 & k_1(1-\cos t) \\ -k_1(1-\cos t) & 0 \end{array} \right]$$

It can be observed that $\exp[\int_0^t A(\tau)d\tau]$ consists of the term $e^{jk_1(1-\cos t)}$ which does not converge to zero as $t \to \infty$; hence $x(t)$ does not converge to zero, implying that $x = 0$ is not an exponentially stable equilibrium point.

(b) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be a Lyapunov function candidate. Note that $V(x)$ is positive definite and decrescent. Then

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = k_1(\sin t)x_1x_2 - k_2x_1^4 - k_1(\sin t)x_1x_2 - k_2x_2^4 = -k_2(x_1^4 + x_2^4)$$

Since $k_2(x_1^4 + x_2^4)$ is positive definite, $\dot{V}$ is negative definite, which implies that $x = 0$ is a uniformly asymptotically stable equilibrium point.

5. (a) Let $f(x_1) = ax_1^2 - (1 - \cos x_1)$ for $|x_1| \leq \frac{\pi}{4}$, where $a > 0$. Note that $f(x_1)$ is an even function, i.e. $f(-x_1) = f(x_1)$, and $f(0) = 0$. We want to find the coefficient a such that either $f(x_1) \geq 0$ or $f(x_1) \leq 0$ for all $|x_1| \leq \frac{\pi}{4}$.

Notice that $f'(x_1) = 2ax_1 - \sin x_1$. If $f'(x_1) \geq 0$ for all $0 \leq x_1 \leq \frac{\pi}{4}$, then $f(x_1) \geq 0$ for all $|x_1| \leq \frac{\pi}{4}$. Similarly, if $f'(x_1) \leq 0$ for all $0 \leq x_1 \leq \frac{\pi}{4}$, then $f(x_1) \leq 0$ for all $|x_1| \leq \frac{\pi}{4}$.

For $0 \leq x_1 \leq \frac{\pi}{4}$, $\sin x_1$ is bounded by two straight lines, i.e. $mx_1 \leq \sin x_1 \leq x_1$, where $m = \frac{1}{\sqrt{2}\frac{\pi}{4}} \approx 0.9$. Hence if $a = \frac{1}{2}$, then $f(x_1) \geq 0$ for all $|x_1| \leq \frac{\pi}{4}$, and if $a = 0.45$, then $f(x_1) \leq 0$ for all $|x_1| \leq \frac{\pi}{4}$.

Based on the previous observation, we have

$$0.45x_1^2 \leq 1 - \cos x_1 \leq 0.5x_1^2, \quad \forall |x_1| < \frac{\pi}{4}$$

Hence

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \leq \frac{1}{2}\|x\|^2 = \alpha_2(\|x\|)$$

On the other hand,

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \geq 0.45x_1^2 + \frac{1}{2}x_2^2 \geq 0.45\|x\|^2 = \alpha_1(\|x\|)$$

We also note that $\alpha_2^{-1}(y) = \sqrt{2y}$ and $\alpha_1^{-1}(y) = \sqrt{\frac{y}{0.45}} \approx 1.49\sqrt{y}$.

(b) The time derivative of V is

$$\begin{aligned} \dot{V} &= \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 = -x_1 \sin x_1 + x_2 \sin x_1 - x_2 \sin x_1 - x_2^2 + g(t)x_2 \\ &\leq -x_1 \sin x_1 - x_2^2 + k|x_2| \end{aligned}$$

From part (a) we see that $mx_1 \leq \sin x_1 \leq x_1$ for $0 \leq x_1 \leq \frac{\pi}{4}$ and $m \approx 0.9$. Hence $mx_1^2 \leq x_1 \sin x_1 \leq x_1^2$ for $|x_1| < \frac{\pi}{4}$. Thus

$$\dot{V} \leq -mx_1^2 - x_2^2 + k|x_2| \leq -m\|x\|^2 + k\|x\| = -m(1-\theta)\|x\|^2 - (m\theta\|x\| - k)\|x\|$$

where $0 < \theta < 1$. If $\|x\| > \frac{k}{m\theta} = \mu$, then $\dot{V} \leq -m(1-\theta)\|x\|^2$, i.e. $\dot{V}$ is negative definite for $\|x\| > \mu$. Since μ should satisfy $\mu < \alpha_2^{-1}(\alpha_1(\frac{\pi}{4})) = \sqrt{0.9}\frac{\pi}{4} = 0.745$, we choose $\theta = 0.9$, and then $\mu = \frac{\frac{\pi}{6}}{m\theta} = 0.6464$.
Thus, the ultimate bound is

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = 1.49\sqrt{0.5}\mu = 0.681$$