Nonlinear System Theory
Solution to Final (2023)

1. (a) Let $u \equiv 0$. Clearly, $(x_1, x_2) = (0, 0)$ is an equilibrium point of the system. Linearize the system around $(0, 0)$, and the corresponding Jacobian matrix is

$$\mathbf{A} = \begin{bmatrix} -2 & h'(0) \\ 0 & -1 - h'(0) \end{bmatrix}$$

Suppose that $h'(0) < 0$. Since h' is continuous at $x_2 = 0$, there exists $\delta > 0$ such that $h'(x_2) < 0, \forall |x_2| < \delta$. This implies that h is decreasing over $|x_2| < \delta$ and thus $h(x_2) < 0$ for $0 < x_2 < \delta$. This is a contradiction to the assumption $x_2 h(x_2) > 0$ for all $x_2 \neq 0$. Hence $h'(0) \geq 0$.

Because $h'(0) \geq 0$, both of the eigenvalues of $\mathbf{A}$, i.e. $\lambda = -2, -1 - h'(0)$, are negative. Therefore $(0, 0)$ is an asymptotically stable equilibrium point.

(b) Choose $V(x_1, x_2) = \frac{1}{2}x_1^2 + \int_0^{x_2} h(\tau)d\tau$ as the storage function. Clearly, V is positive definite. Then

$$\begin{aligned} \dot{V} &= -2x_1^2 + x_1 h(x_2) - x_1 u - x_2 h(x_2) - h^2(x_2) + h(x_2)u \\ &= -\frac{1}{2}(x_1 - h(x_2))^2 - \frac{3}{2}x_1^2 - x_2 h(x_2) - \frac{1}{2}h^2(x_2) + (-x_1 + h(x_2))u \qquad (1) \\ &\leq -\frac{3}{2}x_1^2 - x_2 h(x_2) + yu \end{aligned}$$

Since $\psi(x_1, x_2) = \frac{3}{2}x_1^2 + x_2 h(x_2)$ is positive definite and $yu \geq \dot{V} + \psi(x_1, x_2)$, the system is strictly passive from input u to output y.

(c) Rearrange (1) and we have

$$\dot{V} \leq -\frac{1}{2}y^2 + yu$$

Hence the system is finite-gain $\mathcal{L}_2$ stable from u to y. Furthermore, the $\mathcal{L}_2$ gain is less than or equal to 2.

(d) Note that $\frac{\partial V}{\partial x} = \begin{bmatrix} \alpha x_1 & \beta h(x_2) \end{bmatrix}$. Given $\gamma > 0$, let

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} \alpha x_1 & \beta h(x_2) \end{bmatrix} \begin{bmatrix} -2x_1 + h(x_2) \\ -x_2 - h(x_2) \end{bmatrix} + \frac{1}{2\gamma^2}\left(\begin{bmatrix} \alpha x_1 & \beta h(x_2) \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^2 + \frac{1}{2}x_1^2 \\ &= -2\alpha x_1^2 + \alpha x_1 h(x_2) - \beta x_2 h(x_2) - \beta h^2(x_2) \\ &\quad + \frac{1}{2\gamma^2}\left(\alpha^2 x_1^2 - 2\alpha\beta x_1 h(x_2) + \beta^2 h^2(x_2)\right) + \frac{1}{2}x_1^2 \\ &= -\left(2\alpha - \frac{\alpha^2}{2\gamma^2} - \frac{1}{2}\right)x_1^2 + \alpha\left(1 - \frac{\beta}{\gamma^2}\right)x_1 h(x_2) - \beta x_2 h(x_2) - \beta\left(1 - \frac{\beta}{2\gamma^2}\right)h^2(x_2) \end{aligned}$$

To cancel the sign-indefinite term $x_1h(x_2)$, we choose $\beta = \gamma^2$. Then

$$\mathcal{H} = -\left(2\alpha - \frac{\alpha^2}{2\gamma^2} - \frac{1}{2}\right)x_1^2 - \gamma^2 x_2 h(x_2) - \frac{\gamma^2}{2}h^2(x_2)$$

We require that $2\alpha - \frac{\alpha^2}{2\gamma^2} - \frac{1}{2} > 0$, i.e.

$$\alpha^2 - 4\gamma^2\alpha + \gamma^2 < 0 \tag{2}$$

For (2) to hold for some $\alpha > 0$, the discriminant of the 2nd order polynomial of α must be positive, i.e.

$$4\gamma^4 - \gamma^2 = \gamma^2(2\gamma+1)(2\gamma-1) > 0 \quad \Rightarrow \quad \gamma > \frac{1}{2}$$

Hence, if $2\gamma^2 - \gamma\sqrt{4\gamma^2-1} < \alpha < 2\gamma^2 + \gamma\sqrt{4\gamma^2-1}$, then (2) holds and $\mathcal{H} \leq 0$. Therefore the system is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$ is less than or equal to $\frac{1}{2}$.

(e) From part (d) we know that the system with input u and output $\tilde{y}$ is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain, denoted by γ_1, less than or equal to $\frac{1}{2}$. The feedback controller $C(s)$ is also stable. By the small-gain theorem, if the $\mathcal{L}_2$ gain of $C(s)$, denoted by γ_2, is less than $\frac{1}{\gamma_1}$, then the feedback system is finite-gain $\mathcal{L}_2$ stable. Notice that

$$\gamma_2 = \max_{\omega\in\mathbb{R}} |C(j\omega)| = \lim_{\omega\to\infty} |C(j\omega)| = ka$$

Therefore, if $ka < 2$, then the feedback system is finite-gain $\mathcal{L}_2$ stable.

2. (a) Since $G(s)$ has one unstable pole, the circle criterion implies that $0 < \alpha < \beta$, and the Nyquist plot of $G(s)$ must lie outside the disk $D(\alpha, \beta)$ and encircle it 1 time in the counterclockwise direction. Hence the $D(\alpha, \beta)$ is located in the location as shown in Figure 1. Then $\alpha = \frac{1}{6.14} = 0.1628$ and $\beta = \frac{1}{3.37} = 0.2967$.

(b) By using the loop transformation, the feedback connected system in part (b) can be transformed into the system in Figure 2. It has the same form as the system in part (a) with $\psi(y) = ky + \phi(y)$. Suppose that $\phi \in (\alpha', \beta')$. Then $\psi \in (\alpha' + k, \beta' + k)$. Since the feedback system in part (a) is absolutely stable for $\psi \in (\alpha, \beta)$, we have

$$\begin{aligned} \alpha' + k = \alpha \quad &\Rightarrow \quad \alpha' = \alpha - k = \alpha - \frac{\alpha+\beta}{2} = \frac{\alpha-\beta}{2} \\ \beta' + k = \beta \quad &\Rightarrow \quad \beta' = \beta - k = \beta - \frac{\alpha+\beta}{2} = \frac{\beta-\alpha}{2} \end{aligned}$$

In other words, if $\phi \in (\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2})$, then the feedback connected system in part (b) is absolutely stable.

3. (a) Let $V(\mathbf{e}, \theta) = \mathbf{e}^T\mathbf{P}\mathbf{e} + \theta^T\mathbf{\Gamma}^{-1}\theta$ be the Lyapunov function candidate. Then

$$\begin{aligned} \dot{V} &= \dot{\mathbf{e}}^T\mathbf{P}\mathbf{e} + \mathbf{e}^T\mathbf{P}\dot{\mathbf{e}} + 2\theta^T\mathbf{\Gamma}^{-1}\dot{\theta} \\ &= \mathbf{e}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{e} + 2\mathbf{e}^T\mathbf{P}\mathbf{b}\phi^T(t)\theta - 2\theta^T\phi(t)\mathbf{b}^T\mathbf{P}\mathbf{e} \\ &= -\mathbf{e}^T\mathbf{Q}\mathbf{e} \end{aligned}$$

Hence $\dot{V}$ is negative semidefinite. By Lyapunov stability theorem, $(\mathbf{e}, \theta) = (\mathbf{0}, \mathbf{0})$ is a stable equilibrium point.

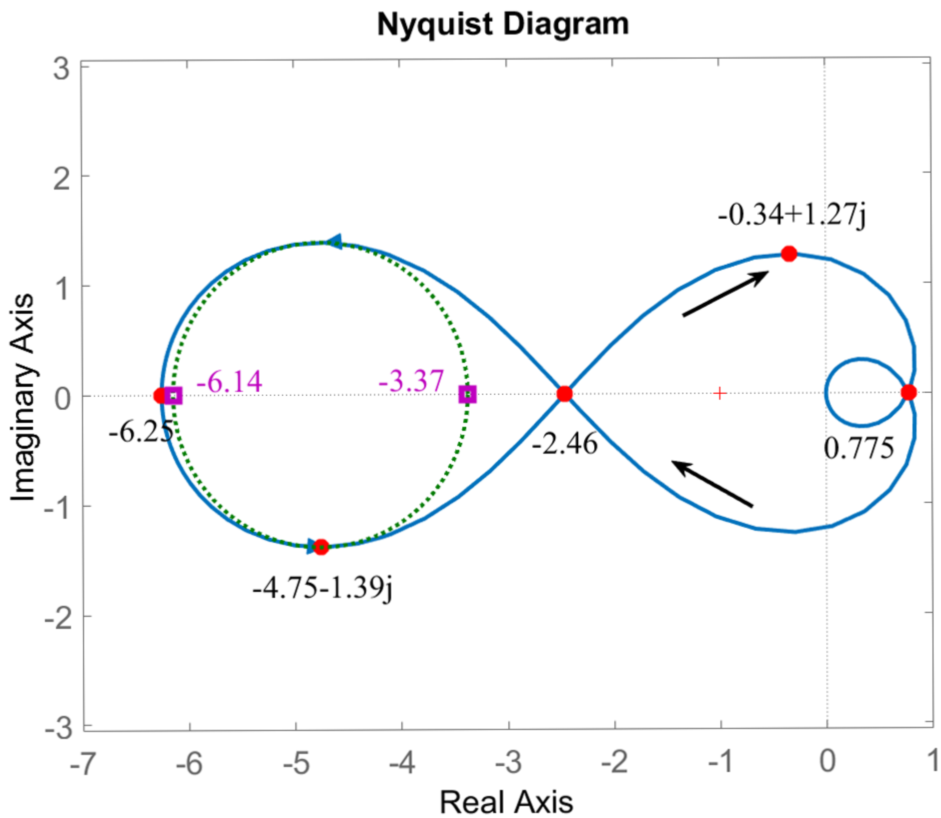

Figure 1: Applying the circle criterion to Problem 2(a)

(b) Since $(\mathbf{e}, \theta) = (\mathbf{0}, \mathbf{0})$ is a stable equilibrium point, both $\mathbf{e}$ and θ are bounded. Then V is bounded. Moreover,

$$\ddot{V} = -2\mathbf{e}^T\mathbf{Q}\dot{\mathbf{e}} = -2\mathbf{e}^T\mathbf{Q}\big(\mathbf{A}\mathbf{e} + \mathbf{b}\phi^T(t)\theta\big)$$

Notice that every term on the right-hand side of the previous equation is bounded, so is $\ddot{V}$. Thus $\dot{V}$ is uniformly continuous. By Barbalat's lemma, $\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} \to \mathbf{0}$ as $t \to \infty$, implying $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$.

4. (a)

$$\begin{aligned} \dot{y} &= -x_2 - \sin x_1 + x_3 \\ \ddot{y} &= x_2 + \sin x_1 - x_3 - (x_2 + u)\cos x_1 + x_2^2 - x_3 - 2u \\ &= \big(1 - \cos x_1 + x_2\big)x_2 + \sin x_1 - 2x_3 - (2 + \cos x_1)u \end{aligned}$$

Since $2 + \cos x_1 > 0$ for all x_1, the relative degree of the system is 2 over $\mathbb{R}^3$.

(b) To determine the zero dynamics of the system, let $y = \dot{y} = \ddot{y} \equiv 0$. This implies

$$x_2 = 0, \quad x_3 = \sin x_1, \quad u = \frac{\sin x_1 - 2x_3}{2 + \cos x_1} = \frac{-\sin x_1}{2 + \cos x_1}$$

Then the 1st differential equation of the system becomes

$$\dot{x}_1 = u = \frac{-\sin x_1}{2 + \cos x_1}$$

For $|x_1| < \pi$, $\dot{x}_1 < 0$ if $x_1 > 0$, and $\dot{x}_1 > 0$ if $x_1 < 0$. Therefore $x_1 = 0$ is an asymptotically stable equilibrium point of the zero dynamics. This shows that the system is minimum phase.

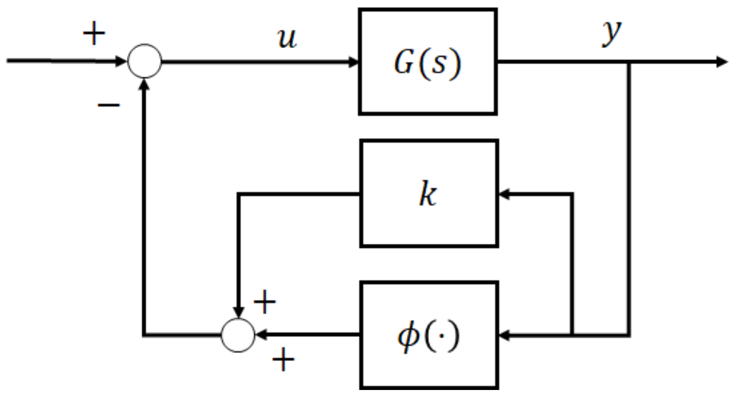

Figure 2: Applying the loop transformation to Problem 2(b)

(c) Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a diffeomorphism satisfying $\phi(0) = 0$ and $\frac{\partial \phi}{\partial x} g = 0$, where $g = [1, 0, -2]^T$. In other words,

$$\frac{\partial \phi}{\partial x_1} - 2\frac{\partial \phi}{\partial x_3} = 0$$

Choose $\phi(x) = 2x_1 + x_3$. Then the state variables for the normal form are

$$\eta = 2x_1 + x_3, \quad \xi_1 = y = x_2, \quad \xi_2 = \dot{y} = -x_2 - \sin x_1 + x_3$$

(d) Note that

$$f(x) = \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 + x_3 \\ x_2^2 - x_3 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then

$$\begin{aligned} ad_f g &= -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 1 & 0 \\ -\cos x_1 & -1 & 1 \\ 0 & 2x_2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos x_1 + 2 \\ -2 \end{bmatrix} \\ [g, ad_f g] &= \begin{bmatrix} 0 & 0 & 0 \\ -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x_1 \\ 0 \end{bmatrix} \end{aligned}$$

Consider the matrix

$$\mathbf{M} = \begin{bmatrix} g & ad_f g & [g, ad_f g] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x_1 + 2 & -\sin x_1 \\ -2 & -2 & 0 \end{bmatrix}$$

Notice that $\text{rank}(\mathbf{M}) = 2$ for $x_1 = 0$; otherwise $\text{rank}(\mathbf{M}) = 3$. This implies $\text{span}\{g, ad_f g(x)\}$ is not involutive for $0 < |x_1| < \pi$.