Nonlinear System Theory
Homework 6

Due date: 5/3/2022

1. Consider the following single-input single-output nonlinear system:

$$
\begin{aligned}
\dot{x}_1 &= -g_1(x_1) + x_2 \\
\dot{x}_2 &= -x_1 - g_2(x_2) + u \\
y &= h(x)
\end{aligned}
$$

   where for $i = 1, 2$, $g_i(x_i)$ satisfies $g_i(0) = 0$ and $x_i g(x_i) \geq k_i x_i^2$ for some $k_i > 0$, $\forall x_i$. u and y are the input and output of the system respectively, and $h(0) = 0$.

   (a) Suppose that $h(x) = x_2$. Show that the nonlinear system is finite-gain $\mathcal{L}_2$ stable with the $\mathcal{L}_2$ gain less than or equal to $\frac{1}{k_2}$

   (b) Suppose that $h(x) = x_1$. Show that the nonlinear system if finite-gain $\mathcal{L}_2$ stable with the $\mathcal{L}_2$ gain less than or equal to $\frac{1}{2\sqrt{k_1 k_2}}$

2. Consider an n-joint robot which satisfies the following dynamic equation:

$$
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + B\dot{q} = \tau
$$

   where $q, \dot{q}, \ddot{q}, \tau \in \mathbb{R}^n$ denote the angle, velocity, acceleration, and torque of the joint, respectively. $M(q), C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ are the inertia matrix and the Coriolis and centrifugal matrix, respectively. $G(q) \in \mathbb{R}^n$ is the gravitational vector and we assume that $G(0) = 0$. $B \in \mathbb{R}^{n \times n}$ is a constant positive semidefinite matrix, denoting the viscous friction coefficient. It is well-known in robotics that the following properties hold for all q and $\dot{q}$.

   - $M(q)$ is symmetric positive definite.
   - $\frac{1}{2}\dot{M}(q) - C(q, \dot{q})$ is skew-symmetric, i.e. $x^T \left[\frac{1}{2}\dot{M}(q) - C(q, \dot{q})\right] x = 0$ for all $x \in \mathbb{R}^n$.
   - There exists a positive semidefinite scalar function $U(q)$ such that $\frac{\partial U}{\partial q} = G(q)$.

   If we take τ and $\dot{q}$ as the input and output of the robotic system, respectively, answer the following questions.

   (a) Suppose that $\lambda_{min}(B) > \frac{1}{2}$, where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue. Show that the system is finite-gain $\mathcal{L}_2$ stable with the $\mathcal{L}_2$ gain less than or equal to $\frac{1}{\sqrt{2\lambda_{min}(B) - 1}}$.

   (b) Show that the system is lossless when $B = 0$.

   (c) Show that the system is output strictly passive when B is positive definite.

3. Consider the feedback control system in Figure 1, where

$$G(s) = \frac{2}{(s+1)(s+2)}, \quad C(s) = \frac{K}{s+3}, \quad u = \mathrm{sat}(u_1) = \begin{cases} u_1, & |u_1| \le 1 \\ \mathrm{sgn}(u_1), & |u_1| > 1 \end{cases}$$

and K is a positive constant gain of the controller. Find an upper bound of K with which the feedback system is $\mathcal{L}_\infty$ stable. Note that it suffices to find a conservative upper bound.

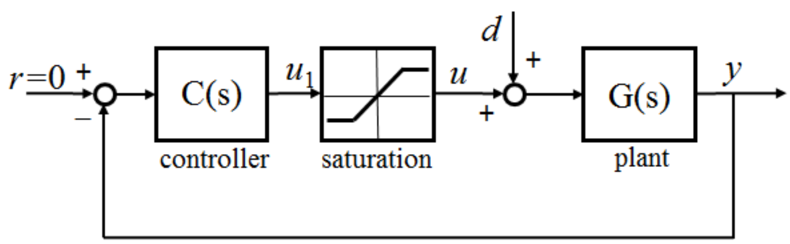

Figure 1: Feedback control system of Problem 3

4. Suppose that a system is input strictly passive with $\psi(u) = \epsilon u$, $\epsilon > 0$ and finite-gain $\mathcal{L}_2$ stable with zero bias, i.e. $\|y_\tau\|_{\mathcal{L}_2} \le \gamma \|u_\tau\|_{\mathcal{L}_2}$, $\gamma > 0$, for all τ. Show that there is a storage function V and positive constants ϵ_1 and δ_1 such that

$$u^T y \ge V + \epsilon_1 u^T u + \delta_1 y^T y$$