Nonlinear System Theory
Solution to Homework 6

1. Since $0 < p_{12} < \frac{ak}{2}$, P is positive definite. Then

$$\begin{aligned}\dot{V} &= kh(x_1)\dot{x}_1 + 2x^T P\dot{x} \\ &= kh(x_1)x_2 + \big(2ap_{12}x_1 + 2p_{12}x_2\big)x_2 + \big(2p_{12}x_1 + kx_2\big)\big(-h(x_1) - ax_2 + u\big) \\ &= 2p_{12}x_2^2 - 2p_{12}x_1h(x_1) + 2p_{12}x_1u - kax_2^2 + kx_2u\end{aligned}$$

Hence

$$\begin{aligned}uy &= \dot{V} + u^2 - 2p_{12}x_2^2 + 2p_{12}x_1h(x_1) - 2p_{12}x_1u + kax_2^2 \\ &= \dot{V} + \big(u - p_{12}x_1\big)^2 - p_{12}^2x_1^2 + 2p_{12}x_1h(x_1) + \big(ka - 2p_{12}\big)x_2^2 \\ &\geq \dot{V} - p_{12}^2x_1^2 + 2\alpha_1p_{12}x_1^2 + \big(ka - 2p_{12}\big)x_2^2 \\ &= \dot{V} + \psi(x)\end{aligned}$$

where $\psi(x) = p_{12}(2\alpha_1 - p_{12})x_1^2 + (ka - 2p_{12})x_2^2$. Since $0 < p_{12} < \min\{2\alpha_1,\ \frac{ak}{2}\}$, $\psi(x)$ is positive definite. Hence the system is strictly passive.

2. First of all, $G(s)$ is Hurwitz if and only if $a_1, a_2 > 0$. This can be easily checked by the Routh test. The real part of $G(j\omega)$ is

$$\mathrm{Re}[G(j\omega)] = \mathrm{Re}\left[\frac{jb_0\omega + b_1}{-\omega^2 + a_2 + ja_1\omega}\right] = \frac{b_1a_2 + (a_1b_0 - b_1)\omega^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2}$$

Therefore $\mathrm{Re}[G(j\omega)] > 0$ for all $\omega \in \mathbb{R}$ if and only if $b_1 > 0$ and $b_0a_1 \geq b_1$. Moreover,

$$\lim_{\omega\to\infty} \omega^2\mathrm{Re}[G(j\omega)] = a_1b_0 - b_1 > 0$$

Combining all conditions we conclude that $G(s)$ is positive real is and only if all coefficients are positive and $b_1 < a_1b_0$.

3. Let u_i, y_i, and V_i, $i = 1, 2$, be the input, output, and storage function of the two systems in parallel. Let $u = u_1 = u_2$ and $y = y_1 + y_2$ be the input and output of the parallel system. Suppose that

$$\begin{aligned}u^Ty_1 &\geq \dot{V}_1 + u^T\varphi_1(u) + y_1^T\rho_1(y_1) + \psi_1(x_1) \\ u^Ty_2 &\geq \dot{V}_2 + u^T\varphi_2(u) + y_2^T\rho_2(y_2) + \psi_2(x_2)\end{aligned}$$

Then if both subsystems are input strictly passive, then $u^T\varphi_i(u) > 0$ for $u \neq 0$ and $\rho_i(y_i) = 0$, $\psi_i(x_i) = 0$ for all y_i and x_i, $i = 1, 2$. If both subsystems are output strictly passive, then $\rho_i(y_i) = \delta_iy_i$, and $\varphi_i(u) = 0$, $\psi_i(x_i) = 0$ for all u and x_i, $i = 1, 2$. Similarly, these equations can represent the cases of passive, and strictly passive systems.

For the parallel system, we have

$$u^T y = u^T(y_1+y_2) \geq \dot{V}_1+\dot{V}_2+u^T(\varphi_1(u)+\varphi_2(u))+y_1^T\rho(y_1)+y_2^T\rho(y_2)+\psi_1(x_1)+\psi_2(x_2)$$

Define $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ to be the storage function of the parallel system. $\varphi(u) = \varphi_1(u) + \varphi_2(u)$, and $\psi(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2)$. Note that $\psi(x_1, x_2)$ is positive definite. If both subsystems are passive (respectively, input strictly passive, and strictly passive), then

$$u^T y \geq \dot{V} + u^T\varphi(u) + \psi(x_1, x_2)$$

In other words, the parallel system is passive (respectively, input strictly passive and strictly passive). If both subsystems are output strictly passive, then

$$u^T y \geq \dot{V} + \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \geq \dot{V} + \delta(y_1^T y_1 + y_2^T y_2) \geq \dot{V} + \frac{\delta}{2} y^T y$$

where $\delta = \min\{\delta_1, \delta_2\} > 0$, and note that $(y_1 + y_2)^T(y_1 + y_2) \leq 2(y_1^T y_1 + y_2^T y_2)$. Therefore, the parallel system is also output strictly passive.

4. (a) Decompose the system into two feedback connected system in Figure 1, where $u_1 = u$, $u_2 = 0$, and

$$\mathcal{H}_1: \begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -x_1^3 - f(x_2) + e_1 \\ y_1 &=& x_2 \end{array}, \qquad \mathcal{H}_2: \begin{array}{rcl} \dot{x}_3 &=& -h(x_3) + e_2 \\ y_2 &=& h(x_3) \end{array}$$

Firstly, we show that $\mathcal{H}_1$ is output strictly passive and zero-state observable. Consider the storage function $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. Then its time derivative is

$$\dot{V}_1 = x_1^3 x_2 + x_2(-x_1^3 - f(x_2) + e_1) = -y_1 f(y_1) + y_1 e_1 \leq -k y_1^2 + y_1 e_1$$

Hence $\mathcal{H}_1$ is output strictly passive. To show that $\mathcal{H}_1$ is zero-state observable, let $e_1 = 0$ and $y_1 = x_2 \equiv 0$. Then $\dot{x}_2 \equiv 0$ and $f(x_2) \equiv 0$. This implies that $x_1 \equiv 0$. Therefore, $\mathcal{H}_1$ is zero-state observable.

Next, we will show that $\mathcal{H}_2$ is strictly passive. Let the storage function be $V_2(x_3) = \int_0^{x_3} h(s)ds$. Then

$$\dot{V}_2 = h(x_3)\dot{x}_3 = -h^2(x_3) + e_2 h(x_3) = -h^2(x_3) + e_2 y_2$$

Note that $h^2(x_3) > 0$ for all $x_3 \neq 0$ and $h^2(0) = 0$; hence $h^2(x_3)$ is a positive definition function of x_3 and $\mathcal{H}_2$ is strictly passive.

Because $\mathcal{H}_1$ is output strictly passive and zero-state observable, and $\mathcal{H}_2$ is strictly passive, $(x_1, x_2, x_3) = (0, 0, 0)$ is an asymptotically stable equilibrium point.

(b) Choose the storage function for the whole system as $V = V_1 + V_2 = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \int_0^{x_3} h(s)ds$. Then

$$\begin{array}{rcl} \dot{V} &=& x_1^3 x_2 - x_1^3 x_2 - x_2 f(x_2) - x_2 h(x_3) + x_2 u + h(x_3)(-h(x_3) + x_2) \\ &=& -x_2 f(x_2) - h^2(x_3) + x_2 u \\ &\leq& -k y^2 + u y \end{array}$$

Hence the system is output strictly passive and therefore it is finite-gain $\mathcal{L}_2$ stable with the $\mathcal{L}_2$-gain less than or equal to $\frac{1}{k}$.

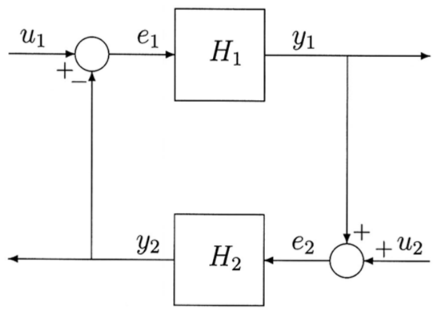

Figure 1: Feedback connection of the nonlinear system in Problem 4

(c) Rewrite the proof for the strict passivity of $\mathcal{H}_2$ in part (a) as follows. Let $V_2(x_3)$ be defined as in part (a). Then

$$\dot{V}_2 = h(x_3)\big[-(1+\Delta(x_3))h(x_3)+e_2\big] = -(1+\Delta(x_3))h^2(x_3)+h(x_3)e_2$$

Clearly, if $\Delta(x_3) > -1$ for all $x_3 \in \mathbb{R}$, then $(1+\Delta(x_3))h^2(x_3) > 0$ for all $x_3 \neq 0$. In other words, if $\Delta(x_3) \in S = (-1, \infty)$, then $\mathcal{H}_2$ is strictly passive and $(x_1, x_2, x_3) = (0, 0, 0)$ is asymptotically stable.