Nonlinear System Theory
Solution to Homework 10

1. (a) Augment the state equation with the integrator $\dot{\sigma} = \psi - \psi_r$. Then the augmented state model is given by

$$\frac{d}{dt}\begin{bmatrix} \psi \\ \dot{\psi} \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-1}{\tau} & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi \\ \dot{\psi} \\ \sigma \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{\tau} \\ 0 \end{bmatrix}\delta + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\psi_r$$

where $\frac{1}{\tau} = (\frac{1}{\tau_0})(\frac{v}{v_0})$ and $\frac{k}{\tau} = (\frac{k_0}{\tau_0})(\frac{v}{v_0})^2$. Let the state feedback control be $u = -K_1(\psi - \psi_r) - K_2\dot{\psi} - K_3\sigma$. Then the closed-loop system is

$$\frac{d}{dt}\begin{bmatrix} \psi \\ \dot{\psi} \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{\tau}K_1 & -\frac{1}{\tau} - \frac{k}{\tau}K_2 & -\frac{k}{\tau}K_3 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi \\ \dot{\psi} \\ \sigma \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{r}K_1 \\ -1 \end{bmatrix}\psi_r$$

The closed-loop characteristic equation is

$$s^3 + \left(\frac{1}{\tau} + \frac{k}{\tau}K_2\right)s^2 + \frac{k}{\tau}K_1 s + \frac{k}{\tau}K_3 = 0 \tag{1}$$

Choose

$$K_1 = 12\frac{\tau}{k}, \quad K_2 = 9\frac{\tau}{k} - \frac{1}{k}, \quad K_3 = 6\frac{\tau}{k}$$

Then (1) becomes $s^3 + 9s^2 + 12s + 6 = 0$, and the eigenvalues are -7.5082, and $-0.7459 \pm 0.4927j$. Furthermore, the transfer function from ψ_r to ψ is

$$\frac{6(2s+1)}{s^3 + 9s^2 + 12s + 6}$$

and the unit step response is shown in Figure 1. It is clear that the specifications on the maximum overshoot and settling time are satisfied.

(b) The gain-scheduled control law is

$$u = -12\left(\frac{\tau_0}{k_0}\right)\left(\frac{v_0}{v}\right)^2(\psi - \psi_r) - \left[9\left(\frac{\tau_0}{k_0}\right)\left(\frac{v_0}{v}\right)^2 - \left(\frac{1}{k_0}\right)\left(\frac{v_0}{v}\right)\right]\dot{\psi} - 6\left(\frac{\tau_0}{k_0}\right)\left(\frac{v_0}{v}\right)^2\sigma$$

Since the scheduling variable is the forward speed of the ship v, not the reference input ψ_r, there is no need to modify it.

2.

$$\begin{aligned} \dot{y} &= x_2 + x_1^2 \\ \ddot{y} &= x_3 + u + 2x_1(x_2 + x_1^2) \end{aligned}$$

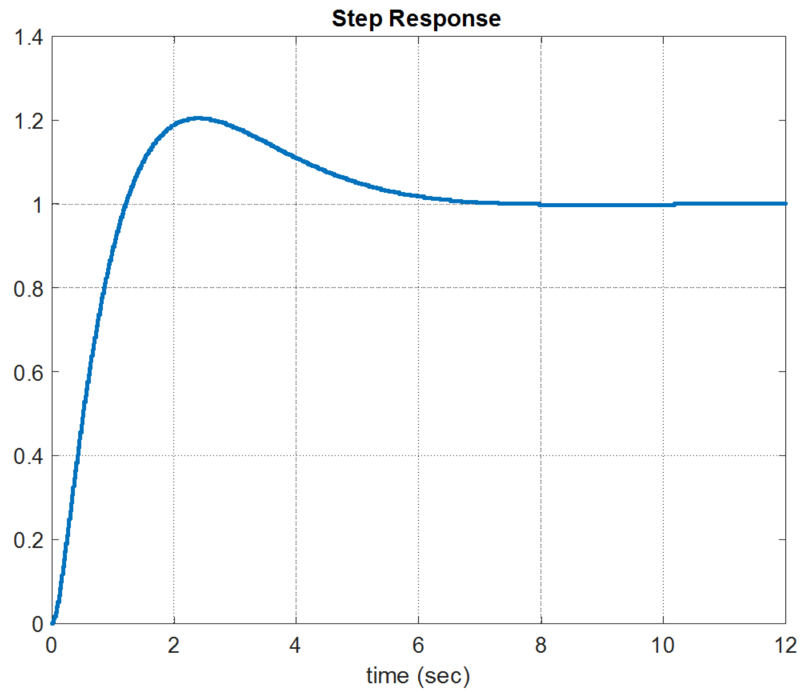

Figure 1: Step Response of Problem 1(a)

Hence the relative degree is two. To check if the system is minimum phase, let $y = \dot{y} = 0$, i.e. $x_1 = x_2 = 0$, and $u = -x_3 - 2x_1(x_2 + x_1^2) = -x_3$. Then the zero dynamics is

$$\dot{x}_3 = -x_3$$

Hence the system is minimum phase. Linearize the input-output relation by choosing

$$u = -x_3 - 2x_1(x_2 + x_1^2) + v$$

Then the linearized system is $\ddot{y} = v$. For y to track r. Define the tracking error as $e = y - r$, and

$$\ddot{e} = \ddot{y} - \ddot{r} = v - \ddot{r}$$

Let

$$v = -k_1 e - k_2 \dot{e} + \ddot{r} = -k_1(y - r) - k_2(\dot{y} - \dot{r}) + \ddot{r}$$

where $k_1, k_2 > 0$. Then $\ddot{e} = -k_1 e - k_2 \dot{e}$ and therefore $e \to 0$ as $t \to \infty$. In summary, the control law is

$$u = -x_3 - 2x_1(x_2 + x_1^2) - k_1(x_1 - r) - k_2(x_2 + x_1^2 - \dot{r}) + \ddot{r}$$