Nonlinear System Theory
Midterm

4/11/2023

1. (26%)Consider the following 2nd-order system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + \tan^{-1} x_1 + x_2 \\ \dot{x}_2 &= -\tan^{-1} x_1 - x_2\end{aligned}$$

   (a) (8%)Use $V(x_1, x_2) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the Lyapunov function candidate to show that $(x_1, x_2) = (0, 0)$ is a stable equilibrium point.

   (b) (8%)Use LaSalle Theorem to show that $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

   (c) (10%)Use another Lyapunov function to directly show that $(x_1, x_2) = (0, 0)$ is a globally asymptotically stable equilibrium point.

2. (24%)Consider the following 2nd-order system

$$\begin{aligned}\dot{x}_1 &= \frac{1}{2}x_1 + x_2 - \tan^{-1} x_1 \\ \dot{x}_2 &= -2x_1x_2 - x_2^2\end{aligned}$$

   (a) (8%)Linearize the nonlinear system around the equilibrium point $(x_1, x_2) = (0, 0)$. Is the linearized system stable in the sense of Lyapunov stability? Why?

   (b) (8%)Show that $S = \{x \in \mathbb{R}^2 | -x_1^2 - x_2 - 1 > 0\}$ is a positively invariant set.

   (c) (8%)Is $(x_1, x_2) = (0, 0)$ a stable equilibrium point of the nonlinear system? Why?

3. (16%)Consider the following system

$$\begin{aligned}\dot{x}_1 &= -(1 + x_1^4)x_1 - 2x_1x_2 \sin x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \sin x_1 + \frac{x_2}{1 + x_2^2}e^{-t}\end{aligned}$$

   Let the initial state be $x(0) = [x_1(0), x_2(0)]^T = [1, 1]^T$, and define $V(x) = \frac{1}{2}x_1^2 + x_2^2$.

   (a) (8%)Find an exponentially decayed function $u(t)$ such that $V(x(t)) \leq u(t)$ for all $t \geq 0$.

   (b) (8%)Find an exponentially decayed function $b(t)$ such that $\|x(t)\|_2 \leq b(t)$ for all $t \geq 0$.

4. (16%)Consider the following system

$$\begin{aligned}\dot{x}_1 &= -\tan\frac{x_1}{2} + x_2 \\ \dot{x}_2 &= -g(t)\sin x_1 - x_2\end{aligned}$$

where $g(t) = 2 + \cos(t)$, $|x_1| < \pi$ and $x_2 \in \mathbb{R}$.

(a) (8%)Define $V(x_1, x_2) = g(t)(1 - \cos x_1) + \frac{1}{2}x_2^2$. Show that V is a positive definite and decrescent function.

(b) (8%)Show that $(x_1, x_2) = (0, 0)$ is a uniformly asymptotically stable equilibrium point.

5. (18%)Consider the following system

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e}(t) + \mathbf{b}\phi^T(t)\theta(t) \\ \dot{\theta}(t) &= -\mathbf{\Gamma}\big(\phi(t)\mathbf{b}^T\mathbf{P}\mathbf{e}(t) + \mathbf{K}\theta(t)\big)\end{aligned}$$

where $\mathbf{e}(t) \in \mathbb{R}^n$, $\theta(t) \in \mathbb{R}^m$, and define $\mathbf{z}(t) = [\mathbf{e}^T(t), \theta^T(t)]^T$ to be the state of the system. Suppose that the initial state is $\mathbf{z}(0) = [\mathbf{e}^T(0), \theta^T(0)]^T$.

In addition, $\phi(t) \in \mathbb{R}^m$ is a known vector of functions. $\mathbf{A}, \mathbf{P} \in \mathbb{R}^{n\times n}$, $\mathbf{\Gamma}, \mathbf{K} \in \mathrm{R}^{m\times m}$, and $\mathbf{b} \in \mathbb{R}^{n\times 1}$ are constant matrices and vectors, where $\mathbf{A}$ is Hurwitz, $\mathbf{P}, \mathbf{\Gamma}, \mathbf{K}$ are symmetric positive definite and $\mathbf{P}$ satisfies the Lyapunov equation: $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$ for some positive definite matrix $\mathbf{Q}$.

(a) (10%)Show that $\mathbf{z} = \mathbf{0}$ is an exponentially stable equilibrium point.

(b) (8%)Find $c, \lambda > 0$ (in terms of the constant matrices and vectors of the system and the initial state) such that $\|\mathbf{z}(t)\| \le ce^{-\lambda t}$, for all $t \ge 0$.