# Nonlinear System Theory
# Homework 9

Due date: 5/24/2022

1. Repeat Problem 1 of Homework 8 by using the Popov criterion.
   *Hint: For unstable $G(s)$, you need to apply the loop transformation to convert $G(s)$ to a stable system while maintaining an equivalent feedback loop.*

   *Remark: You can compare the results of Problem 1 of Homework 8 with Problem 3, so the differences between the circle criterion and the Popov criterion will be clear.*

2. Consider the following nonlinear system

$$
\begin{aligned}
\dot{e}_1(t) &= e_2(t) \\
\dot{e}_2(t) &= -\tilde{a}(t)\phi(t) - k_1 e_1(t) - k_2 e_2(t) \\
\dot{\tilde{a}}(t) &= \gamma e_2(t)\phi(t)
\end{aligned}
$$

   where $k_1, k_2, \gamma$ are positive constants, and $\phi(t)$ is a bounded function of time.

   (a) Show that $(e_1, e_2, \tilde{a}) = (0, 0, 0)$ is a globally stable equilibrium point.

   (b) Show that $e_2 \to 0$ as $t \to \infty$.

3. Consider the inverted pendulum system in Figure 1. $M$ and $m$ are the mass of the cart and the pendulum, respectively. $L$ is the length of the pendulum. $I$ is the moment of inertia of the pendulum with respect to its center of gravity. $k$ is the friction coefficient of the cart. $y$ is the displacement of the pendulum pivot, $\theta$ is the angular rotation of the pendulum (measured clockwise), and $g$ is the gravitational acceleration. $F$ is the external force applied to the cart. The differential equations of the inverted pendulum are

$$
\begin{aligned}
(I + mL^2)\ddot{\theta} &= mgL\sin\theta - mL\ddot{y}\cos\theta \\
(M + m)\ddot{y} &= F - mL\big(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\big) - k\dot{y}
\end{aligned}
$$

   (a) Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as state variables and $u = F$ as control input, write down the state equation.

   (b) Show that the equilibrium points of the unforced system are $(x_1, x_2, x_3, x_4) = (n\pi, 0, \bar{x}_3, 0)$, where $n$ is any integer and $\bar{x}_3$ can be any real number.

   (c) Show that the equilibrium point in the vertical position, i.e. $n$ is an even number and $\bar{x}_3$ is any real number in part (b), is unstable. Without loss of generality, you can consider the equilibrium point $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$.

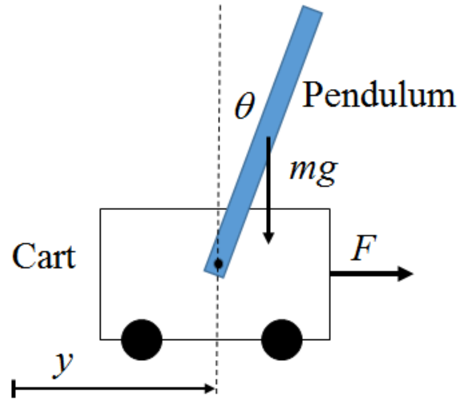


Figure 1: Inverted Pendulum in Problem 3

(d) Let $m = 0.1$ kg, $M = 1$ kg, $k = 0.1$ N/(m-sec), $I = 0.008$ kg $-$ m$^2$. $g = 9.81$ m/sec$^2$, and $L = 0.5$ m. Suppose that all state variables are measurable. Design a state feedback control law based on the linearized model to stabilize the pendulum in the upright position ($\theta = 0$). Verify your design by simulation. Your controller should be able to keep the pendulum in the upright position ($\theta = 0$) even when the initial pendulum angle is as large as 30 degree, i.e. $\theta(0) = 30°$. Moreover, the force applied to the cart should be less than or equal to 10 N ($|F| \leq 10$), and the cart position should be less than 2 meter ($|y| \leq 2$). Present your design and simulation results.

*Remark: In your simulation, you should use the nonlinear inverted pendulum model as the plant, not the linearized one.*