Nonlinear System Theory
Solution to Homework 4

1. (a) The equilibrium point is

$$x_1 = 0, \quad -x_2^3 - x_2 = -x_2(x_1^2 + 1) = 0 \Rightarrow x_2 = 0$$

Hence $x = 0$ is the unique equilibrium point.

(b) The Jacobian matrix around $x = 0$ is

$$\left[\begin{array}{cc} -1 & 0 \\ x_2^4 + x_2^2 + 2x_1x_2 & 4x_1x_2^3 - 3x_2^2 + 2x_1x_2 - 1 + x_1^2 \end{array}\right]\Bigg|_{x=0} = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right]$$

Since both eigenvalues of the Jacobian matrix have negative real parts, $x = 0$ is asymptotically stable.

(c) Define $V = x_1x_2$. Then

$$\dot{V} = -x_1x_2 + (x_1x_2 - 1)x_1x_2^3 + (x_1x_2 - 1 + x_1^2)x_1x_2$$

Note that $\dot{V}|_{x_1x_2=2} = 4x_2^2 > 0$. This implies that V is increasing at the boundary of Γ, pushing the state trajectory staying in Γ. Hence Γ is a positively invariant set.

(d) $x = 0$ is not globally asymptotically stable since trajectories starting in Γ do not converge to $x = 0$.

2. If $r_1 \geq r_2$, then $r_1 + r_2 \leq 2r_1$. Hence

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) \leq \alpha(2r_1) + \alpha(2r_2)$$

On the other hand, if $r_2 \geq 2r_1$, then $r_1 + r_2 \leq 2r_2$. Hence

$$\alpha(r_1 + r_2) \leq \alpha(2r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

Combine both cases and we have

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

3. Consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^Tx$. Then $V(x)$ is positive definite and raidally unbounded. Furthermore,

$$\dot{V} = -ax^T\big(I + S(x) + xx^T\big)x = -ax^Tx - ax^TS(x)x - a(x^Tx)x^Tx = -a(1 + \|x\|^2)\|x\|^2$$

Since $\dot{V}$ is negative definite, we conclude that $x = 0$ is globally asymptotically stable.