Nonlinear System Theory
Solution to Midterm

1. (a) It is easy to verify that $(x_1, x_2) = (0, 0)$ is indeed an equilibrium point. Also notice that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is positive definite; hence V is a positive definition function. Since $V = x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2$, its time derivative is

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + \dot{x}_1 x_2 + x_1\dot{x}_2 + x_2\dot{x}_2 \\ &= -2x_1^2 + 2x_1 \tan^{-1} x_1 + 2x_1x_2 - x_1x_2 + x_2 \tan^{-1} x_1 + x_2^2 - x_1 \tan^{-1} x_1 - x_1x_2 \\ &\quad -x_2 \tan^{-1} x_1 - x_2^2 \\ &= -2x_1^2 + x_1 \tan^{-1} x_1 \\ &= -x_1^2 - x_1\big(x_1 - \tan^{-1} x_1\big) \end{aligned}$$

Let $f(x_1) = x_1 - \tan^{-1} x_1$. Then $f(0) = 0$ and $f'(x_1) = 1 - \frac{1}{1+x_1^2} > 0$ for all $x_1 \neq 0$. This implies that $f(x_1)$ is a strictly increasing function. Therefore, $f(x_1) = x_1 - \tan^{-1} x_1 > f(0) = 0$ for $x_1 > 0$ and $f(x_1) = x_1 - \tan^{-1} x_1 < f(0) = 0$ for $x_1 < 0$. Furthermore, $x_1 f(x_1) > 0$ for $x_1 \neq 0$. As a result, $\dot{V} \le 0$ for all x_1, x_2 and thus $\dot{V}$ is negative semidefinite. By Lyapunov theorem, $(x_1, x_2) = (0, 0)$ is a stable equilibrium point.

(b) For $\dot{V} = 0$, we have $x_1 = 0$. If $x_1 \equiv 0$, we have $\dot{x}_1 \equiv 0$ and thus $x_2 \equiv 0$. This implies that $(x_1, x_2) = (0, 0)$ is the only solution for $\dot{V} = 0$. By LaSalle theorem, $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

(c) Choose the following Lyapunov function candidate

$$V_1(x_1, x_2) = \int_0^{x_1} \tan^{-1} \tau d\tau + \frac{1}{2}x_2^2$$

Clearly V_1 is positive definite and radially unbounded. Then

$$\begin{aligned} \dot{V}_1 &= \dot{x}_1 \tan^{-1} x_1 + x_2\dot{x}_2 \\ &= -x_1 \tan^{-1} x_1 + (\tan^{-1} x_1)^2 + x_2 \tan^{-1} x_1 - x_2 \tan^{-1} x_1 - x_2^2 \\ &= -\tan^{-1} x_1(x_1 - \tan^{-1} x_1) - x_2^2 \end{aligned}$$

Notice that $\tan^{-1} x_1(x_1 - \tan^{-1} x_1) > 0$ for all $x_1 \neq 0$ and $\tan^{-1} x_1(x_1 - \tan^{-1} x_1) = 0$ for $x_1 = 0$. Hence $\dot{V}_1$ is negative definite and therefore, $(x_1, x_2) = (0, 0)$ is a globally asymptotically stable equilibrium point.

2. (a) The Jacobain matrix of the linearized system is

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} - \frac{1}{1+x_1^2} & 1 \\ -2x_2 & -2x_1 - 2x_2 \end{bmatrix}_{(x_1,x_2)=(0,0)} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

Hence the eigenvalues of $\mathbf{A}$ are $-\frac{1}{2}$ and 0. Since both eigenvalues have non-positive real parts and the algebraic multiplicity of the zero eigenvalue is 1, the linearized system is stable in the sense of Lyapunov stability.

(b) Define $h(x_1, x_2) = -x_1^2 - x_2 - 1$. Then $h(x_1, x_2) > 0$ for all $(x_1, x_2) \in S$. The time derivative of h for all $(x_1, x_2) \in S$ is

$$\dot{h} = -2x_1\dot{x}_1 - \dot{x}_2 = -x_1^2 - 2x_1x_2 + 2x_1 \tan^{-1} x_1 + 2x_1x_2 + x_2^2$$

Because $x_1 \tan^{-1} x_1 \geq 0$ for all x_1 and $-x_1^2 > x_2 + 1$ for all $(x_1, x_2) \in S$, we have

$$\dot{h} > x_2 + 1 + x_2^2 \geq \frac{3}{4} > 0, \quad \forall (x_1, x_2) \in S$$

Thus $h(x_1, x_2) > 0$ and $\dot{h}(x_1, x_2) > 0$ for all $(x_1, x_2) \in S$. This implies that once the state is in S, h keeps increasing and is always positive; hence the state will not leave S. Then we conclude that S is a positively invariant set.

(c) Let $B_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 1\}$ be the unit circle on the phase plane, and $V(x_1, x_2) = -x_1^2 - x_2$. Define $U = \{x \in B_1 | V(x_1, x_2) > 0\}$. Clearly U is non-empty since there exists $(x_1, x_2) = (0, -\varepsilon) \in U$ for arbitrarily small $\varepsilon > 0$. Besides, $V(0, 0) = 0$. Then

$$\dot{V} = -2x_1\dot{x}_1 - \dot{x}_2 = -x_1^2 - 2x_1x_2 + 2x_1 \tan^{-1} x_1 + 2x_1x_2 + x_2^2 = -x_1(x_1 - 2\tan^{-1} x_1) + x_2^2$$

Define $f(x_1) = x_1 - 2\tan^{-1} x_1$. Then $f(0) = 0$ and $f'(x_1) = 1 - \frac{2}{1+x_1^2} = \frac{-1+x_1^2}{1+x_1^2} < 0$ for all $|x_1| < 1$. This implies that $f(x_1)$ is a strictly decreasing function in $(-1, 1)$. In other words, $f(x_1) < f(0) = 0$ for $x_1 \in (0, 1)$ and $f(x_1) > f(0) = 0$ for $x_1 \in (-1, 0)$, or equivalently, $x_1 f(x_1) < 0$ for $|x_1| < 1$ and $x_1 \neq 0$. Therefore, $\dot{V} > 0$ for $(x_1, x_2) \in U$. By Chetaev's theorem, $(x_1, x_2) = (0, 0)$ is an unstable equilibrium point.

3. (a) The time derivative of V is

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + 2x_2\dot{x}_2 = -(1 + x_1^4)x_1^2 - 2x_1^2x_2 \sin x_1 - 2x_2^2 + 2x_1^2x_2 \sin x_1 + \frac{2x_2^2}{1 + x_2^2}e^{-t} \\ &\leq -x_1^2 - 2x_2^2 + 2e^{-t} = -2V + 2e^{-t}\end{aligned}$$

Define $\dot{u} = -2u + 2e^{-t}$ and $u(0) = V(x(0)) = \frac{1}{2} + 1 = \frac{3}{2}$. Then $u(t) = c_1e^{-2t} + c_2e^{-t}$, where c_1 and c_2 are coefficients to be determined. Differentiating u results in

$$\dot{u} = -2c_1e^{-2t} - c_2e^{-t} = -2(c_1e^{-2t} + c_2e^{-t}) + 2e^{-t} \Rightarrow -c_2 = -2c_2 + 2 \Rightarrow c_2 = 2$$

On the other hand, $u(0) = c_1 + c_2 = c_1 + 2 = \frac{3}{2} \Rightarrow c_1 = -\frac{1}{2}$. Hence by the comparison lemma,

$$V(x(t)) \leq u(t) = -\frac{1}{2}e^{-2t} + 2e^{-t}, \quad \forall t \geq 0$$

(b)

$$\|x(t)\|_2^2 = x_1^2(t) + x_2^2(t) \leq x_1^2(t) + 2x_2^2(t) = 2V(x(t)) \leq 2u(t) = -e^{-2t} + 4e^{-t}, \quad \forall t \geq 0$$

Then

$$\|x(t)\|_2 \leq \sqrt{-e^{-2t} + 4e^{-t}} \leq e^{-t} + 2e^{-\frac{t}{2}} = b(t)$$

Remark: You can choose either $b(t) = e^{-t} + 2e^{-\frac{t}{2}}$ *or* $b(t) = \sqrt{-e^{-2t} + 4e^{-t}}$ *as your answer.*

4. (a) Note that $1 \leq g(t) = 2+\cos(t) \leq 3$ for all t and $1-\cos x_1 \geq 0$ for all x_1. Therefore

$$(1-\cos x_1) + \frac{1}{2}x_2^2 \leq V(x_1,x_2) = g(t)(1-\cos x_1) + \frac{1}{2}x_2^2 \leq 3(1-\cos x_1) + \frac{1}{2}x_2^2$$

where $W_1(x_1,x_2) = (1-\cos x_1) + \frac{1}{2}x_2^2$ and $W_2(x_1,x_2) = 3(1-\cos x_1) + \frac{1}{2}x_2^2$ are positive definite for $|x_1| < \pi$ and $x_2 \in \mathbb{R}$. Therefore V is positive definite and decrescent.

(b) The time derivative of V in part (a) is

$$\begin{aligned}
\dot{V} &= g(t)\dot{x}_1 \sin x_1 + \dot{g}(t)(1-\cos x_1) + x_2\dot{x}_2 \\
&= -g(t)\left(\tan\frac{x_1}{2} - x_2\right)\sin x_1 + \dot{g}(t)(1-\cos x_1) - g(t)x_2 \sin x_1 - x_2^2
\end{aligned}$$

According to the properties of triangular functions: $\tan\frac{x_1}{2} = \frac{\sin\frac{x_1}{2}}{\cos\frac{x_1}{2}}$, $\sin x_1 = 2\sin\frac{x_1}{2}\cos\frac{x_1}{2}$, and $1-\cos x_1 = 2\sin^2\frac{x_1}{2}$, we have

$$\dot{V} = -2\sin^2\frac{x_1}{2}\big(g(t)-\dot{g}(t)\big) - x_2^2$$

Since $g(t)-\dot{g}(t) = 2+\cos(t)+\sin(t)$ for all t, there exists $c > 0$ such that $g(t)-\dot{g}(t) > c$ for all t. Therefore,

$$\dot{V} < -2c\sin^2\frac{x_1}{2} - x_2^2 = -W_3(x_1,x_2)$$

where $W_3(x_1,x_2) = 2c\sin^2\frac{x_1}{2} + x_2^2$ is positive definite for $|x_1| < \pi$ and $x_2 \in \mathbb{R}$. Thus $(x_1,x_2) = (0,0)$ is a uniformly asymptotically stable equilibrium point.

5. (a) Define $V(\mathbf{z}) = \mathbf{e}^T\mathbf{P}\mathbf{e} + \theta^T\mathbf{\Gamma}^{-1}\theta$. Then

$$V \leq \lambda_{max}(\mathbf{P})\|\mathbf{e}\|^2 + \lambda_{max}(\mathbf{\Gamma}^{-1})\|\theta\|^2 \leq \max\{\lambda_{max}(\mathbf{P}), \lambda_{max}(\mathbf{\Gamma}^{-1})\}\|\mathbf{z}\|^2 = k_2\|\mathbf{z}\|^2$$

where

$$k_2 = \max\{\lambda_{max}(\mathbf{P}), \lambda_{max}(\mathbf{\Gamma}^{-1})\} = \max\left\{\lambda_{max}(\mathbf{P}), \frac{1}{\lambda_{min}(\mathbf{\Gamma})}\right\} \tag{1}$$

$\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the maximum and minimum eigenvalues of a matrix, respectively.

Similarly,

$$k_1\|\mathbf{z}\|^2 = \min\{\lambda_{min}(\mathbf{P}), \lambda_{min}(\mathbf{\Gamma}^{-1})\}\|\mathbf{z}\|^2 \leq V$$

where

$$k_1 = \min\{\lambda_{min}(\mathbf{P}), \lambda_{min}(\mathbf{\Gamma}^{-1})\} = \min\left\{\lambda_{min}(\mathbf{P}), \frac{1}{\lambda_{max}(\mathbf{\Gamma})}\right\} \tag{2}$$

Moreover,

$$\begin{aligned}
\dot{V} &= \mathbf{e}^T\mathbf{P}\big(\mathbf{A}\mathbf{e} + \mathbf{b}\phi^T\theta\big) + \big(\mathbf{A}\mathbf{e} + \mathbf{b}\phi^T\theta\big)^T\mathbf{P}\mathbf{e} - 2\theta^T\big(\phi\mathbf{b}^T\mathbf{P}\mathbf{e} + \mathbf{K}\theta\big) \\
&= \mathbf{e}^T\big(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\big)\mathbf{e} + 2\theta^T\phi\mathbf{b}^T\mathbf{P}\mathbf{e} - 2\theta^T\phi\mathbf{b}^T\mathbf{P}\mathbf{e} - 2\theta^T\mathbf{K}\theta \\
&= -\mathbf{e}^T\mathbf{Q}\mathbf{e} - 2\theta^T\mathbf{K}\theta \\
&\leq -\lambda_{min}(\mathbf{Q})\|\mathbf{e}\|^2 - 2\lambda_{min}(\mathbf{K})\|\theta\|^2 \\
&\leq -\min\{\lambda_{min}(\mathbf{Q}), 2\lambda_{min}(\mathbf{K})\}\|\mathbf{z}\|^2 \\
&= -k_3\|\mathbf{z}\|^2
\end{aligned}$$

where

$$k_3 = \min\{\lambda_{min}(\mathbf{Q}), 2\lambda_{min}(\mathbf{K})\} \tag{3}$$

Therefore, $\mathbf{z} = \mathbf{0}$ is an exponentially stable equilibrium point.

(b) Since $V \leq k_2\|\mathbf{z}\|^2$, we have $\|\mathbf{z}\|^2 \geq \frac{V}{k_2}$. Therefore,

$$\dot{V} \leq -k_3\|\mathbf{z}\|^2 \leq -\frac{k_3}{k_2}V$$

By the comparison lemma, we have

$$V(\mathbf{z}(t)) \leq V(\mathbf{z}(0))e^{-\frac{k_3}{k_2}t}, \quad \forall t \geq 0$$

Since $k_1\|\mathbf{z}(t)\|^2 \leq V(\mathbf{z}(t))$,

$$\|\mathbf{z}(t)\| \leq \sqrt{\frac{V(\mathbf{z}(0))}{k_1}}e^{-\frac{k_3}{2k_2}t} = ce^{-\lambda t}, \quad \forall t \geq 0$$

where

$$c = \sqrt{\frac{V(\mathbf{z}(0))}{k_1}}, \quad \lambda = \frac{k_3}{2k_2}$$

and $V(\mathbf{z}(0)) = \mathbf{e}^T(0)\mathbf{P}\mathbf{e}(0) + \theta^T(0)\mathbf{\Gamma}^{-1}\theta(0)$, k_1, k_2, k_3 are defined in (1)-(3).