Note that $\dot{V}(x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$ for all $x \in D$ and all $t \ge t_0$. Therefore, any trajectory starting at (t_0, x_0), where $x_0 \in \{x \in B_r |\ W_2(x) \le c\} \subset \Omega_{t_0,c}$ stays in $\Omega_{t,c}$ for all $t \ge t_0$. Moreover

$$V(t, x(t)) \le V(t_0, x(t_0)), \quad \forall t \ge t_0$$

By Lemma 2, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \le W_1(x) \le V(t,x) \le W_2(x) \le \alpha_2(\|x\|), \quad \forall t \ge t_0$$

後面証明有用到，标起来好找

Hence

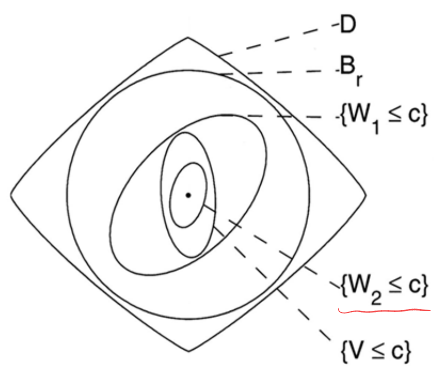
$$\|x(t)\| \le \alpha_1^{-1}\big(V(t,x)\big) \le \alpha_1^{-1}\big(V(t_0, x(t_0))\big) \le \alpha_1^{-1}\big(\alpha_2(\|x(t_0)\|)\big), \quad \forall t \ge t_0$$

By Lemma 1 we see that $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function. The inequality $\|x(t)\| \le \alpha_1^{-1}\big(\alpha_2(\|x(t_0)\|)\big)$ and Lemma 4 show that $x = 0$ is uniformly stable.

Q.E.D.

若t随時间流逝，$\Omega_{t,c}$ 会越来越小，

若 Ω_t 的面積 $\to 0$ as $t \to \infty$ $\Rightarrow$ 对於某特定的 C，總有一天 Ω_t 会掉到 B_c 内，使 C 不再保証 $\forall \|x_0\| \le c \Rightarrow \|x(t)\| \le \varepsilon$



表示，无論 t 怎么變，V 都会在 W_2 外側

$\Rightarrow$ 取 B_δ 在 W_2 内，即可保証 $V \le c$ 永遠不会縮小縮到 B_δ 内

Figure 1: Geometric representation of sets in the proof of Lyapunov stability theorem for nonautonomous systems

Theorem 2. *Suppose the assumptions of Theorem 1 are satisfied with inequality (6) strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x)$$

与 theorem 差在，一个在 $\forall x$，$\dot{V} \le 0$
一个是 $\forall x \neq 0$，$\dot{V} < 0$

$\forall t \ge 0$ and $\forall x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then $x = 0$ is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r |\ W_2(x) \le c\}$ satisfies

再加上 radially unbounded

$$\|x(t)\| \le \beta\big(\|x(t_0)\|, t - t_0\big), \quad t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

Proof: Continuing with the proof of Theorem 1, we know that trajectories starting in $\{x \in B_r |\ W_2(x) \le c\}$ stay in $\{x \in B_r |\ W_1(x) \le c\}$ for all $t \ge t_0$. By Lemma 2, there exists a class $\mathcal{K}$ function α_3, defined on $[0, r]$, such that

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x) \le -\alpha_3(\|x\|)$$

大家的証明流程大多都是 ① 由 $\dot{V} =$ (与x有关) ≤ 0 or a negative definite function
② 用一些不等式，将 $\dot{V}$ 化成 $\dot{V} \le$ (与V有关)
③ 用 comprasion lemma 去得 V 的 upper bound
④ 再用一些不等式，把 V 的 upper bound 化成 class $\mathcal{KL}$ function，$t \to \infty$ $V \to 0$, QED.

Using the inequality

$$V \le \alpha_2(\|x\|) \Leftrightarrow \alpha_2^{-1}(V) \le \|x\| \Leftrightarrow \alpha_3(\alpha_2^{-1}(V)) \le \alpha_3(\|x\|)$$

we see that V satisfies the differential inequality

$$\underline{\dot{V} \le -\alpha_3(\alpha_2^{-1}(V)) \triangleq -\alpha(V)}$$

用 comparison lemma

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0, r]$. Without loss of generality, we can assume that α is locally Lipschitz. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\underline{\dot{y} = -\alpha(y), \quad y(t_0) = V(t_0, x(t_0)) \ge 0}$$

By the comparison lemma and Lemma 3, there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, r] \times [0, \infty)$ such that

$$V(t, x(t)) \le \sigma\big(V(t_0, x(t_0)), t - t_0\big), \quad \forall V(t_0, x(t_0)) \in [0, c]$$

Therefore, any solution starting in $\{x \in B_r | \, W_2(x) \le c\}$ satisfies the inequality

$\because \alpha_1(\|x\|) \le W_1(x) \le V(t,x)$

$\Rightarrow \|x\| \le \alpha_1^{-1}(V(t,x))$

$$\begin{aligned} \|x(t)\| &\le \alpha_1^{-1}\big(V(t, x(t))\big) \le \alpha_1^{-1}\big(\sigma(\underline{V(t_0, x(t_0))}, t - t_0)\big) \\ &\le \alpha_1^{-1}\big(\sigma(\underline{\alpha_2(\|x_0\|)}, t - t_0)\big) \triangleq \beta(\|x(t_0)\|, t - t_0) \end{aligned}$$

$V(t_0, x(t_0)) \le \alpha_2(\|x_0\|)$

Since β is a class $\mathcal{KL}$ function (Lemma 1), $x = 0$ is uniformly asymptotically stable (Lemma 4). If $D = \mathbb{R}^n$, the functions α_1, α_2, and α_3 are defined on $[0, \infty)$. Hence α, and consequently β, are independent of c. As $W_1(x)$ is radially unbounded, c can be chosen arbitrarily large to include any initial state in $\{W_2(x) \le c\}$. Thus (4) holds for any initial state, showing that the origin is globally uniformly asymptotically stable.

Theorem 3. *Let $x = 0$ be an equilibrium point for (1) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$, Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\underline{k_1\|x\|^a} \le V(t, x) \le \underline{k_2\|x\|^a}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -k_3\|x\|^a$$

若 W_1, W_2 是 $\|x\|^a$ 型式
$\Rightarrow$ system 会是 exponentially stable
就是一个 stable 的特例

$\forall t \ge 0$ and $\forall x \in D$, where k_1, k_2, k_3, and a are positive constants. Then $x = 0$ is exponentially stable. If the assumptions hold globally, then $x = 0$ is globally exponentially stable.

可以先凑出 quadratic form，再用
$x^T Q x \le \lambda_{max}\|x\|^2$ 去化出 $k_2\|a\|^a$ 即可
同理 $k_1\|x\|^2$

Proof: See the text book.

Example 3. *Consider the system*

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \le g(t) \le k \quad \textit{and} \quad \dot{g}(t) \le g(t), \quad \forall t \ge 0$$

Taking $V(t, x) = x_1^2 + (1 + g(t))x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \le V(t, x) \le x_1^2 + (1 + k)x_2^2, \quad \forall x \in \mathbb{R}^2$$

Hence, $V(t,x)$ is positive definite, decrescent, and radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

Using the inequality

$$2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2$$

we obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq -x^TQx$$

where Q is positive definite; therefore $\dot{V}(t,x)$ is negative definite. Thus all the assumptions of Theorem 2 are satisfied globally with positive definite quadratic functions W_1, W_2, and W_3.

会是 exponentially stable

Note that

$$\|x\|_2^2 \leq V(t,x) \leq (1+k)\|x\|_2^2$$

and

$$\dot{V}(t,x) \leq -x^TQx \leq -\lambda_{min}(Q)\|x\|_2^2$$ negative definite

We see that the conditions of Theorem 3 are satisfied globally with $a = 2$. Hence the origin is globally exponentially stable.

3 Linear Time-Varying Systems and Linearization

Consider the following linear time-varying system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0 \tag{7}$$

where $A(t) \in \mathbb{R}^{n\times n}$ for all $t \geq t_0$. The solution of (7) can be represented by means of the *state transition matrix* $\Phi(t,t_0) \in \mathbb{R}^n$:

$$x(t) = \Phi(t,t_0)x_0$$

Therefore the stability of the equilibrium point $x = 0$ can be determined by $\Phi(t,t_0)$.

Theorem 4. *The equilibrium point $x = 0$ of (7) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality*

$$\|\Phi(t,t_0)\| \leq ke^{-\lambda(t-t_0)}, \quad \forall t \geq t_0$$

for some positive constants k and λ.

Proof: See the text book.

Remark 2.

1. *Theorem 4 implies that for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability, since if $x = 0$ is uniformly asymptotically stable, then*

$$\|x(t)\| \leq \|\Phi(t,t_0)\|\|x_0\| \leq k\|x_0\|e^{-\lambda(t-t_0)}$$

Hence $x = 0$ is also exponentially stable.

(观念：一个持續存在的 disturbance，系統是否有能力加 feedback 使 state still bounded)？待讀

4 Boundedness and Ultimate Boundedness

In this section, we will use Lyapunov analysis method to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin. Firstly, we give definitions about various kinds of boundedness.

Definition. The solutions to (1) are

- *uniformly bounded* if there exists a positive constant c, independent of $t_0 > 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that

☆ boundedness：先给定 I.C. 的 ball $\xrightarrow{\text{去找}}$ $x(t)$ 的 ball

stability：先給 $\|x(t)\| \leq \varepsilon$ 的 ball $\xrightarrow{\text{去找}}$ I.C 的 δ ball

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq \beta, \quad \forall t \geq t_0 \tag{9}$$

- *globally uniformly bounded* if (9) holds for arbitrarily large a.
- *uniformly ultimately bounded with ultimate bound b* if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that

ultimate 是保証在一 finite time 後，会 bounded 在一个 β 之中

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq b, \quad \underline{\forall t \geq t_0 + T} \tag{10}$$

而 $t_0 \sim t_0+T$ 之间，他不保証会不会在 β 内

- *globally uniformly ultimately bounded* if (10) holds for arbitrarily large a.

In the case of autonomous systems, we may drop the word "uniformly" since the solution depends only on $t - t_0$.

Theorem 7. *Let $D \in \mathbb{R}^n$ be a domain that contains the origin and $V : \overset{\text{time}}{\underline{[0, \infty)}} \times \overset{x}{\underline{D}} \to \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1\big(\|x\|\big) \leq V(t, x) \leq \alpha_2\big(\|x\|\big)$$

可能只收到 $\|x\| = \mu$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \underline{\forall \|x\| \geq \mu > 0}$$

不一定会收到 $x=0$，但至少永遠不会再跑出 μ 之外了

$\forall t \geq 0$ and $\forall x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}\big(\alpha_1(r)\big)$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}\big(\alpha_1(r)\big)$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of (1) satisfies

$$\|x(t)\| \leq \beta\big(\|x(t_0)\|, t - t_0\big), \qquad \forall t_0 \leq t \leq t_0 + T \tag{11}$$

$$\|x(t)\| \leq \underbrace{\alpha_1^{-1}\big(\alpha_2(\mu)\big)}_{\text{ultimate bound}}, \qquad \forall t \geq t_0 + T \tag{12}$$

Moreover, if $D = \mathbb{R}^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (11) and (12) hold for any initial state $x(t_0)$, with no restriction on how large μ is.

Proof: Let $\rho = \alpha_1(r)$. Since $\mu < \alpha_2^{-1}\big(\alpha_1(r)\big)$ and $\|x(t_0)\| \leq \alpha_2^{-1}\big(\alpha_1(r)\big)$ we have $\alpha_2(\mu) < \rho$ and $\alpha_2\big(\|x(t_0)\|\big) \leq \rho$. Let $\eta = \alpha_2(\mu)$ and define $\Omega_{t,\eta} = \{x \in B_r |\; V(t, x) \leq \eta\}$ and $\Omega_{t,\rho} = \{x \in B_r |\; V(t, x) \leq \rho\}$. Then

$$\begin{aligned}
\|x(t)\| \leq \mu \;&\Rightarrow\; V(t, x(t)) \leq \alpha_2(\|x(t)\|) \leq \alpha_2(\mu) = \eta \Rightarrow x \in \Omega_{t,\eta} \\
V(t, x(t)) \leq \eta \;&\Rightarrow\; \alpha_1\big(\|x\|\big) \leq \eta < \rho \Rightarrow x \in \{x \in B_r |\; \alpha_1(\|x\|) \leq \eta\} \\
\alpha_1\big(\|x\|\big) \leq \rho \;&\Leftrightarrow\; \|x\| \leq \alpha_1^{-1}(\rho) = r \Leftrightarrow x \in B_r
\end{aligned}$$