Nonlinear System Theory
Solution to Homework 5

1. (a) The linearized system is $\dot{x} = A(t)x$, where

$$A(t) = \begin{bmatrix} -3k_2x_1^2 & k_1 \sin t \\ -k_1 \sin t & -3k_2x_2^2 \end{bmatrix}_{(x_1=0,x_2=0)} = \begin{bmatrix} 0 & k_1 \sin t \\ -k_1 \sin t & 0 \end{bmatrix}$$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be a Lyapunov function candidate for the linearized system. Then

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = k_1(\sin t)x_1x_2 - k_1(\sin t)x_1x_2 = 0$$

This shows that solutions starting on the surface $V(x) = c$ remain on that surface for all t. Hence $x = 0$ is not an exponentially stable equilibrium point of the linearized system.

Alternatively, we can solve the linearized system directly. The solution to $\dot{x} = A(t)x$ is

$$x(t) = \exp\left[\int_0^t A(\tau)d\tau\right]x(0)$$

and

$$\int_0^t A(\tau)d\tau = \begin{bmatrix} 0 & \int_0^t k_1 \sin\tau d\tau \\ -\int_0^t k_1 \sin\tau d\tau & 0 \end{bmatrix} = \begin{bmatrix} 0 & k_1(1-\cos t) \\ -k_1(1-\cos t) & 0 \end{bmatrix}$$

It can be observed that $\exp[\int_0^t A(\tau)d\tau]$ consists of the term $e^{jk_1(1-\cos t)}$ which does not converge to zero as $t \to \infty$; hence $x(t)$ does not converge to zero, implying that $x = 0$ is not an exponentially stable equilibrium point.

(b) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be a Lyapunov function candidate. Note that $V(x)$ is positive definite and decrescent. Then

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = k_1(\sin t)x_1x_2 - k_2x_1^4 - k_1(\sin t)x_1x_2 - k_2x_2^4 = -k_2(x_1^4 + x_2^4)$$

Since $k_2(x_1^4 + x_2^4)$ is positive definite, $\dot{V}$ is negative definite, which implies that $x = 0$ is a uniformly asymptotically stable equilibrium point.

2. (a) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be a Lyapunov function candidate. Then

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -2x_1^2 + x_1x_2 - x_1x_2 + x_2^2 - ax_2^4 = -2x_1^2 - x_2^2(ax_2^2 - 1)$$

Let $\theta > 0$; we can rewrite the expression of $\dot{V}$ as

$$\dot{V} = -2x_1^2 - \theta x_2^2 - x_2^2\big(ax_2^2 - (1+\theta)\big)$$

Note that

$$\dot{V} \leq -2x_1^2 - \theta x_2^2, \quad \text{if } \|x\| \geq |x_2| > \sqrt{\frac{1+\theta}{a}} \quad \text{for any } \theta > 0$$

Hence $x(t)$ is ultimately bounded.

(b) There exist class $\mathcal{K}$ functions α_1 and α_2 such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$. Since $V(x) = \frac{1}{2}\|x\|_2^2$, we can choose $\alpha_1(\|x\|) = \alpha_2(\|x\|) = V(x)$. Let $\mu = \sqrt{\frac{1+\theta}{a}}$. Then the ultimate bound is $b = \alpha_1^{-1}\big(\alpha_2(\mu)\big) = \mu = \sqrt{\frac{1+\theta}{a}}$.

3. (a) The Jacobian matrix around $x = 0$ is

$$A = \begin{bmatrix} -1 & -2x_2 \\ \delta(t) + x_2 & -2 + x_1 \end{bmatrix}\Bigg|_{x=0} = \begin{bmatrix} -1 & 0 \\ \delta(t) & -2 \end{bmatrix}$$

Now we show that $x = 0$ is an exponentially stable equilibrium point of the linearized system $\dot{x} = A(t)x$. Let $V(x) = \frac{a}{2}x_1^2 + \frac{1}{2}x_2^2$ be a Lyapunov function candidate for some $a > 0$. Then

$$\begin{aligned} \dot{V} &= -ax_1^2 + x_2\big(\delta(t)x_1 - 2x_2\big) \\ &\leq -ax_1^2 + k|x_1||x_2| - 2x_2^2 \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} a & -\frac{k}{2} \\ -\frac{k}{2} & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \end{aligned}$$

Choose $a > \frac{k^2}{8}$; then $Q = \begin{bmatrix} a & -\frac{k}{2} \\ -\frac{k}{2} & 2 \end{bmatrix}$ is positive definite and therefore $\dot{V}$ is negative definite. This implies that $x = 0$ is an exponentially stable equilibrium point of the linearized system.

(b) Consider the nonlinear system and let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Then

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1x_2^2 + \delta(t)x_1x_2 - 2x_2^2 + x_1x_2^2 + x_2\cos(t) \\ &\leq -x_1^2 + |x_1||x_2| - 2x_2^2 + x_2\cos(t) \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} + x_2\cos(t) \\ &\leq -\lambda_{min}\|x\|_2^2 + \|x\|_2 \\ &= -\|x\|_2\big(\lambda_{min}\|x\|_2 - 1\big) \end{aligned}$$

where λ_{min} is the minimum eigenvalue of $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$, which is $\lambda_{min} = \frac{3-\sqrt{2}}{2} = 0.7929$. Hence $\dot{V} < 0$ for $\|x\|_2 > \frac{1}{\lambda_{min}} = \mu = 1.2612$. Therefore, the nonlinear system is uniformly ultimately bounded.

(c) Choose $\alpha_1(\|x\|) = \alpha_2(\|x\|) = \frac{1}{2}\|x\|^2 = V(x)$. $\alpha_1(\|x\|)$ and $\alpha_2(\|x\|)$ are class $\mathcal{K}$ functions and $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$. Then the ultimate bound is $\alpha_1^{-1}(\alpha_2(\mu)) = \mu = 1.2612$.