

# Nonlinear System Theory

## Homework 3

Due date: 3/22/2022

1. Consider an  $n$ -joint robot which satisfies the following dynamic equation:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + B\dot{q} = \tau \quad (1)$$

where  $q, \dot{q}, \ddot{q}, \tau \in \mathbb{R}^n$  denote the joint angle, joint velocity, joint acceleration, and joint torque, respectively.  $M(q), C(q, \dot{q}) \in \mathbb{R}^{n \times n}$  are the inertia matrix and the Coriolis and centrifugal matrix, respectively.  $G(q) \in \mathbb{R}^n$  is the gravitational vector.  $B = \text{diag}(b_1, b_2, \dots, b_n) \in \mathbb{R}^{n \times n}$ , where  $b_i > 0$ , for  $i = 1, 2, \dots, n$ , is the viscous friction coefficient. It is well-known in robotics that the following properties hold for all  $q$  and  $\dot{q}$ .

- $M(q)$  is symmetric positive definite.
- $\frac{1}{2}\dot{M}(q) - C(q, \dot{q})$  is skew-symmetric, i.e.  $x^T [\frac{1}{2}\dot{M}(q) - C(q, \dot{q})]x = 0$  for all  $x \in \mathbb{R}^n$ .

Suppose that the desired joint angle  $q_d \in \mathbb{R}^n$  is a constant vector, i.e. we are considering the set-point control problem, and the following *PD control law with gravity compensation* is implemented:

$$\tau = K_P \tilde{q} + K_D \dot{\tilde{q}} + G(q) \quad (2)$$

where  $\tilde{q} = q_d - q$ ;  $K_P, K_D \in \mathbb{R}^{n \times n}$  are symmetric positive definite.

- (a) Show that  $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$  is the unique equilibrium point of the closed-loop system consisting of (1) and (2).
- (b) Use  $V(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_P \tilde{q}$  as the Lyapunov function candidate to show that the equilibrium point is stable.
- (c) Use LaSalle Theorem to show that the equilibrium point is asymptotically stable.

2. Show that the origin of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{aligned}$$

is globally asymptotically stable.

3. Consider the following scalar differential equation

$$\dot{x} = -(1 + x^2) \tan^{-1} x + r(t), \quad x(0) = x_0$$

- (a) Let  $r(t) \equiv 0$ . Check whether the equilibrium point  $x = 0$  is (asymptotically) stable or unstable.
- (b) Let  $r(t) = e^{-2t}$ . Find an upper bound  $y(t)$  such that  $x(t) \leq y(t)$  for all  $t \geq 0$ .

4. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_1\left(\frac{\pi}{2} - x_1\right) + \frac{x_1 \sin x_2}{10} \\ \dot{x}_2 &= x_1 \cos x_1 - \frac{x_2^3}{10}\end{aligned}$$

- (a) Show that  $(x_1, x_2) = (\frac{\pi}{2}, 0)$  is an asymptotically stable equilibrium point.
- (b) Use Chetaev's theorem to show that  $(x_1, x_2) = (0, 0)$  is an unstable equilibrium point.