Nonlinear System Theory
Homework 4

Due date: 4/7/2023

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= (x_1 x_2 - 1)x_2^3 + (x_1 x_2 - 1 + x_1^2)x_2 \end{aligned}$$

   (a) Show that $x = 0$ is the unique equilibrium point.
   (b) Show, by using linearization, that $x = 0$ is asymptotically stable.
   (c) Show that $\Gamma = \{x \in \mathbb{R}^2 | x_1 x_2 \geq 2\}$ is a positively invariant set.
   (d) Is $x = 0$ globally asymptotically stable?

2. Let α be a class $\mathcal{K}$ function on $[0,\, a)$. Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \quad \forall r_1, r_2 \in \big[0,\, a/2\big)$$

3. Consider the system

$$\dot{x} = -a\big(I + S(x) + xx^T\big)x$$

   where $x \in \mathbf{R}^n$, $a > 0$ is a constant, I is the $n \times n$ identity matrix, and $S(x)$ is a state-dependent skew symmetric matrix, i.e. $v^T S(x) v = 0$ for all $v, x \in \mathbf{R}^n$. Show that $x = 0$ is globally asymptotically stable.

4. Consider the following system

$$\begin{aligned} \dot{x}_1 &= k_1(\sin t)x_2 - k_2 x_1^3 \\ \dot{x}_2 &= -k_1(\sin t)x_1 - k_2 x_2^3 \end{aligned}$$

   where $k_1, k_2 > 0$.

   (a) Show that the linearized system around $x = 0$ is NOT exponentially stable.
   (b) Show that $x = 0$ is a uniformly asymptotically stable equilibrium point.

5. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 + g(t) \end{aligned}$$

   for $\|x\| < \frac{\pi}{4}$, where $x = [x_1, x_2]^T$ and $|g(t)| < k < \frac{\pi}{6}$ for some positive constant k. Define $V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2$ to be a Lyapunov function candidate.

   (a) Find class $\mathcal{K}$ functions α_1 and α_2 such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$.
   (b) Show that the system is ultimately bounded and find an ultimate bound.
   Remark: Please give the numerical value of the ultimate bound. Since the answer is not unique, please illustrate how you choose the value.