Nonlinear System Theory
Homework 7

Due date: 5/16/2023

1. Consider the following 2nd-order system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= h(x_1) - 2x_2 + u \\ y &= x_2 - x_1 \end{aligned}$$

   where u and y are the input and output, respectively, and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$.

   (a) Use the storage function with the form $V(x) = \frac{1}{2}x^T P x + \int_0^{x_1} h(\tau)d\tau$ to show that the system is output strictly passive, where $x = [x_1, x_2]^T$ and $P \in \mathbb{R}^{2\times 2}$ is symmetric positive definite. Then find an upper bound of its $\mathcal{L}_2$ gain.

   (b) Show by the Hamilton-Jacobi inequality that the $\mathcal{L}_2$-gain of the system is upper bounded by 2.

   (c) Let $u = 0$. Show that $(x_1, x_2) = (0, 0)$ is an asymptotically stable equilibrium point.

   (d) Consider the output feedback integral control law $u(t) = r(t) - \int_0^t y(\tau)d\tau$, where r is the reference input. Show that the closed-loop system from r to y is finite-gain $\mathcal{L}_2$ stable and find an upper bound of its $\mathcal{L}_2$ gain.

2. Consider the feedback connected system in Figure 1. Let

$$H_1 : \left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - x_2 + 3e_1 \\ y_1 &= x_2 - e_1 \end{aligned} \right. \quad \text{and} \quad H_2 : \left\{ \begin{aligned} \dot{x}_3 &= -kx_3 + e_2 \\ y_2 &= x_3 \end{aligned} \right.$$

   where $k > 2$ and $x_1 h(x_1) > 0$ for all $x_1 \neq 0$.

   (a) Show that the feedback connected system is finite-gain $\mathcal{L}_2$ stable.

   (b) When $u_1 = u_2 = 0$, show that $(x_1, x_2, x_3) = (0, 0, 0)$ is an asymptotically stable equilibrium point.

3. Consider the feedback system in Figure 2 and let $G(s) = \frac{2s}{s^2+s+1}$.

   (a) Show that the feedback system is absolutely stable for $\psi \in [0,\ 1]$.

   (b) Show that the system has no limit cycles with $\psi(y) = \mathrm{sat}(y)$, where $\mathrm{sat}(\cdot)$ denotes the saturation function.

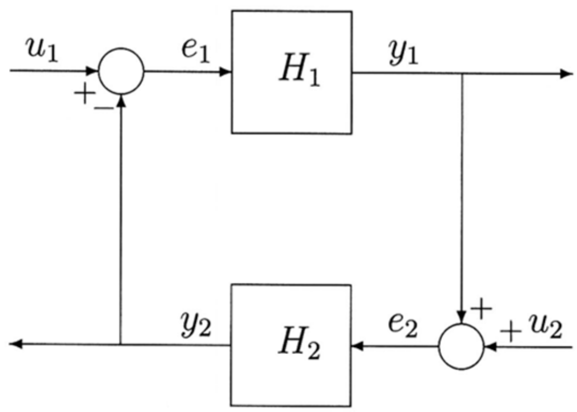

Figure 1: Feedback connection of the nonlinear system in Problem 2

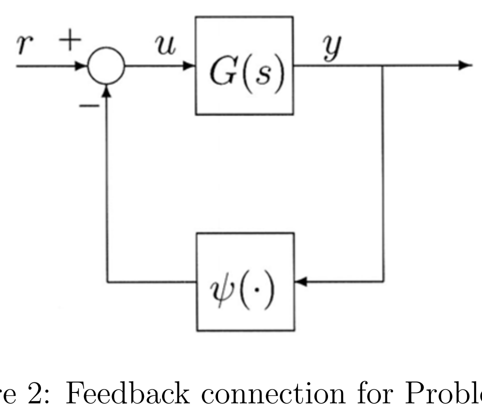

Figure 2: Feedback connection for Problem 3.