Nonlinear System Theory
Solution to Final (2022)

1. (a) Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. P is positive definite if and only if $p_1 > 0$ and $p_1 p_3 > p_2^2$. Then

$$\begin{aligned}
\dot{V} &= (p_1x_1 + p_2x_2)\dot{x}_1 + (p_2x_1 + p_3x_2)\dot{x}_2 + h(x_1)\dot{x}_1 \\
&= (p_1x_1 + p_2x_2 + h(x_1))(-x_1 - x_2) + (p_2x_1 + p_3x_2)(h(x_1) - 2x_2 + u) \\
&= -p_1x_1^2 - (p_1 + 3p_2)x_1x_2 - (p_2 + 2p_3)x_2^2 + h(x_1)(-x_1 - x_2 + p_2x_1 + p_3x_2) \\
&\quad +(p_2x_1 + p_3x_2)u
\end{aligned}$$

Choose $p_3 = 1$, $p_2 = -1$, and $p_1 = 2$. Then P is positive definite, and

$$\begin{aligned}
\dot{V} &= -2x_1^2 + x_1x_2 - x_2^2 - 2x_1h(x_1) + (x_2 - x_1)u \\
&\leq -\frac{1}{2}x_1^2 + x_1x_2 - \frac{1}{2}x_2^2 + (x_2 - x_1)u \\
&\leq -\frac{1}{2}(x_2 - x_1)^2 + (x_2 - x_1)u
\end{aligned}$$

Therefore $uy \geq \dot{V} + \frac{1}{2}y^2$, i.e. the system is output strictly passive. Moreover, it is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$ gain is less than or equal to 2.

(b) Let V be the storage function in part (a) and $W = \alpha V$, where $\alpha > 0$. $f(x) = \begin{bmatrix} -x_1 - x_2 \\ h(x_1) - 2x_2 \end{bmatrix}$ and $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\frac{\partial W}{\partial x} = \alpha x^T P + \alpha \begin{bmatrix} h(x_1) & 0 \end{bmatrix} = \alpha \begin{bmatrix} 2x_1 - x_2 + h & -x_1 + x_2 \end{bmatrix}$$

and the Hamilton-Jacobi inequality is

$$\begin{aligned}
&\alpha\big[(2x_1 - x_2 + h)(-x_1 - x_2) + (-x_1 + x_2)(h - 2x_2)\big] + \frac{\alpha^2}{2\gamma^2}(-x_1 + x_2)^2 + \frac{1}{2}(x_2 - x_1)^2 \\
= \;& \alpha\big(-2x_1^2 + x_1x_2 - x_2^2 - 2x_1h(x_1)\big) + \frac{1}{2}\left(\frac{\alpha^2}{\gamma^2} + 1\right)(x_2 - x_1)^2 \\
\leq \;& -\frac{1}{2}\left(\alpha - \frac{\alpha^2}{\gamma^2} - 1\right)(x_2 - x_1)^2 \leq 0
\end{aligned}$$

The Hamilton-Jacobi inequality holds if $\alpha - \frac{\alpha^2}{\gamma^2} - 1 > 0$ for some $\alpha > 0$ and $\gamma > 0$. This implies that

$$\gamma^2 \geq \frac{\alpha^2}{\alpha - 1} = A(\alpha), \quad \alpha > 1$$

Notice that the derivative of $A(\alpha)$ w.r.t. α vanishes at $\alpha = 2$, which is the minimum point of $A(\alpha)$. Therefore

$$\gamma \geq \left.\frac{\alpha}{\sqrt{\alpha - 1}}\right|_{\alpha=2} = 2$$

Hence the $\mathcal{L}_2$ gain of the system is upper bounded by γ, and the smallest γ is 2.

(c) When $u = 0$, $(x_1, x_2) = (0, 0)$ is an equilibrium point of the system. Suppose that $y \equiv 0$, which implies $x_1 \equiv x_2$, and thus $\dot{x}_1 \equiv \dot{x}_2$. Therefore

$$\dot{x}_1 = -x_1 - x_2 = -2x_1 = \dot{x}_2 = h(x_1) - 2x_2 \Rightarrow h(x_1) = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0$$

This means that the system is zero-state observable. Since the system is output strictly passive and zero-state observable, $(0, 0)$ is an asymptotically stable equilibrium point.

(d) Let $v(t) = \int_0^t y(\tau)d\tau$. Then the state space representation of the integrator is

$$\dot{x}_3 = y \qquad v = x_3$$

Choose $V_c = V + \frac{1}{2}x_3^2$ as the storage function of the feedback system, where V is defined in part (a). Then

$$\dot{V}_c = \dot{V} + x_3\dot{x}_3 \leq -\frac{1}{2}y^2 + uy + vy = -\frac{1}{2}y^2 + ry$$

Hence $ry \geq \dot{V}_c + \frac{1}{2}y^2$, and the system is output strictly passive. Therefore, it is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$-gain is upper bounded by 2.

2. (a) Since $G(s)$ has two unstable poles, the Nyquist plot of $G(j\omega)$ should lie outside the disk $D(\alpha, \beta)$ and encircle it twice in the counterclockwise direction. Hence we locate the center of $D(\alpha, \beta)$ at $(-0.3845, 0)$ with radius 0.3845 (see the green dotted circle in Figure 1). Therefore, $\alpha = \frac{1}{0.3845 \times 2} = 1.30$, and $\beta = \infty$. Hence ψ belongs to the sector $(1.30, \infty)$.

(b) Because $G(s)$ is not stable, the Popov criterion or the circle criterion for $\alpha < 0$ cannot be applied directly to the feedback system. Perform loop transformation and the equivalent system is shown in Figure 2, where $a > 0$. Suppose that $G(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials. Then from Figure 2 we have

$$\tilde{G}(s) = \frac{G(s)}{1 + aG(s)} = \frac{N(s)}{D(s) + aN(s)} = \frac{3s^2 + 9s + 6}{s^3 + 3as^2 + (9a - 7)s + 6(1 + a)}$$

Choose $a = 2$, then $\tilde{G}(s) = \frac{3s^2+9s+6}{s^3+6s^2+11s+18}$, which is stable.

The Problem 2(b) does not require you to find the sector; however, for completeness, this solution continues to find the sector based on the equivalent system.

The Nyquist plot of $\tilde{G}(j\omega)$ is shown in Figure 3, which lies entirely inside the circle intersecting the real-axis at 0 and 1.3. Hence if $\tilde{\psi} \in [-\frac{1}{1.3}, \infty) = [-0.77, \infty)$, then the feedback system is absolutely stable. In other words, if $\psi \in [-0.77 + a, \infty) = [1.23, \infty)$, then the feedback system is absolutely stable.

3. (a) Let the storage function of H_1 be $V_1 = \int_0^{x_1} h(\tau)d\tau + \frac{1}{2}x_2^2$. Then

$$\begin{aligned} \dot{V}_1 &= h(x_1)x_2 + x_2(-h(x_1) - x_2 + 3e_1) = -(y_1 + e_1)^2 + 3(y_1 + e_1)e_1 \\ &= -y_1^2 + 2e_1^2 + y_1e_1 \end{aligned}$$

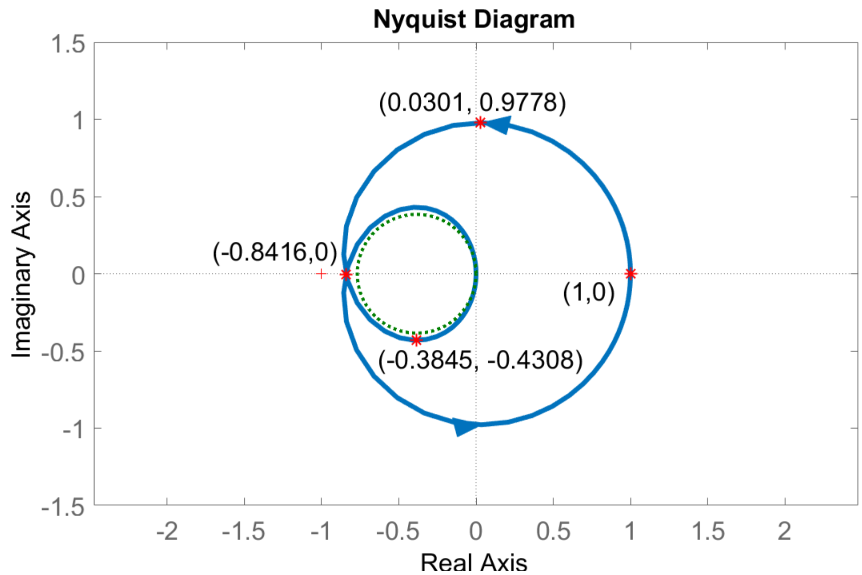

Figure 1: Circle criterion for Problem 2(a)

On the other hand, let $V_2 = \frac{1}{2}x_3^2$ be the storage function of H_2. Then

$$\dot{V}_2 = x_3(-kx_3 + e_2) = -ky_2^2 + y_2 e_2$$

Since $e_1 y_1 = \dot{V}_1 - 2e_1^2 + y_1^2$, $e_2 y_2 = \dot{V}_2 + ky_2^2$ and $k > 2$, the feedback system is finite-gain $\mathcal{L}_2$ stable (by Theorem 2 of Lecture 8).

(b) Perform loop transformation of the feedback connected system as shown in Figure 4. Note that $\tilde{y}_1 = y_1 + e_1$ and $\tilde{e}_2 = e_2 - y_2$. Then the state space representation of $\tilde{H}_1$ and $\tilde{H}_2$ is

$$\tilde{H}_1 : \left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) - x_2 + 3e_1 \\ \tilde{y}_1 & = & x_2 \end{array} \right. \quad \text{and} \quad \tilde{H}_2 : \left\{ \begin{array}{rcl} \dot{x}_3 & = & -(k-1)x_3 + \tilde{e}_2 \\ y_2 & = & x_3 \end{array} \right.$$

Take V_1 and V_2 in part (a). Then

$$\dot{V}_1 = -\tilde{y}_1^2 + 3\tilde{y}_1 e_1$$

Hence $\tilde{H}_1$ is output strictly passive. Moreover, if $e_1 = 0$ and $\tilde{y}_1 = x_2 \equiv 0$, then $x_1 \equiv 0$. This implies that $\tilde{H}_1$ is zero-state observable.

On the other hand,

$$\dot{V}_2 = -(k-1)x_3^2 + y_2 \tilde{e}_2$$

Since $k - 1 > 0$, $\tilde{H}_2$ is strictly passive. Therefore, $(x_1, x_2, x_3) = (0, 0, 0)$ is an asymptotically stable equilibrium point of the feedback connected system.

4. (a)

$$\begin{array}{rcl} \dot{y} & = & \dot{x}_1 = -x_1^3 + x_2 \\ \ddot{y} & = & -3x_1^2(-x_1^3 + x_2) + x_2^2 + u \end{array}$$

Hence the relative degree is 2 over $\mathbb{R}^3$.

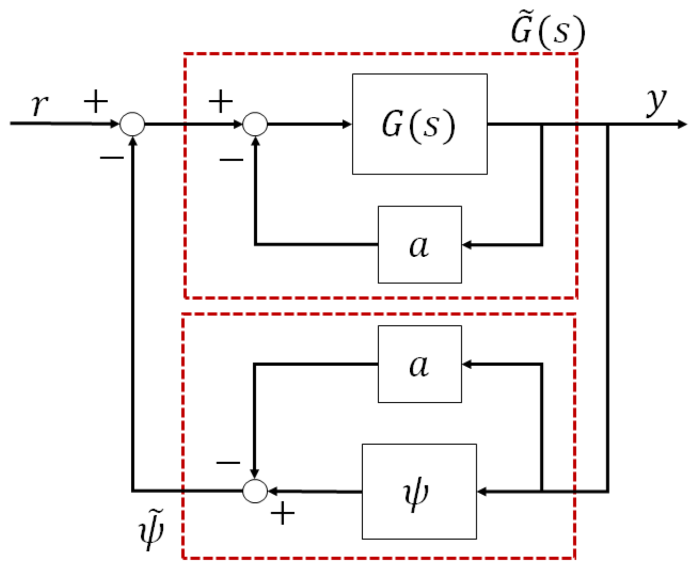

Figure 2: Loop Transformation for Problem 2(b)

(b) To find the zero dynamics, let $y = x_1 \equiv 0$. Then $\dot{y} = \dot{x}_1 \equiv 0$ and therefore $x_2 \equiv 0$, $\dot{x}_2 \equiv 0$, and $u = 0$. As a result, the zero dynamics of the system is $\dot{x}_3 = -\sin x_3$. Let $V(x_3) = 1 - \cos x_3$ for $|x_3| < \pi$. Thus $V(x_3)$ is positive definite. $\dot{V}_3 = -(\sin x_3)^2 < 0$. We conclude that $x_3 = 0$ is an asymptotically stable equilibrium point of the zero dynamics $\dot{x}_3 = -\sin x_3$. In other words, the system is minimum phase.

(c) Let $\eta = \phi(x)$, which should satisfy $\phi(0) = 0$ and $\frac{\partial \phi}{\partial x} g = 0$, where $g = [0, 1, 0]^T$. Therefore $\frac{\partial \phi}{\partial x_2} = 0$. We can choose $\eta = \phi(x) = x_3$. Moreover,

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1^3 + x_2 \end{bmatrix}$$

Then $x_2 = \xi_2 + \xi_1^3$, and the normal form of the system is

$$\begin{aligned}
\dot{\eta} &= -\sin \eta + \xi_2 + \xi_1^3 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -3x_1^2(-x_1^3 + x_2) + x_2^2 + u = -3\xi_1^2 \xi_2 + (\xi_2 + \xi_1^3)^2 + u \\
y &= \xi_1
\end{aligned}$$

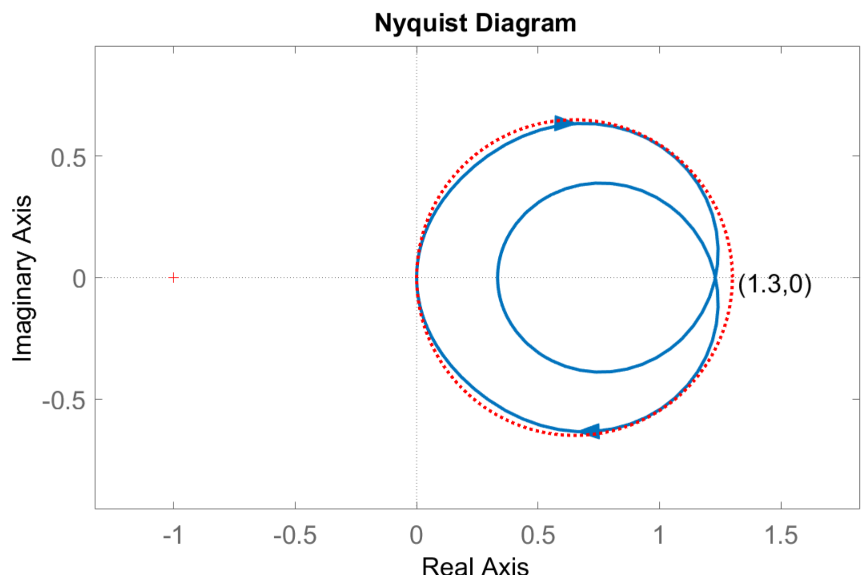

Figure 3: Nyquist plot of $\tilde{G}(j\omega)$ for Problem 2(b)

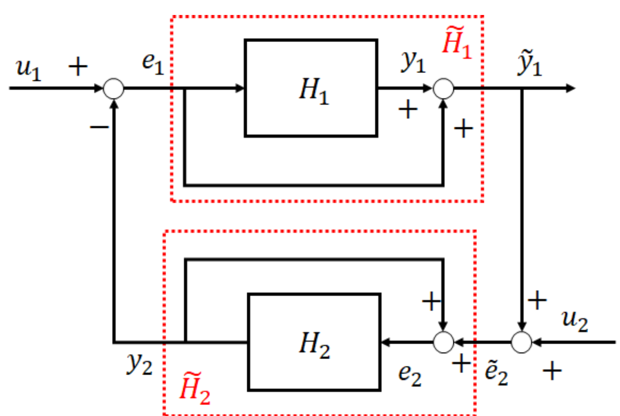

Figure 4: Loop Transformation of the Feedback Connected System in Problem 3

(d) Let $f(x) = \begin{bmatrix} -x_1^3 + x_2 \\ x_2^2 \\ -\sin x_3 + x_2 \end{bmatrix}$ and $g = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then

$$\begin{aligned}
ad_f g &= [f, g] = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} -3x_1^2 & 1 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 1 & -\cos x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2x_2 \\ 1 \end{bmatrix} \\
ad_f^2 g &= [f, ad_f g] \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_1^3 + x_2 \\ x_2^2 \\ -\sin x_3 + x_2 \end{bmatrix} + \begin{bmatrix} -3x_1^2 & 1 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 1 & -\cos x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2x_2 \\ 1 \end{bmatrix} \\
& \begin{bmatrix} -3x_1^2 + 2x_2 \\ 2x_2^2 \\ 2x_2 - \cos x_3 \end{bmatrix} \\
[g, ad_f g] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}
\end{aligned}$$

Note that

$$\begin{aligned}
\det \mathcal{G}(x) &= \det \left[\begin{array}{ccc} g & ad_f g & ad_f^2 g \end{array} \right] = \det \left[\begin{array}{ccc} 0 & -1 & -3x_1^2 + 2x_2 \\ 1 & -2x_2 & 2x_2^2 \\ 0 & -1 & 2x_2 - \cos x_3 \end{array} \right] \\
&= 3x_1^2 - 2x_2 + 2x_2 - \cos x_3 = 3x_1^2 - \cos x_3
\end{aligned}$$

Hence $\mathcal{G}(x)$ is full rank on $D_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 3x_1^2 \neq \cos x_3\}$. On the other hand, the distribution $\mathcal{D} = \text{span}\{g, ad_f g\}$ is nonsingular and

$$\text{rank} \left[\begin{array}{ccc} g & ad_f g & [g, ad_f g] \end{array} \right] = \text{rank} \left[\begin{array}{ccc} 0 & -1 & 0 \\ 1 & -2x_2 & -2 \\ 0 & -1 & 0 \end{array} \right] = 2$$

Hence the distribution $\mathcal{D}$ is involutive in D_0. Consequently, the system is input-state linearizable.

(e) h should satisfy

$$L_g h = 0, \quad L_g L_f h = 0, \quad L_g L_g^2 h \neq 0$$

Notice that $L_g h = \frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$, i.e. h is independent of x_2, so are $\frac{\partial h}{\partial x_1}$ and $\frac{\partial h}{\partial x_3}$. On the other hand,

$$L_g L_f h = \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_3} f_3 \right) g = \frac{\partial h}{\partial x_1} \frac{\partial f_1}{\partial x_2} + \frac{\partial h}{\partial x_3} \frac{\partial f_3}{\partial x_2} = \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_3} = 0$$

Choose $\tilde{y} = h(x) = x_1 - x_3$. Then

$$\begin{aligned}
\dot{\tilde{y}} &= \dot{x}_1 - \dot{x}_3 = -x_1^3 + x_2 + \sin x_3 - x_2 = -x_1^3 + \sin x_3 \\
\ddot{\tilde{y}} &= -3x_1^2(-x_1^3 + x_2) + \cos x_3(-\sin x_3 + x_2) \\
&= 3x_1^5 - 3x_1^2 x_2 - \sin x_3 \cos x_3 + x_2 \cos x_3 \\
\tilde{y}^{(3)} &= (15x_1^4 - 6x_1 x_2)(-x_1^3 + x_2) - (3x_1^2 - \cos x_3)(x_2^2 + u) \\
&\quad + (-\cos^2 x_3 + \sin^2 x_3 - x_2 \sin x_3)(-\sin x_3 + x_2)
\end{aligned}$$

Hence the system has relative degree 3 w.r.t. $\tilde{y} = x_1 - x_3$ over the domain $D_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 3x_1^2 \neq \cos x_3\}$.