Nonlinear System Theory
Solution to Homework 1

1. Note that $\bar{\alpha}_f = \frac{\bar{F}_{yf}}{C_{\alpha f}} = 0.0436$ and $\bar{\alpha}_r = \frac{\bar{F}_{yr}}{C_{\alpha r}} = 0.0404$. We consider different cases to find the equilibrium points.

   - Case I: $|\alpha_f| \le \bar{\alpha}_f$ and $|\alpha_r| \le \bar{\alpha}_r$
     The dynamic equations are

$$
\begin{aligned}
\dot{\beta} &= \frac{2}{Mv_x}\left[C_{\alpha f}\left(\delta_f - \beta - \frac{rl_f}{v_x}\right) + C_{\alpha r}\left(-\beta + \frac{rl_r}{v_x}\right)\right] - r \\
\dot{r} &= \frac{2}{I_z}\left[C_{\alpha f}\left(\delta_f - \beta - \frac{rl_f}{v_x}\right)l_f - C_{\alpha r}\left(-\beta + \frac{rl_r}{v_x}\right)l_r\right]
\end{aligned}
$$

     Rearrange the equation and we have

$$
\begin{aligned}
\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} -\frac{2(C_{\alpha f}+C_{\alpha r})}{Mv_x} & \frac{-2(C_{\alpha f}l_f - C_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{-2(C_{\alpha f}l_f - C_{\alpha r}l_r)}{I_z} & -\frac{2(C_{\alpha f}l_f^2 + C_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{2C_{\alpha f}}{Mv_x} \\ \frac{2C_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\
&= \underbrace{\begin{bmatrix} -17.7183 & -0.9247 \\ 5.1756 & -15.8140 \end{bmatrix}}_{\mathbf{A}_1} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} 0.9275 \\ 5.6595 \end{bmatrix}}_{\mathbf{B}_1}
\end{aligned}
$$

     The equilibrium point is

$$
\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = -\mathbf{A}_1^{-1}\mathbf{B}_1 = \begin{bmatrix} 0.0331 \\ 0.3687 \end{bmatrix}
$$

     Then the corresponding tire slip angles are

$$
\alpha_f^* = -\beta^* - \frac{r^* l_f}{v_x} = 0.0214, \quad \alpha_r^* = -\beta^* + \frac{r^* l_r}{v_x} = 0.0199
$$

     Note that the assumptions of this case ($|\alpha_f| \le \bar{\alpha}_f$ and $|\alpha_r| \le \bar{\alpha}_r$) hold. The eigenvalues of $\mathbf{A}_1$ are $-16.7661 \pm 1.9696j$; hence the equilibrium point is a stable focus.

   - Case II: $|\alpha_f| > \bar{\alpha}_f$ and $|\alpha_r| \le \bar{\alpha}_r$
     Follow the same argument and the dynamic equation is

$$
\begin{aligned}
\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{2(C'_{\alpha f} - C_{\alpha r})}{Mv_x} & \frac{2(C'_{\alpha f}l_f + C_{\alpha r}l_r)}{Mv_x^2} - 1 \\ \frac{2(C'_{\alpha f}l_f + C_{\alpha r}l_r)}{I_z} & \frac{2(C'_{\alpha f}l_f^2 - C_{\alpha r}l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} -\frac{2C'_{\alpha f}}{Mv_x} \\ \frac{-2C'_{\alpha f}l_f}{I_z} \end{bmatrix} \delta_f \\
&\quad + 2\begin{bmatrix} \frac{1}{Mv_x} \\ \frac{l_f}{I_z} \end{bmatrix} (\bar{F}_{yf} + C'_{\alpha f}\bar{\alpha}_f)\mathrm{sgn}(\alpha_f) \\
&= \underbrace{\begin{bmatrix} -6.8775 & 0.0373 \\ 71.3256 & -9.9438 \end{bmatrix}}_{\mathbf{A}_2} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0185 \\ -0.1132 \end{bmatrix} \pm \begin{bmatrix} 0.4732 \\ 2.8872 \end{bmatrix}}_{\mathbf{B}_2}
\end{aligned}
$$

The equilibrium points are

$$\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = \begin{bmatrix} 0.0703 \\ 0.7836 \end{bmatrix} \text{ for } \alpha_f > 0, \text{ or } \begin{bmatrix} -0.0761 \\ -0.8475 \end{bmatrix} \text{ for } \alpha_f < 0$$

For $(\beta^*, r^*) = (0.0703, 0.7836)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-0.0526, 0.0422)$, which violate the assumption that $\alpha_f^* > 0$.

For $(\beta^*, r^*) = (-0.0761, -0.8475)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (0.2386, -0.0456)$, which violate the assumption that $\alpha_f^* < 0$.

Therefore, no equilibrium point is found for this case.

- Case III: $|\alpha_f| \leq \bar{\alpha}_f$ and $|\alpha_r| > \bar{\alpha}_r$
  Follow the same argument and the dynamic equation is

$$\begin{aligned} \begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} -\frac{2(C_{\alpha f} - C'_{\alpha r})}{M v_x} & \frac{-2(C_{\alpha f} l_f + C'_{\alpha r} l_r)}{M v_x^2} - 1 \\ \frac{-2(C_{\alpha f} l_f + C'_{\alpha r} l_r)}{I_z} & -\frac{2(C_{\alpha f} l_f^2 - C'_{\alpha r} l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{2C_{\alpha f}}{M v_x} \\ \frac{2C_{\alpha f} l_f}{I_z} \end{bmatrix} \delta_f \\ &\quad +2 \begin{bmatrix} \frac{1}{M v_x} \\ \frac{-l_r}{I_z} \end{bmatrix} (\bar{F}_{yr} + C'_{\alpha r} \bar{\alpha}_r) \mathrm{sgn}(\alpha_r) \\ &= \underbrace{\begin{bmatrix} -10.4864 & -1.9635 \\ -66.2535 & -5.5539 \end{bmatrix}}_{\mathbf{A}_3} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} 0.9275 \\ 5.6595 \end{bmatrix} \pm \begin{bmatrix} 0.2923 \\ -2.8872 \end{bmatrix}}_{\mathbf{B}_3} \end{aligned}$$

  The equilibrium points are

$$\begin{bmatrix} \beta^* \\ r^* \end{bmatrix} = \begin{bmatrix} -0.0185 \\ 0.7202 \end{bmatrix} \text{ for } \alpha_r > 0, \text{ or } \begin{bmatrix} 0.1845 \\ -0.6617 \end{bmatrix} \text{ for } \alpha_r < 0$$

  For $(\beta^*, r^*) = (-0.0185, 0.7202)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (0.0419, 0.1220)$, which satisfy the assumptions that $|\alpha_f| \leq \bar{\alpha}_f$ and $\alpha_r > \bar{\alpha}_r > 0$.

  For $(\beta^*, r^*) = (0.1845, -0.6617)$, the tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-0.0385, -0.2795)$, which satisfy the assumptions that $|\alpha_f| \leq \bar{\alpha}_f$ and $\alpha_r < -\bar{\alpha}_r < 0$.

  Hence both $(\beta^*, r^*) = (-0.0185, 0.7202)$ and $(\beta^*, r^*) = (0.1845, -0.6617)$ are equilibrium points. They correspond to the same Jacobian matrix $\mathbf{A}_3$, whose eigenvalues are -19.6895 and 3.6491. In other words, the equilibrium points in Case III are saddle points and they are unstable.

- Case IV: $|\alpha_f| > \bar{\alpha}_f$ and $|\alpha_r| > \bar{\alpha}_r$
  Follow the same argument and the dynamic equation is

$$\begin{aligned} \begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{2(C'_{\alpha f} + C'_{\alpha r})}{M v_x} & \frac{2(C'_{\alpha f} l_f - C'_{\alpha r} l_r)}{M v_x^2} - 1 \\ \frac{2(C'_{\alpha f} l_f - C'_{\alpha r} l_r)}{I_z} & \frac{2(C'_{\alpha f} l_f^2 + C'_{\alpha r} l_r^2)}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \frac{-2C'_{\alpha f}}{M v_x} \\ \frac{-2C'_{\alpha f} l_f}{I_z} \end{bmatrix} \delta_f \\ &\quad \pm 2 \begin{bmatrix} \frac{1}{M v_x} \\ \frac{l_f}{I_z} \end{bmatrix} (\bar{F}_{yf} + C'_{\alpha f} \bar{\alpha}_f) \mathrm{sgn}(\alpha_f) \pm 2 \begin{bmatrix} \frac{1}{M v_x} \\ \frac{-l_r}{I_z} \end{bmatrix} (\bar{F}_{yr} + C'_{\alpha r} \bar{\alpha}_r) \mathrm{sgn}(\alpha_r) \\ &= \underbrace{\begin{bmatrix} 0.3544 & -1.0015 \\ -0.1035 & 0.3163 \end{bmatrix}}_{\mathbf{A}_4} \begin{bmatrix} \beta \\ r \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0185 \\ -0.1132 \end{bmatrix} \pm \begin{bmatrix} 0.4732 \\ 2.8872 \end{bmatrix} \pm \begin{bmatrix} 0.2923 \\ -2.8872 \end{bmatrix}}_{\mathbf{B}_4} \end{aligned}$$

For $\alpha_f > 0$ and $\alpha_r > 0$, the equilibrium point is $(\beta^*, r^*) = (-14.6091, -4.4234)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (15.0889, 13.9737)$, which satisfy the assumption that $\alpha_f > \bar{\alpha}_f > 0$ and $\alpha_r > \bar{\alpha}_r > 0$.

For $\alpha_f < 0$ and $\alpha_r < 0$, the equilibrium point is $(\beta^*, r^*) = (42.9594, 14.4176)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-44.1516, -40.8885)$, which satisfy the assumption that $\alpha_f < -\bar{\alpha}_f$ and $\alpha_r < -\bar{\alpha}_r$.

For $\alpha_f > 0$ and $\alpha_r < 0$, the equilibrium point is $(\beta^*, r^*) = (-680.1881, -240.5115)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (701.6183, 645.6410)$, which violate the assumption that $\alpha_f > \bar{\alpha}_f > 0$ and $\alpha_r < -\bar{\alpha}_r$.

For $\alpha_f < 0$ and $\alpha_r > 0$, the equilibrium point is $(\beta^*, r^*) = (708.5384, 250.5057)$. The tire slip angles are $(\alpha_f^*, \alpha_r^*) = (-730.6811, -672.5558)$, which violate the assumption that $\alpha_f < -\bar{\alpha}_f$ and $\alpha_r > \bar{\alpha}_r > 0$.

Therefore, there are two equilibrium points for case IV: $(\beta^*, r^*) = (-14.6091, -4.4234)$ and $(\beta^*, r^*) = (42.9594, 14.4176)$. Since the eigenvalues of $\mathbf{A}_4$ are 0.6579, and 0.0128. These two equilibrium points are unstable nodes.

2. (a) The loop transfer function is

$$L(s) = \frac{K}{s} G(s) = \frac{500K}{s(s^2 + 60s + 500)}$$

The root locus is shown in Figure 1.

(b) For $K = 60$, the characteristic equation of the closed-loop system is

$$0 = s^3 + 60s^2 + 500s + 30000 = (s + 60)(s^2 + 500)$$

Hence the closed-loop poles are $p = -60,$ and $\pm 10\sqrt{5}j$. The step response and the state trajectory are shown in Figure 2 and Figure 3, respectively.

(c) The step response and the state trajectory are shown in Figure 4 and Figure 5, respectively. The state trajectory exhibits the property of limit cycle.

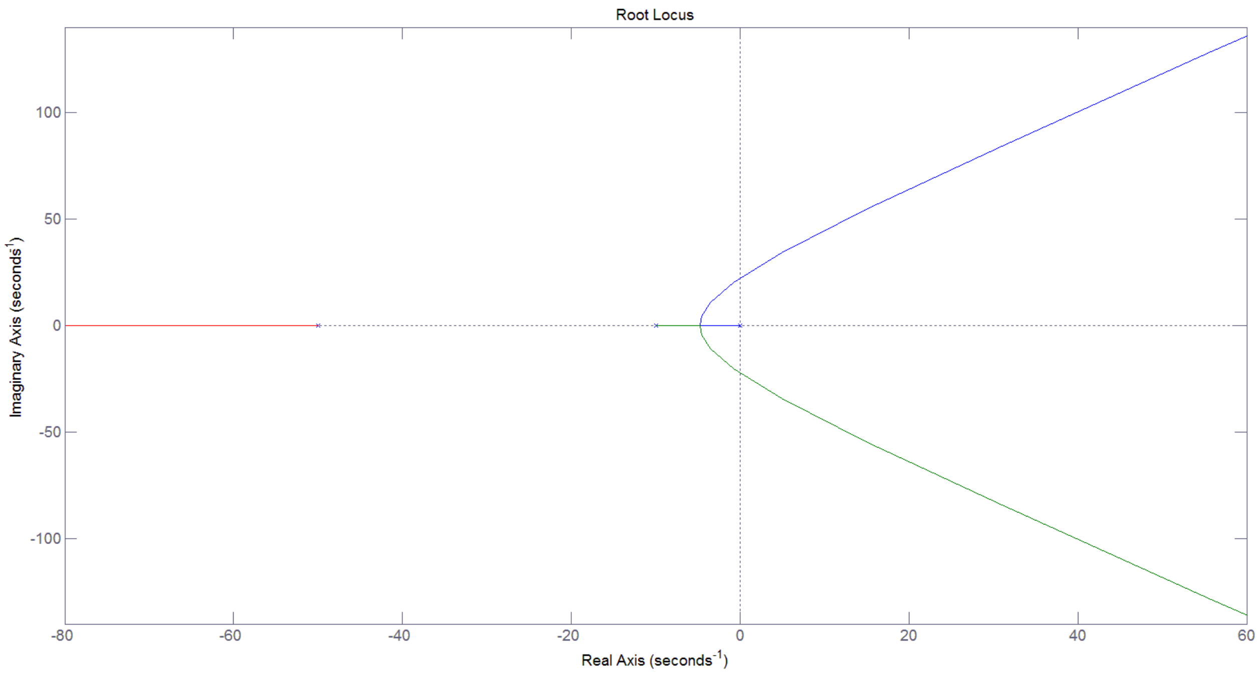

Figure 1: Root locus of the system in Problem 2

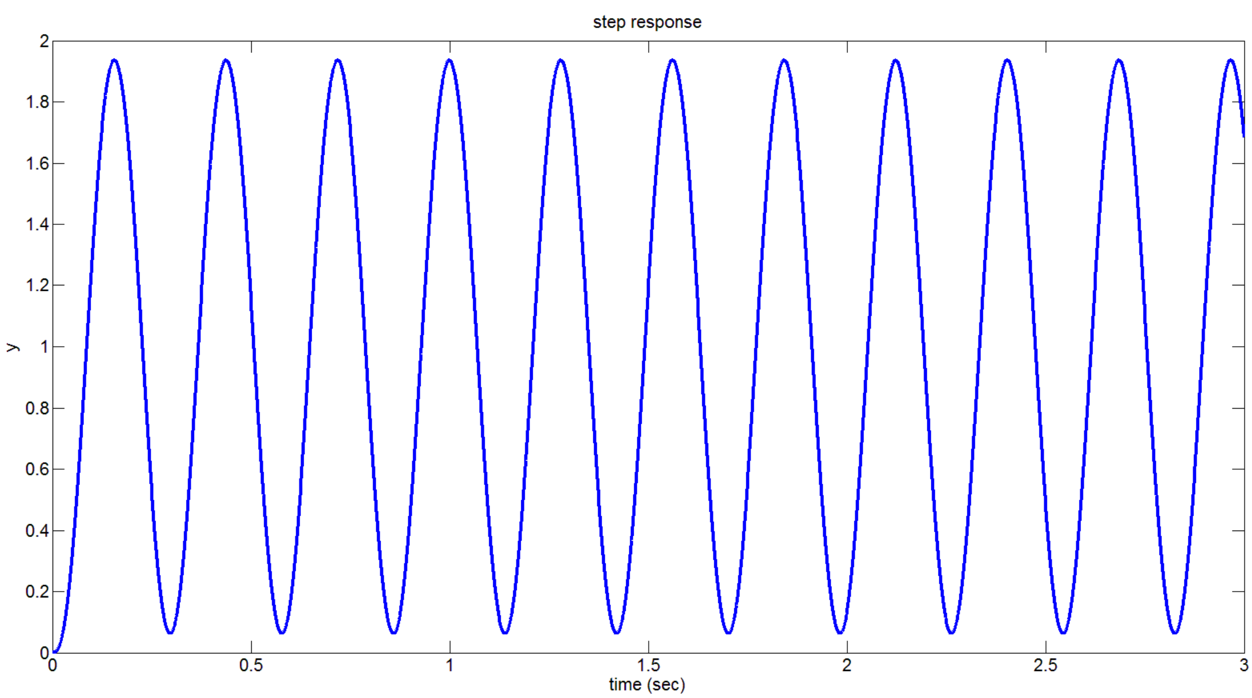

Figure 2: Step response of the system in Problem 2

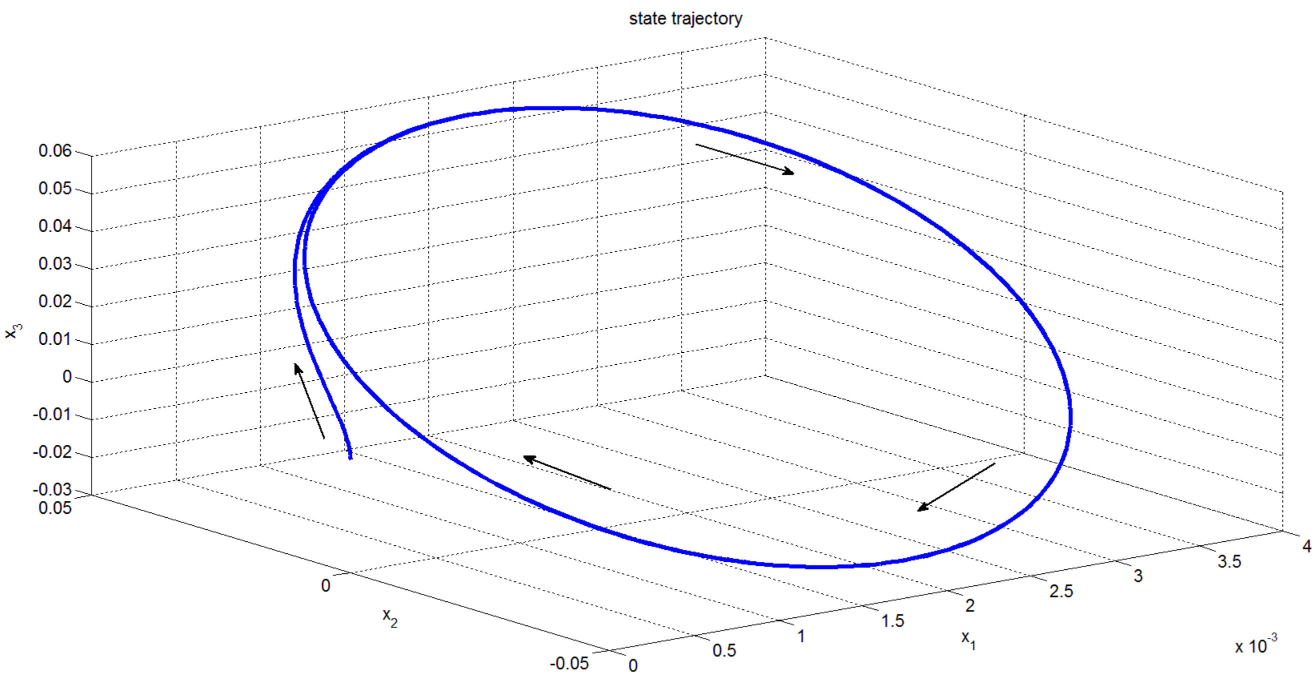

Figure 3: state trajectory of the system in Problem 2

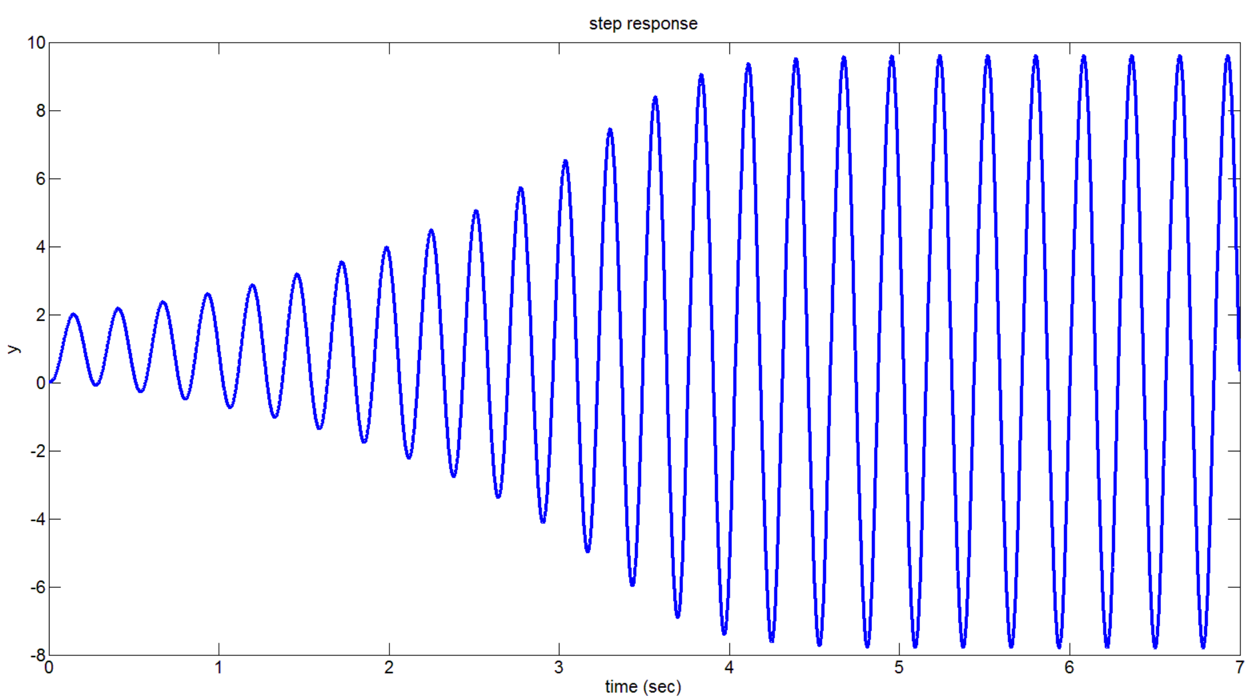

Figure 4: Step response of the system in Problem 2

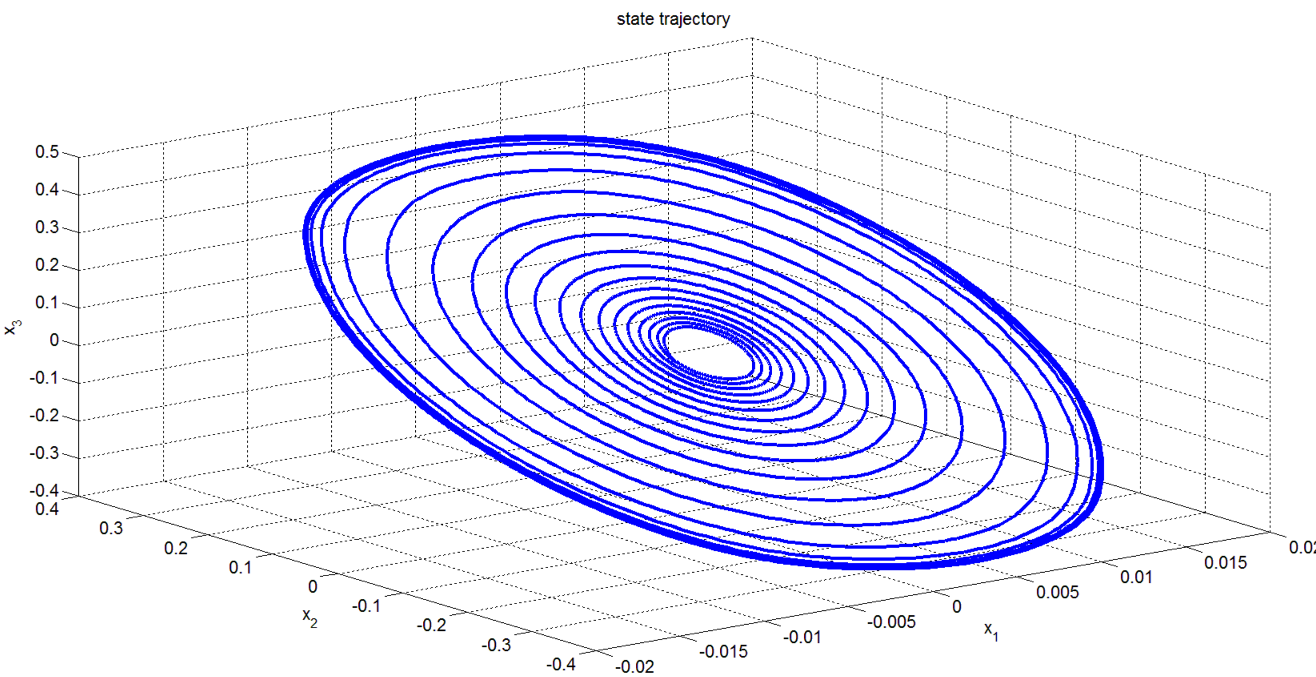

Figure 5: State trajectory of the system in Problem 2