Nonlinear System Theory
Solution to Homework 7

1. (a) Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. P is positive definite if and only if $p_1 > 0$ and $p_1 p_3 > p_2^2$. Then

$$\begin{aligned} \dot{V} &= (p_1 x_1 + p_2 x_2)\dot{x}_1 + (p_2 x_1 + p_3 x_2)\dot{x}_2 + h(x_1)\dot{x}_1 \\ &= (p_1 x_1 + p_2 x_2 + h(x_1))(-x_1 - x_2) + (p_2 x_1 + p_3 x_2)(h(x_1) - 2x_2 + u) \\ &= -p_1 x_1^2 - (p_1 + 3p_2)x_1 x_2 - (p_2 + 2p_3)x_2^2 + h(x_1)(-x_1 - x_2 + p_2 x_1 + p_3 x_2) \\ &\quad +(p_2 x_1 + p_3 x_2)u \end{aligned}$$

Choose $p_3 = 1$, $p_2 = -1$, and $p_1 = 2$. Then P is positive definite, and

$$\begin{aligned} \dot{V} &= -2x_1^2 + x_1 x_2 - x_2^2 - 2x_1 h(x_1) + (x_2 - x_1)u \\ &\le -\frac{1}{2}(x_2 - x_1)^2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_2^2 + (x_2 - x_1)u \\ &\le -\frac{1}{2}(x_2 - x_1)^2 + (x_2 - x_1)u \end{aligned}$$

Therefore $uy \ge \dot{V} + \frac{1}{2}y^2$, i.e. the system is output strictly passive. Moreover, it is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$ gain is less than or equal to 2.

(b) Let V be the storage function in part (a) and $W = \alpha V$, where $\alpha > 0$. $f(x) = \begin{bmatrix} -x_1 - x_2 \\ h(x_1) - 2x_2 \end{bmatrix}$ and $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\frac{\partial W}{\partial x} = \alpha x^T P + \alpha \begin{bmatrix} h(x_1) & 0 \end{bmatrix} = \alpha \begin{bmatrix} 2x_1 - x_2 + h & -x_1 + x_2 \end{bmatrix}$$

and the Hamilton-Jacobi inequality is

$$\begin{aligned} &\alpha\big[(2x_1 - x_2 + h)(-x_1 - x_2) + (-x_1 + x_2)(h - 2x_2)\big] + \frac{\alpha^2}{2\gamma^2}(-x_1 + x_2)^2 + \frac{1}{2}(x_2 - x_1)^2 \\ = \;& \alpha\big(-2x_1^2 + x_1 x_2 - x_2^2 - 2x_1 h(x_1)\big) + \frac{1}{2}\left(\frac{\alpha^2}{\gamma^2} + 1\right)(x_2 - x_1)^2 \\ \le \;& -\frac{1}{2}\left(\alpha - \frac{\alpha^2}{\gamma^2} - 1\right)(x_2 - x_1)^2 \le 0 \end{aligned}$$

The Hamilton-Jacobi inequality holds if $\alpha - \frac{\alpha^2}{\gamma^2} - 1 > 0$ for some $\alpha > 0$ and $\gamma > 0$. This implies that

$$\gamma^2 \ge \frac{\alpha^2}{\alpha - 1} = A(\alpha), \quad \alpha > 1$$

Notice that the derivative of $A(\alpha)$ w.r.t. α vanishes at $\alpha = 2$, which is the minimum point of $A(\alpha)$. Therefore

$$\gamma \geq \left. \frac{\alpha}{\sqrt{\alpha - 1}} \right|_{\alpha=2} = 2$$

Hence the $\mathcal{L}_2$ gain of the system is upper bounded by γ, and the smallest γ is 2.

(c) When $u = 0$, $(x_1, x_2) = (0, 0)$ is an equilibrium point of the system. Suppose that $y \equiv 0$, which implies $x_1 \equiv x_2$, and thus $\dot{x}_1 \equiv \dot{x}_2$. Therefore

$$\dot{x}_1 = -x_1 - x_2 = -2x_1 = \dot{x}_2 = h(x_1) - 2x_2 \Rightarrow h(x_1) = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0$$

This means that the system is zero-state observable. Since the system is output strictly passive and zero-state observable, $(0, 0)$ is an asymptotically stable equilibrium point.

(d) Let $v(t) = \int_0^t y(\tau) d\tau$. Then the state space representation of the integrator is

$$\dot{x}_3 = y \qquad v = x_3$$

Choose $V_c = V + \frac{1}{2}x_3^2$ as the storage function of the feedback system, where V is defined in part (a). Then

$$\dot{V}_c = \dot{V} + x_3\dot{x}_3 \leq -\frac{1}{2}y^2 + uy + vy = -\frac{1}{2}y^2 + ry$$

Hence $ry \geq \dot{V}_c + \frac{1}{2}y^2$, and the system is output strictly passive. Therefore, it is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$-gain is upper bounded by 2.

2. (a) Let the storage function of H_1 be $V_1 = \int_0^{x_1} h(\tau)d\tau + \frac{1}{2}x_2^2$. Then

$$\begin{aligned} \dot{V}_1 &= h(x_1)x_2 + x_2(-h(x_1) - x_2 + 3e_1) = -(y_1 + e_1)^2 + 3(y_1 + e_1)e_1 \\ &= -y_1^2 + 2e_1^2 + y_1 e_1 \end{aligned}$$

On the other hand, let $V_2 = \frac{1}{2}x_3^2$ be the storage function of H_2. Then

$$\dot{V}_2 = x_3(-kx_3 + e_2) = -ky_2^2 + y_2 e_2$$

Since $e_1 y_1 = \dot{V}_1 - 2e_1^2 + y_1^2$, $e_2 y_2 = \dot{V}_2 + ky_2^2$ and $k > 2$, the feedback system is finite-gain $\mathcal{L}_2$ stable (by Theorem 2 of Lecture 8).

(b) Perform loop transformation of the feedback connected system as shown in Figure 1. Note that $\tilde{y}_1 = y_1 + e_1$ and $\tilde{e}_2 = e_2 - y_2$. Then the state space representation of $\tilde{H}_1$ and $\tilde{H}_2$ is

$$\tilde{H}_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) - x_2 + 3e_1 \\ \tilde{y}_1 = x_2 \end{cases} \quad \text{and} \quad \tilde{H}_2 : \begin{cases} \dot{x}_3 = -(k-1)x_3 + \tilde{e}_2 \\ y_2 = x_3 \end{cases}$$

Take V_1 and V_2 in part (a). Then

$$\dot{V}_1 = -\tilde{y}_1^2 + 3\tilde{y}_1 e_1$$

Hence $\tilde{H}_1$ is output strictly passive. Moreover, if $e_1 = 0$ and $\tilde{y}_1 = x_2 \equiv 0$, then $x_1 \equiv 0$. This implies that $\tilde{H}_1$ is zero-state observable.

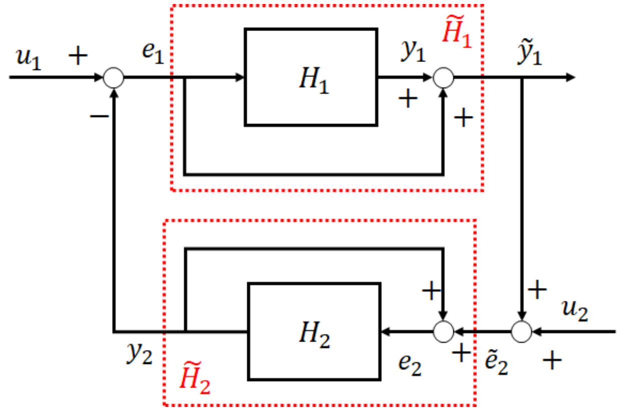

Figure 1: Loop Transformation of the Feedback Connected System in Problem 2

On the other hand,

$$\dot{V}_2 = -(k-1)x_3^2 + y_2\tilde{e}_2$$

Since $k - 1 > 0$, $\tilde{H}_2$ is strictly passive. Therefore, $(x_1, x_2, x_3) = (0, 0, 0)$ is an asymptotically stable equilibrium point of the feedback connected system.

3. (a) $G(s)$ is stable and its Nyquist plot is shown in Figure 2. Since $G(j\omega)$ is on the right-hand side of the vertical line passing through $(-1, 0)$, the feedback system is absolutely stable for the sector [0, 1].

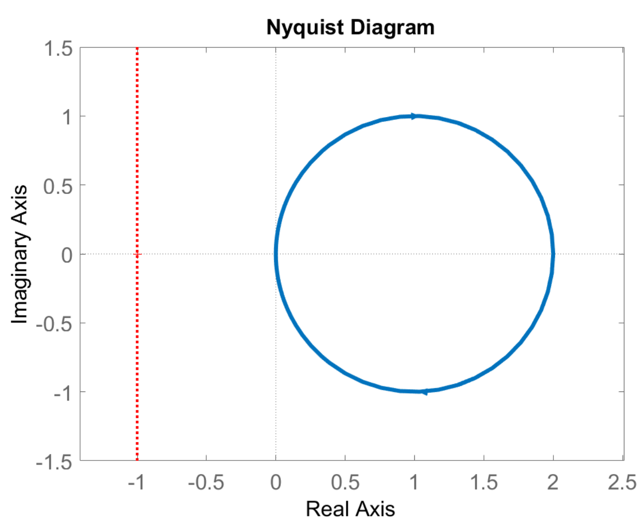

Figure 2: Nyquist Plot for Problem 3

(b) Note that $\psi(y) = \text{sat}(y) \in [0, 1]$. Because the feedback system is absolutely stable for the sector [0, 1], $x = 0$ is globally asymptotically stable. In other words, starting from any initial state $x(0) \in \mathbb{R}^2$, the state trajectory converges to 0 as $t \to \infty$. Therefore, there cannot exist periodic solutions, and cannot have limit cycles.