Nonlinear System Theory
Homework 1

Due date: 3/14/2023

Use Matlab to solve the following problems.

1. The lateral motion of a 4-wheel vehicle can be represented by a single-track model (a.k.a. the *bicycle model*), which merges the left-sided and right-sided wheels into single one. Let β denote the *sideslip angle* of the vehicle, which is the angle between the velocity of the vehicle's center of gravity (C.G.) and the longitudinal direction of the vehicle (see Figure 1(a)). r is the *yaw rate* of the vehicle. Suppose that the *front steering angle* δ_f is small, and the *longitudinal velocity* of the vehicle's C.G., v_x, is a constant. Then the dynamic equations of the vehicle's lateral motion are

$$
\begin{aligned}
\dot{\beta} &= \frac{2}{M v_x}\left(F_{yf} + F_{yr}\right) - r \\
\dot{r} &= \frac{2}{I_z}\left(F_{yf} l_f - F_{yr} l_r\right)
\end{aligned}
$$

where M and I_z are the mass and moment of inertial (with respect to the yaw motion) of the vehicle. l_f and l_r are the distances between the vehicle's C.G. to the front and rear axles, respectively (see Figure 1(a)). F_{yf} and F_{yr} are the lateral tire forces of the front and rear tires, respectively. A simplified tire model represents the lateral tire force as a function of the *tire slip angle* as follows:

$$
F_{yi}(\alpha_i) = \begin{cases} C_{\alpha i}\alpha_i, & |\alpha_i| \le \bar{\alpha}_i \\ \bar{F}_{yi}\mathrm{sgn}(\alpha_i) - C'_{\alpha i}(\alpha_i - \mathrm{sgn}(\alpha_i)\bar{\alpha}_i), & |\alpha_i| > \bar{\alpha}_i \end{cases}, \qquad i = f, r
$$

where α_i is the tire slip angle, which is the angle between the velocity of the wheel center and the wheel plane (see Figure 1(a)). They can be expressed as

$$
\begin{aligned}
\alpha_f &= \delta_f - \beta - \frac{r l_f}{v_x} \\
\alpha_r &= -\beta + \frac{r l_r}{v_x}
\end{aligned}
$$

$C_{\alpha i}, C'_{\alpha i} > 0$ are the slope of F_{yi} w.r.t. α_i, where $C_{\alpha i}$ is called the *cornering stiffness*. $\bar{F}_{yi} = C_{\alpha i}\bar{\alpha}_i$ is the maximum lateral tire force, which takes place at the tire slip angle $\bar{\alpha}_i$. A typical graph of the lateral tire force is shown in Figure 1(b). The parameter values of the bicycle model considered in Problem 1 are given in Table 1. Set $\delta_f = 0.0873$ rad, and $v_x = 11.11$ m/s.

   (a) Find all equilibrium points of the bicycle model.

Table 1: Parameters of the bicycle model for Problem 1

M	1310 kg	$C_{\alpha f}$	77350 N/rad	$C'_{\alpha f}$	1547 N/rad	l_f	0.986 m	$\bar{F}_{yr}$	3376 N
I_z	2352 kg-m^2	$C_{\alpha r}$	51600 N/rad	$C'_{\alpha r}$	1032 N/rad	l_r	1.596 m	$\bar{F}_{yr}$	2086 N

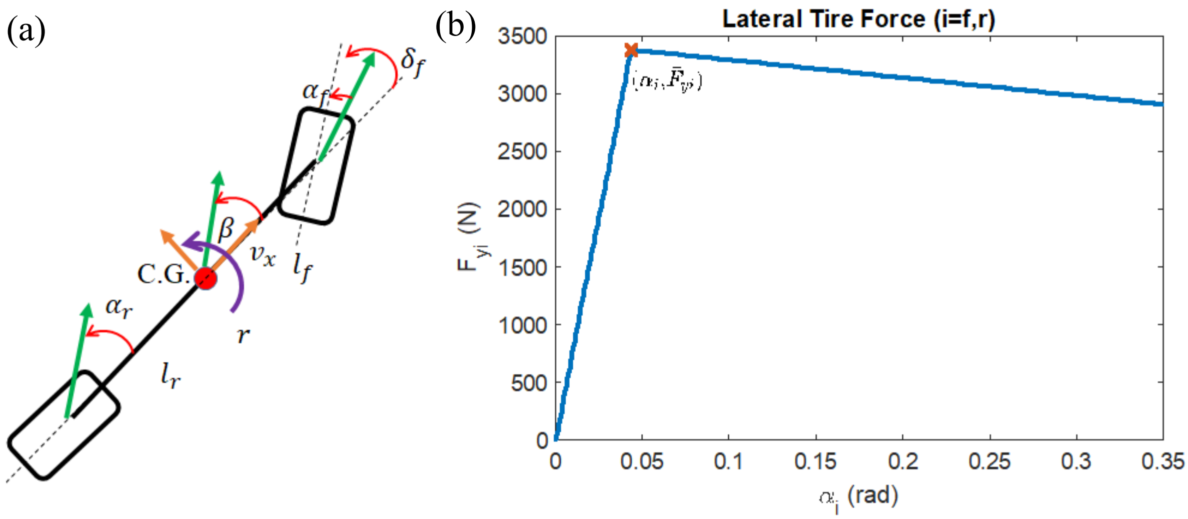

Figure 1: (a) Bicycle Model (b) Lateral Tire Force

(b) Determine stability of each equilibrium point.

2. Consider the feedback system in Figure 2, where G is a linear time-invariant system with the following state model

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -500x_1 - 60x_2 + u \\
y &= 500x_1
\end{aligned}
$$

The integral controller (I-controller) can be represented by

$$\dot{x}_3 = e, \quad u_1 = Kx_3, \quad \text{where } K > 0$$

The saturation function is

$$u = \text{sat}(u_1) = \begin{cases} u_1, & |u_1| \le 20 \\ 20 \cdot \text{sgn}(u_1), & |u_1| > 20 \end{cases}$$

Suppose $x_1(0) = x_2(0) = x_3(0) = 0$. Answer the following questions.

(a) Neglect the saturation function temporarily, i.e. let $u = u_1$. Draw the root locus of the system. (Hint: use the function "rlocus" in Matlab.)

(b) Neglect the saturation function and let $K = 60$. Find the closed-loop poles. Use Matlab/Simulink to simulate the step response of this system. Plot the step response $y(t)$ and the state trajectory in the $x_1 - x_2 - x_3$ space.

(c) Let $K = 70$ and consider the saturation function. Use Matlab/Simulink to simulate the step response of this system. Plot the step response $y(t)$ and the state

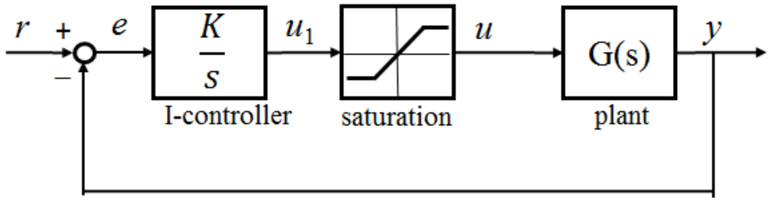

Figure 2: Feedback system of Problem 2

trajectory in the $x_1 - x_2 - x_3$ space. Identify the type of nonlinear characteristics of the state trajectory.

Note: The simulation time should be long enough to show the complete behavior of the system.