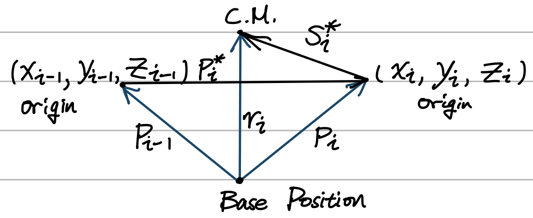
Linearity Velocity: $V_{i+1} = W_{i+1}\times P_{i+1}^* + V_i \longrightarrow \dot V_{i+1} = \dot W_{i+1}\times P_{i+1}^* + W_{i+1}\times(W_{i+1}\times P_{i+1}^*) + \dot V_{i+1}$

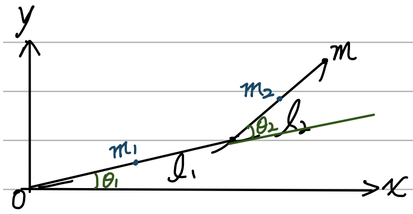
Newton's equation: $F_i = \frac{d}{dt}(m_i\cdot V_{ci}) = m_i\cdot \dot V_{ci}$ V_{ci} is the velocity of center of mass (C.M.)

Euler's equation: $N_i = \frac{d}{dt}(J_i\cdot W_i) = J_i\cdot\dot W_i + W_i\times(J_i\cdot W_i)$

$$\begin{cases} f_i = F_i + f_{i+1} \\ n_i = N_i + n_{i+1} + P_i^*\times f_{i+1} + (P_i^*+S_i)\times F_i \end{cases}$$

$$\begin{cases} \tau_i = n_i^t \vec{Z}_{i-1}, & R \\ \tau_i = f_i^t \vec{Z}_{i-1}, & P \end{cases}$$





For the 2-link robot manipulator example in the class with CM shift to the middle of each link and with a load m held at the top of the second link.

$W_{1z} = \dot\theta_1$ $\dot V_1 = \dot W_1\times P_1^* + W_1\times(W_1\times P_1^*) + \dot V_0$ where $\dot V_0 = [0\quad g\quad 0]^T$

$\dot W_{1z} = \ddot\theta_1 \Rightarrow \dot V_1 = \dot W_1\times[l_1C_1\quad l_1S_1\quad 0]^T + W_1\times\{W_1\times[l_1C_1\quad l_1S_1\quad 0]^T\} + \dot V_0$

$= [-l_1S_1\ddot\theta_1 - l_1C_1\dot\theta_1^2\quad l_1C_1\ddot\theta_1 - l_1S_1\dot\theta_1^2 + g\quad 0]^T$

$V_{ci} = W_i\times S_i + V_i \longrightarrow \dot V_{c1} = \dot W_1\times S_1 + W_1\times(W_1\times S_1) + \dot V_1$, where $S_1 = -\frac{1}{2}P_1^*$

$$\dot V_{c1} = \begin{bmatrix} -\frac12 l_1S_1\ddot\theta_1 - \frac12 l_1C_1\dot\theta_1^2 \\ \frac12 l_1C_1\ddot\theta_1 - \frac12 l_1S_1\dot\theta_1^2 + g \\ 0 \end{bmatrix} \quad F_1 = m_1\dot V_{c1},\quad J_1 = \begin{bmatrix} \frac14 m_1l_1^2 & 0 & 0 \\ 0 & \frac14 m_1l_1^2 & 0 \\ 0 & 0 & \frac14 m_1l_1^2 \end{bmatrix} \quad N_1 = J_1\dot W_1 + W_2\times(J_1W_1)$$

$W_2 = \dot\theta_1 + \dot\theta_2$ $\dot W_2 = \ddot\theta_1 + \ddot\theta_2$

$\dot V_2 = \dot W_2\times P_2^* + W_2\times[W_2\times P_2^*] + \dot V_1$

$$= \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1+\ddot\theta_2 \end{bmatrix}\times\begin{bmatrix} l_2C_{12} \\ l_2S_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1+\dot\theta_2 \end{bmatrix}\times\left(\begin{bmatrix} 0 \\ 0 \\ \dot\theta_1+\dot\theta_2 \end{bmatrix}\times\begin{bmatrix} l_2C_{12} \\ l_2S_{12} \\ 0 \end{bmatrix}\right)\times\dot V_1 = \begin{bmatrix} -l_2S_{12}(\ddot\theta_1+\ddot\theta_2) - l_2C_{12}(\dot\theta_1+\dot\theta_2)^2 - l_1S_1\ddot\theta_1 - l_1C_1\dot\theta_1^2 \\ l_2C_{12}(\ddot\theta_1+\ddot\theta_2) - l_2S_{12}(\dot\theta_1+\dot\theta_2)^2 + l_1C_1\ddot\theta_1 - l_1S_1\dot\theta_1^2 + g \\ 0 \end{bmatrix}$$

$$\dot V_{2c} = \dot W_2\times S_2 + W_2\times(W_2\times S_2) + \dot V_2 = \begin{bmatrix} -\frac12 l_2S_{12}(\ddot\theta_1+\ddot\theta_2) - \frac12 l_2C_{12}(\dot\theta_1+\dot\theta_2)^2 - l_1S_1\ddot\theta_1 - l_1C_1\dot\theta_1^2 \\ \frac12 l_2C_{12}(\ddot\theta_1+\ddot\theta_2) - \frac12 l_2S_{12}(\dot\theta_1+\dot\theta_2)^2 + l_1C_1\ddot\theta_1 - l_1S_1\dot\theta_1^2 + g \\ 0 \end{bmatrix}$$

where $S_2 = -\frac12 P_2^*$

$$F_2 = m_2\dot V_{c2},\quad J = \begin{bmatrix} \frac14 m_2l_2^2 & 0 & 0 \\ 0 & \frac14 m_2l_2^2 & 0 \\ 0 & 0 & \frac14 m_2l_2^2 \end{bmatrix},\quad N_2 = J_2\dot W_2 + W_2\times(J_2W_2)$$

$f_2 = F_2 + f_3$, where $f_3 = [0\quad mg\quad 0]^T$

$\Longrightarrow n_2 = n_3 + P_2^*\times f_3 + (P_2^*+S_2)\times F_2 + N_2$, $n_3 = 0$ (zero vector)

$$= \begin{bmatrix} l_2C_{12} \\ l_2S_{12} \\ 0 \end{bmatrix}\times\begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix} + \begin{bmatrix} \frac12 l_2C_{12} \\ \frac12 l_2S_{12} \\ 0 \end{bmatrix}\times m_2\dot V_{c2} + N_2$$

due to $N_2 \rightarrow \tau_2 = n_2^t \vec{Z}_1$, $\vec{Z}_1 = [0\quad 0\quad 1]^T$ $f_1 = F_1 + f_2 = m_1\dot V_{c1} + m_2\dot V_{c2} + [0\quad mg\quad 0]^T$

$n_1 = n_2 + P_1^*\times f_2 + (P_1^*+S_1)\times F_1 + N_1$

$$= n_2 + \begin{bmatrix} l_1C_1 \\ l_1S_1 \\ 0 \end{bmatrix}\times\left(m_2\dot V_{c2} + \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}\right) + \begin{bmatrix} \frac12 l_1C_1 \\ \frac12 l_1S_1 \\ 0 \end{bmatrix}\times m_1\dot V_{c1} + N_1$$

$$
= n_2 + \begin{bmatrix} l_1 C_1 \\ l_1 S_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{2} m_2 l_2 S_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - \frac{1}{2} m_2 l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 - m_2 l_1 S_1 \ddot{\theta}_1 - m_2 l_1 C_1 \dot{\theta}_1^2 \\ \frac{1}{2} m_2 l_2 C_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - \frac{1}{2} m_2 l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 C_1 \ddot{\theta}_1 - m_2 l_1 S_1 \dot{\theta}_1^2 + (m_2 - m_1) g \\ 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} \frac{1}{2} l_1 C_1 \\ \frac{1}{2} l_1 S_1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} m_1 l_1 S_1 \ddot{\theta}_1 - \frac{1}{2} m_1 l_1 C_1 \dot{\theta}_1^2 \\ \frac{1}{2} m_1 l_1 C_1 \ddot{\theta}_1 - \frac{1}{2} m_1 l_1 S_1 \dot{\theta}_1^2 + mg \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} m_1 l_1^2 \ddot{\theta}_1 \end{bmatrix}
$$

$\tau_1 = n_1^t \vec{z}_0$, $\vec{z}_0 = [0 \quad 0 \quad 1]^T$

在此系統內將 load m 以 $-y$ 方向的力 mg 替代，得 $\tau_1 = \tau_2$，從方程式的推導中可以看到 load m 以一個力 f_3 代表，因此若從方程式回推系統的動態結構，並無法區別 f_3 是 load m or external force $-mg$.

$\dot{W}_0 = 0$ $\quad W_1 = [0 \quad 0 \quad \dot{\theta}_1]^T$ $\quad W_2 = [0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2]^T$ $\quad P_1^* = [l_1 C_1 \quad l_1 S_1 \quad 0]^T$

$\dot{W}_1 = [0 \quad 0 \quad \ddot{\theta}_1]^T$ $\quad \dot{W}_2 = [0 \quad 0 \quad \ddot{\theta}_1 + \ddot{\theta}_2]^T$ $\quad P_2^* = [l_2 C_{12} \quad l_2 S_{12} \quad 0]^T$

$$
S_1^* = -\frac{1}{2} P_1^* = \begin{bmatrix} -\frac{1}{2} l_1 C_1 \\ -\frac{1}{2} l_1 S_1 \\ 0 \end{bmatrix} \qquad S_2^* = -\frac{1}{2} P_2^* = \begin{bmatrix} -\frac{1}{2} l_2 C_{12} \\ -\frac{1}{2} l_2 S_{12} \\ 0 \end{bmatrix} \qquad f_3 = \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}
$$