Assignment 4

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***Abstract*—This document determines the value of k for which the given equation represents a pair of straight lines.**

1 Problem

For what value of k does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.1}$$

represent a pair of straight lines?

2 Solution

Equation (1.1) can also be written as

$$\mathbf{x}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mathbf{x} + \begin{pmatrix} d & e \end{pmatrix} \mathbf{x} + f = 0 \tag{2.1}$$

$$\begin{aligned} \mathbf{V} &= \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \\ \mathbf{u} &= \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \\ f &= -2 \end{aligned} \tag{2.2}$$

Block Matrix

$$= \begin{pmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{pmatrix} \tag{2.3}$$

Determinant of the Block Matrix

$$\Delta = \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} \tag{2.4}$$

If the equation (1.1) represents a pair of straight lines then the Determinant is zero

$$\Delta = 0 \tag{2.5}$$

$$\implies \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} = 0 \tag{2.6}$$

$$\implies 6 \times (6 - 25/4) - k/2(-k - 5) + 2(5k/4 + 6) = 0 \tag{2.7}$$

$$\implies k^2 + 10k + 21 = 0 \tag{2.8}$$

$$\implies \boxed{k = -3} \tag{2.9}$$

$$\implies \boxed{k = -7} \tag{2.10}$$

Substituting k=-3 in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.11}$$

Equation (2.2) can be represented as

$$\begin{aligned} \mathbf{V} &= \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \\ \mathbf{u} &= \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \\ f &= -2 \end{aligned} \tag{2.12}$$

To find the separate equations of the straight lines we will use spectral decomposition.
Characteristic equation of $\mathbf{V}$ is given by:

$$\begin{aligned} \left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 6 - \lambda & -3/2 \\ -3/2 & -3 - \lambda \end{vmatrix} = 0 \\ \implies \lambda^2 - 3\lambda - 81/4 = 0 \end{aligned} \tag{2.13}$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3 + 3\sqrt{10}}{2}, \lambda_2 = \frac{3 - 3\sqrt{10}}{2} \tag{2.14}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned} \mathbf{Vp} &= \lambda\mathbf{p} \\ \implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} &= 0 \end{aligned} \tag{2.15}$$

For $\lambda_1 = \frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} \end{pmatrix} \tag{2.16}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V} - \lambda\mathbf{I})$

$$\begin{aligned} &\begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \\ \xleftrightarrow{R_1 \to \frac{2}{9-3\sqrt{10}} R_1} &\begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \\ \xleftrightarrow{R_1 \to 3/2R_1 + R_2} &\begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.17}$$

So we get,

$$x_1 + (3 + \sqrt{10})x_2 = 0 \tag{2.18}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{1}{\sqrt{20 + 6\sqrt{10}}} \begin{pmatrix} -(3 + \sqrt{10}) \\ 1 \end{pmatrix} \tag{2.19}$$

Similarly for $\lambda_2 = \frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2 = \frac{1}{\sqrt{20-6\sqrt{10}}}\begin{pmatrix}-(3-\sqrt{10})\\1\end{pmatrix} \tag{2.20}$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{2.21}$$

$$\implies \mathbf{D} = \begin{pmatrix}\frac{3+3\sqrt{10}}{2} & 0\\0 & \frac{3-3\sqrt{10}}{2}\end{pmatrix} \tag{2.22}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix} \tag{2.23}$$

$$\implies \mathbf{P} = \begin{pmatrix}\frac{-(3+\sqrt{10})}{\sqrt{20+6\sqrt{10}}} & \frac{-(3-\sqrt{10})}{\sqrt{20-6\sqrt{10}}}\\\frac{1}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20-6\sqrt{10}}}\end{pmatrix} \tag{2.24}$$

We know that

$$\begin{pmatrix}\sqrt{|\lambda_1|} & \sqrt{|\lambda_2|}\end{pmatrix}\mathbf{P}^T(\mathbf{x}-\mathbf{c}) = 0 \tag{2.25}$$

where $\mathbf{c}$ is the point of intersection of the lines Let (α,β) be the point of intersection of the lines

$$\begin{gathered}\mathbf{Vc} = -\mathbf{u}\\\begin{pmatrix}6 & -3/2\\-3/2 & -3\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = -\begin{pmatrix}2\\5/2\end{pmatrix}\\\implies \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}-1/9\\8/9\end{pmatrix}\end{gathered} \tag{2.26}$$

Substituting values in (2.25)

$$\begin{gathered}\begin{pmatrix}\sqrt{\frac{3+3\sqrt{10}}{2}} & \pm\sqrt{\frac{3-3\sqrt{10}}{2}}\end{pmatrix}\times\\\begin{pmatrix}-\frac{3+\sqrt{10}}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20+6\sqrt{10}}}\\-\frac{3-\sqrt{10}}{\sqrt{20-6\sqrt{10}}} & -\frac{1}{\sqrt{20-6\sqrt{10}}}\end{pmatrix}\times\\\begin{pmatrix}x+1/9\\y-8/9\end{pmatrix} = 0\end{gathered} \tag{2.27}$$

Simplifying (2.27) we get

$$\begin{gathered}3x-3y+3 = 0 \; and \; 2x+y-2/3 = 0\\\boxed{(3x-3y+3)(2x+y-2/3) = 0}\end{gathered} \tag{2.28}$$

Similarly substituting k=-7 in (1.1)

$$\mathbf{x}^T\begin{pmatrix}6 & -7/2\\-7/2 & -3\end{pmatrix}\mathbf{x} + \begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} - 2 = 0 \tag{2.29}$$

Equation (2.2) can be represented as

$$\begin{gathered}\mathbf{V} = \begin{pmatrix}6 & -7/2\\-7/2 & -3\end{pmatrix}\\\mathbf{u} = \begin{pmatrix}2\\5/2\end{pmatrix}\\f = -2\end{gathered} \tag{2.30}$$

To find the separate equations of the straight lines we will use spectral decomposition.

Characteristic equation of $\mathbf{V}$ is given by:

$$\begin{gathered}\left|V-\lambda\mathbf{I}\right| = \begin{vmatrix}6-\lambda & -7/2\\-7/2 & -3-\lambda\end{vmatrix} = 0\\\implies \lambda^2 - 3\lambda - 121/4 = 0\end{gathered} \tag{2.31}$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3+\sqrt{130}}{2}, \lambda_2 = \frac{3-3\sqrt{130}}{2} \tag{2.32}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\begin{gathered}\mathbf{Vp} = \lambda\mathbf{p}\\\implies (\mathbf{V}-\lambda\mathbf{I})\mathbf{p} = 0\end{gathered} \tag{2.33}$$

For $\lambda_1 = \frac{3+\sqrt{130}}{2}$

$$(\mathbf{V}-\lambda_1\mathbf{I}) = \begin{pmatrix}\frac{9-\sqrt{130}}{2} & -7/2\\-7/2 & \frac{-9-\sqrt{130}}{2}\end{pmatrix} \tag{2.34}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V}-\lambda\mathbf{I})$

$$\begin{gathered}\begin{pmatrix}\frac{9-\sqrt{130}}{2} & -7/2 & 0\\-7/2 & \frac{-9-\sqrt{130}}{2} & 0\end{pmatrix}\\\xleftrightarrow{R_1\rightarrow\frac{2}{9-\sqrt{130}}R_1}\begin{pmatrix}1 & \frac{9+\sqrt{130}}{7} & 0\\-7/2 & \frac{-9-\sqrt{130}}{2} & 0\end{pmatrix}\\\xleftrightarrow{R_1\rightarrow 7/2R_1+R_2}\begin{pmatrix}1 & \frac{9+\sqrt{130}}{7} & 0\\0 & 0 & 0\end{pmatrix}\end{gathered} \tag{2.35}$$

So we get,

$$x_1 + (\frac{9+\sqrt{130}}{7})x_2 = 0 \tag{2.36}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{7}{\sqrt{260+18\sqrt{130}}}\begin{pmatrix}-\frac{9+\sqrt{130}}{7})\\1\end{pmatrix} \tag{2.37}$$

Similarly for $\lambda_2 = \frac{3-\sqrt{130}}{2}$

$$\mathbf{p}_2 = \frac{7}{\sqrt{260-18\sqrt{130}}}\begin{pmatrix}-\frac{9-\sqrt{130}}{7}\\1\end{pmatrix} \tag{2.38}$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{2.39}$$

$$\implies \mathbf{D} = \begin{pmatrix}\frac{3+\sqrt{130}}{2} & 0\\0 & \frac{3-\sqrt{130}}{2}\end{pmatrix} \tag{2.40}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix} \tag{2.41}$$

$$\implies \mathbf{P} = \begin{pmatrix}-\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}}\\\frac{7}{\sqrt{260+18\sqrt{10}}} & \frac{7}{\sqrt{260-18\sqrt{130}}}\end{pmatrix} \tag{2.42}$$

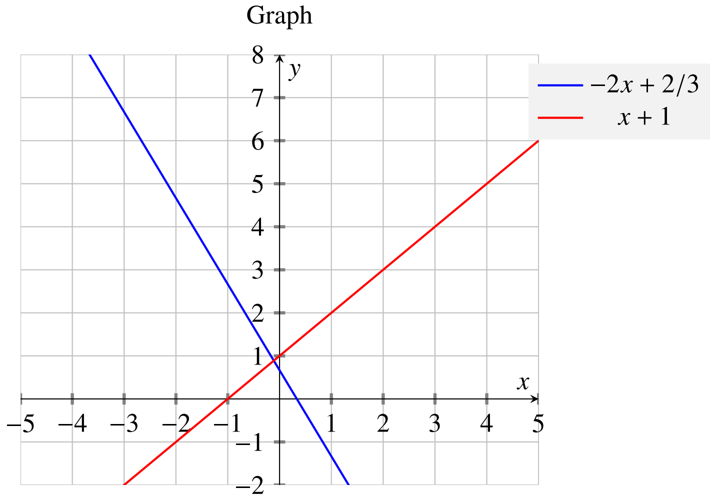

Fig. 1: This is a plot of pair of straight lines when k=-3

We know that

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \tag{2.43}$$

where $\mathbf{c}$ is the point of intersection of the lines Let (α, β) be the point of intersection of the lines

$$\mathbf{Vc} = -\mathbf{u}$$

$$\begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.44}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1/11 \\ 8/11 \end{pmatrix}$$

Substituting values in (2.43)

$$\begin{aligned} &\begin{pmatrix} \sqrt{\frac{3+\sqrt{130}}{2}} & \pm\sqrt{\frac{3-\sqrt{130}}{2}} \end{pmatrix} \times \\ &\begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & \frac{7}{\sqrt{260+18\sqrt{130}}} \\ -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \times \\ &\begin{pmatrix} x - 1/11 \\ y - 8/11 \end{pmatrix} = 0 \end{aligned} \tag{2.45}$$

Simplifying (2.45) we get

$$\begin{aligned} 2x - 3y + 2 = 0 \ and \ 3x + y - 1 = 0 \\ \boxed{(2x - 3y + 2)(3x + y - 1) = 0} \end{aligned} \tag{2.46}$$

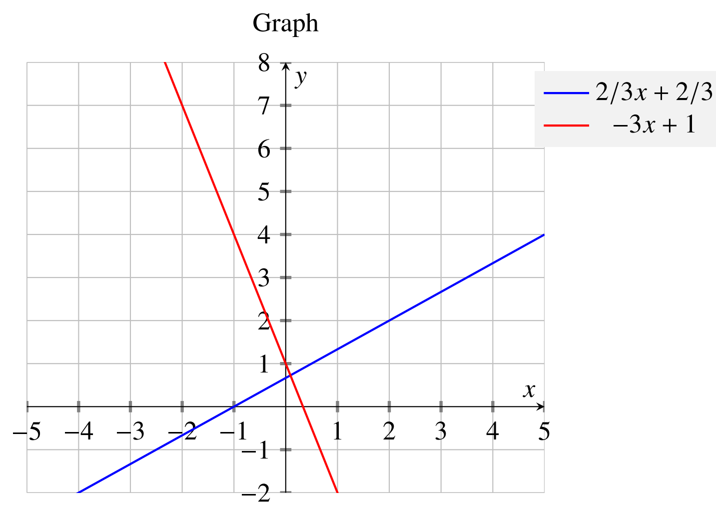

Fig. 2: This is a plot of pair of straight lines when k=-7