

# Assignment 4

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**Abstract**—This document determines the value of  $k$  for which the given equation represents a pair of straight lines.

## 1 PROBLEM

For what value of  $k$  does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \mathbf{x} + (4 \quad 5) \mathbf{x} - 2 = 0 \quad (1.1)$$

represent a pair of straight lines?

## 2 SOLUTION

Equation (1.1) can also be written as

$$\mathbf{x}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mathbf{x} + (d \quad e) \mathbf{x} + f = 0 \quad (2.1)$$

$$\mathbf{V} = \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \quad (2.2)$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.3)$$

$$f = -2 \quad (2.4)$$

Block Matrix

$$= \begin{pmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{pmatrix} \quad (2.5)$$

Determinant of the Block Matrix

$$\Delta = \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} \quad (2.6)$$

If the equation (1.1) represents a pair of straight lines then the Determinant is zero

$$\Delta = 0 \quad (2.7)$$

$$\Rightarrow \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} = 0 \quad (2.8)$$

$$\Rightarrow 6 \times (6 - 25/4) - k/2(-k - 5) + 2(5k/4 + 6) = 0 \quad (2.9)$$

$$\Rightarrow k^2 + 10k + 21 = 0 \quad (2.10)$$

$$\Rightarrow \boxed{k = -3} \quad (2.11)$$

$$\Rightarrow \boxed{k = -7} \quad (2.12)$$

Substituting  $k=-3$  in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + (4 \quad 5) \mathbf{x} - 2 = 0 \quad (2.13)$$

Equation (2.12) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \quad (2.14)$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.15)$$

$$f = -2 \quad (2.16)$$

To find the separate equations of the straight lines we will use spectral decomposition.

Characteristic equation of  $\mathbf{V}$  is given by:

$$|V - \lambda \mathbf{I}| = \begin{vmatrix} 6 - \lambda & -3/2 \\ -3/2 & -3 - \lambda \end{vmatrix} = 0 \quad (2.17)$$

$$\Rightarrow \lambda^2 - 3\lambda - 81/4 = 0 \quad (2.18)$$

The Eigen Values of  $\mathbf{V}$  are:

$$\lambda_1 = \frac{3 + 3\sqrt{10}}{2}, \lambda_2 = \frac{3 - 3\sqrt{10}}{2} \quad (2.19)$$

Let  $\mathbf{p}_1$  and  $\mathbf{p}_2$  be the Eigen vector corresponding to  $\lambda_1$  and  $\lambda_2$  respectively

Eigen vector  $\mathbf{p}$  is given as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.20)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.21)$$

For  $\lambda_1 = \frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} \end{pmatrix} \quad (2.22)$$

To find  $\mathbf{p}_1$  Use Augmented Matrix of  $(\mathbf{V} - \lambda_1 \mathbf{I})$

$$\begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \quad (2.23)$$

$$\xleftrightarrow{R_1 \rightarrow \frac{2}{9-3\sqrt{10}} R_1} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \quad (2.24)$$

$$\xleftrightarrow{R_1 \rightarrow 3/2 R_1 + R_2} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.25)$$

So we get,

$$x_1 + (3 + \sqrt{10})x_2 = 0 \quad (2.26)$$

Therefore, Eigen Vector corresponding to  $\lambda_1$

$$\mathbf{p}_1 = \frac{1}{\sqrt{20+6\sqrt{10}}} \begin{pmatrix} -(3+\sqrt{10}) \\ 1 \end{pmatrix} \quad (2.27)$$

Similarly for  $\lambda_2 = \frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2 = \frac{1}{\sqrt{20-6\sqrt{10}}} \begin{pmatrix} -(3-\sqrt{10}) \\ 1 \end{pmatrix} \quad (2.28)$$

We know that  $\mathbf{V} = \mathbf{PDP}^T$  where  $\mathbf{P}$  and  $\mathbf{V}$  are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.29)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{3+3\sqrt{10}}{2} & 0 \\ 0 & \frac{3-3\sqrt{10}}{2} \end{pmatrix} \quad (2.30)$$

Hence the rotation matrix  $\mathbf{P}$  is

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.31)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \frac{-(3+\sqrt{10})}{\sqrt{20+6\sqrt{10}}} & \frac{-(3-\sqrt{10})}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \quad (2.32)$$

We know that

$$(\sqrt{|\lambda_1|} \quad \sqrt{|\lambda_2|}) \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \quad (2.33)$$

where  $\mathbf{c}$  is the point of intersection of the lines Let  $(\alpha, \beta)$  be the point of intersection of the lines

$$\mathbf{Vc} = -\mathbf{u} \quad (2.34)$$

$$\begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.35)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1/9 \\ 8/9 \end{pmatrix} \quad (2.36)$$

Substituting values in (2.33)

$$\left( \sqrt{\frac{3+3\sqrt{10}}{2}} \quad \pm \sqrt{\frac{3-3\sqrt{10}}{2}} \right) \begin{pmatrix} \frac{3+\sqrt{10}}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20+6\sqrt{10}}} \\ -\frac{3-\sqrt{10}}{\sqrt{20-6\sqrt{10}}} & -\frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \begin{pmatrix} x+1/9 \\ y-8/9 \end{pmatrix} = 0 \quad (2.37)$$

Simplifying (2.37) we get

$$3x - 3y + 3 = 0 \text{ and } 2x + y - 2/3 = 0 \quad (2.38)$$

$$(3x - 3y + 3)(2x + y - 2/3) = 0 \quad (2.39)$$

Similarly substituting  $k=-7$  in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \mathbf{x} + (4 \quad 5) \mathbf{x} - 2 = 0 \quad (2.40)$$

Equation (2.2) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \quad (2.41)$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.42)$$

$$f = -2 \quad (2.43)$$

To find the separate equations of the straight lines we will use spectral decomposition.

Characteristic equation of  $\mathbf{V}$  is given by:

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 6-\lambda & -7/2 \\ -7/2 & -3-\lambda \end{vmatrix} = 0 \quad (2.44)$$

$$\Rightarrow \lambda^2 - 3\lambda - 121/4 = 0 \quad (2.45)$$

The Eigen Values of  $\mathbf{V}$  are:

$$\lambda_1 = \frac{3 + \sqrt{130}}{2}, \lambda_2 = \frac{3 - 3\sqrt{130}}{2} \quad (2.46)$$

Let  $\mathbf{p}_1$  and  $\mathbf{p}_2$  be the Eigen vector corresponding to  $\lambda_1$  and  $\lambda_2$  respectively

Eigen vector  $\mathbf{p}$  is given as:

$$\mathbf{Vp} = \lambda \mathbf{p} \quad (2.47)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \quad (2.48)$$

For  $\lambda_1 = \frac{3+\sqrt{130}}{2}$

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 \\ -7/2 & \frac{-9-\sqrt{130}}{2} \end{pmatrix} \quad (2.49)$$

To find  $\mathbf{p}_1$  Use Augmented Matrix of  $(\mathbf{V} - \lambda \mathbf{I})$

$$\begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix} \quad (2.50)$$

$$\xrightarrow{R_1 \rightarrow \frac{2}{9-\sqrt{130}} R_1} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix} \quad (2.51)$$

$$\xrightarrow{R_1 \rightarrow 7/2 R_1 + R_2} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ 0 & \frac{7}{7} & 0 \end{pmatrix} \quad (2.52)$$

So we get,

$$x_1 + \left( \frac{9 + \sqrt{130}}{7} \right) x_2 = 0 \quad (2.53)$$

Therefore, Eigen Vector corresponding to  $\lambda_1$

$$\mathbf{p}_1 = \frac{7}{\sqrt{260 + 18\sqrt{130}}} \begin{pmatrix} -\frac{9+\sqrt{130}}{7} \\ 1 \end{pmatrix} \quad (2.54)$$

Similarly for  $\lambda_2 = \frac{3-\sqrt{130}}{2}$

$$\mathbf{p}_2 = \frac{7}{\sqrt{260 - 18\sqrt{130}}} \begin{pmatrix} -\frac{9-\sqrt{130}}{7} \\ 1 \end{pmatrix} \quad (2.55)$$

We know that  $\mathbf{V} = \mathbf{PDP}^T$  where  $\mathbf{P}$  and  $\mathbf{V}$  are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.56)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{3+\sqrt{130}}{2} & 0 \\ 0 & \frac{3-\sqrt{130}}{2} \end{pmatrix} \quad (2.57)$$

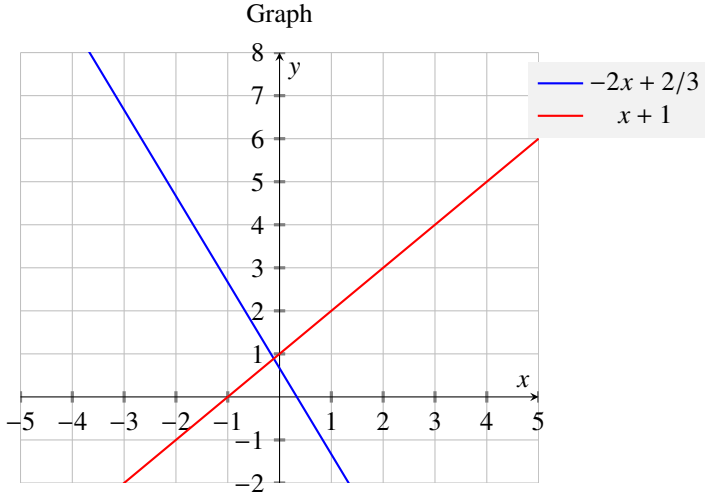


Fig. 1: This is a plot of pair of straight lines when  $k=-3$

Hence the rotation matrix  $P$  is

$$P = (p_1 \quad p_2) \quad (2.58)$$

$$\Rightarrow P = \begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} \\ \frac{7}{\sqrt{260+18\sqrt{130}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \quad (2.59)$$

We know that

$$(\sqrt{|\lambda_1|} \quad \sqrt{|\lambda_2|}) P^T (x - c) = 0 \quad (2.60)$$

where  $c$  is the point of intersection of the lines Let  $(\alpha, \beta)$  be the point of intersection of the lines

$$Vc = -u \quad (2.61)$$

$$\begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.62)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1/11 \\ 8/11 \end{pmatrix} \quad (2.63)$$

Substituting values in (2.33)

$$\left( \sqrt{\frac{3+\sqrt{130}}{2}} \pm \sqrt{\frac{3-\sqrt{130}}{2}} \right) \begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & \frac{7}{\sqrt{260+18\sqrt{130}}} \\ -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \begin{pmatrix} x - 1/11 \\ y - 8/11 \end{pmatrix} \quad (2.64)$$

Simplifying (2.37) we get

$$2x - 3y + 2 = 0 \text{ and } 3x + y - 1 = 0 \quad (2.65)$$

$$(2x - 3y + 2)(3x + y - 1) = 0 \quad (2.66)$$

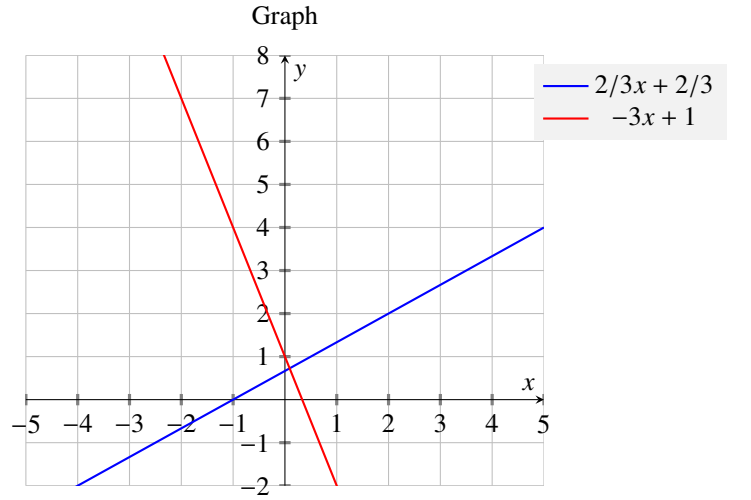


Fig. 2: This is a plot of pair of straight lines when  $k=-7$