Assignment 4

Shweta Verma

***Abstract*—This document determines the value of k for which the given equation represents a pair of straight lines.**

1 Problem

For what value of k does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.1}$$

represent a pair of straight lines?

2 Solution

Equation (1.1) can also be written as

$$\mathbf{x}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mathbf{x} + \begin{pmatrix} d & e \end{pmatrix} \mathbf{x} + f = 0 \tag{2.1}$$

$$\mathbf{V} = \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \tag{2.2}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.3}$$

$$f = -2 \tag{2.4}$$

Block Matrix

$$= \begin{pmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{pmatrix} \tag{2.5}$$

Determinant of the Block Matrix

$$\Delta = \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} \tag{2.6}$$

If the equation (1.1) represents a pair of straight lines then the Determinant is zero

$$\Delta = 0 \tag{2.7}$$

$$\implies \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} = 0 \tag{2.8}$$

$$\implies 6 \times (6 - 25/4) - k/2(-k - 5) + 2(5k/4 + 6) = 0 \tag{2.9}$$

$$\implies k^2 + 10k + 21 = 0 \tag{2.10}$$

$$\implies \boxed{k = -3} \tag{2.11}$$

$$\implies \boxed{k = -7} \tag{2.12}$$

Substituting k=-3 in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.13}$$

Equation (2.12) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \tag{2.14}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.15}$$

$$f = -2 \tag{2.16}$$

The pair of Straight lines can be given by

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = 0 \tag{2.17}$$

Using (2.12) and (2.16)

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 \tag{2.18}$$

Comparing both sides,

$$\mathbf{n_1} c_2 + \mathbf{n_2} c_1 = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.19}$$

$$c_1 c_2 = -2 \tag{2.20}$$

The slope of the lines can be given by the roots of the polynomial:

$$cm^2 + bm + a = 0 \tag{2.21}$$

$$\implies m_i = \frac{-b \pm \sqrt{-det(V)}}{c} \tag{2.22}$$

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.23}$$

Substituting values in (2.21) and (2.22)

$$-3m^2 - 3m + 6 = 0 \tag{2.24}$$

$$\implies m_i = \frac{\frac{-3}{2} \pm \frac{9}{2}}{3} \tag{2.25}$$

$$\implies m_1 = 1, m_2 = -2 \tag{2.26}$$

Therefore,

$$\mathbf{n_1} = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.27}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.28}$$

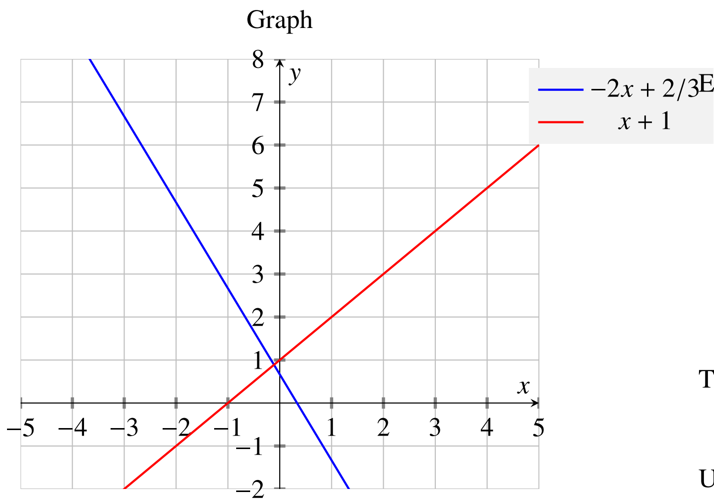

Fig. 1: This is a plot of pair of straight lines when k=-3

we know that

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.29}$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \tag{2.30}$$

$$\implies k_1 k_2 = -3 \tag{2.31}$$

Taking $k_1 = -3$ and $k_2 = 1$ we get

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.32}$$

$$\mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.33}$$

Now to find $c_1$and$c_2$using (2.19)

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.34}$$

$$\begin{pmatrix} 3 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.35}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 3 & 2 & -4 \\ -3 & 1 & -5 \end{pmatrix} \xleftrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 3 & 2 & -4 \\ 0 & 3 & -9 \end{pmatrix} \tag{2.36}$$

$$\xleftrightarrow{R_2 \to R_2/3} \begin{pmatrix} 3 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.37}$$

$$\implies \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{2.38}$$

$$\implies c_1 = -3, c_2 = 2/3 \tag{2.39}$$

Substituting (2.32),(2.33) and (2.39) in (2.17)
The equation of the lines:

$$\boxed{(3x - 3y + 3)(2x + y - 2/3) = 0} \tag{2.40}$$

Similarly, substituting k = -7 in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.41}$$

Equation (2.41) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \tag{2.42}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.43}$$

$$f = -2 \tag{2.44}$$

The pair of Straight lines can be given by

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = 0 \tag{2.45}$$

Using (2.12) and (2.16)

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 \tag{2.46}$$

Comparing both sides,

$$\mathbf{n_1} c_2 + \mathbf{n_2} c_1 = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.47}$$

$$c_1 c_2 = -2 \tag{2.48}$$

The slope of the lines can be given by the roots of the polynomial:

$$cm^2 + bm + a = 0 \tag{2.49}$$

$$\implies m_i = \frac{-b \pm \sqrt{-det(V)}}{c} \tag{2.50}$$

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.51}$$

Substituting values in (2.49)

$$-3m^2 - 7m + 6 = 0 \tag{2.52}$$

$$\implies m_i = \frac{\frac{-7}{2} \pm \frac{11}{2}}{3} \tag{2.53}$$

$$\implies m_1 = 2/3, m_2 = -3 \tag{2.54}$$

Therefore,

$$\mathbf{n_1} = k_1 \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} \tag{2.55}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.56}$$

we know that

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.57}$$

$$k_1 \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \\ -3 \end{pmatrix} \tag{2.58}$$

$$\implies k_1 k_2 = -3 \tag{2.59}$$

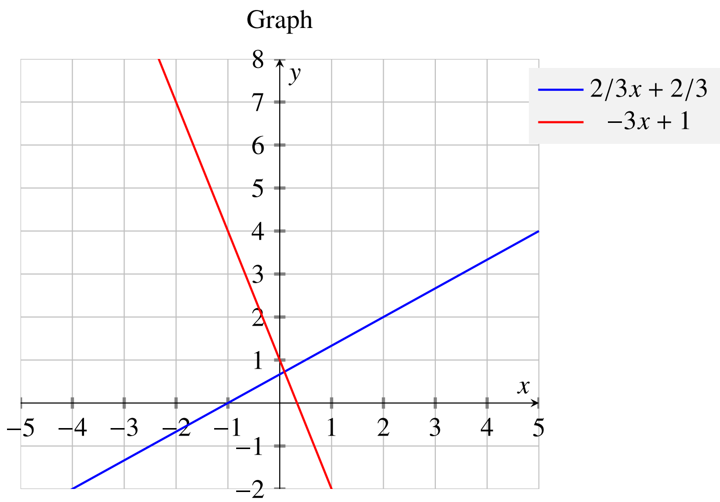

Fig. 2: This is a plot of pair of straight lines when k=-7

Taking $k_1 = -3$ and $k_2 = 1$ we get

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.60}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.61}$$

Now to find c_1 and $c_2$using (2.19)

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.62}$$

$$\begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{2.63}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 2 & 3 & -4 \\ -3 & 1 & -5 \end{pmatrix} \xleftrightarrow{R_2 \to 3R_1 + 2R_2} \begin{pmatrix} 2 & 3 & -4 \\ 0 & 11 & -22 \end{pmatrix} \tag{2.64}$$

$$\xleftrightarrow{R_2 \to R_2/11} \begin{pmatrix} 2 & 3 & -4 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.65}$$

$$\implies \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{2.66}$$

$$\implies c_1 = -2, c_2 = 1 \tag{2.67}$$

Substituting (2.32),(2.33) and (2.39) in (2.17)
The equation of the lines:

$$\boxed{(2x - 3y + 2)(3x + y - 1) = 0} \tag{2.68}$$