Assignment 4

Shweta Verma

***Abstract*—This document determines the value of k for which the given equation represents a pair of straight lines.**

1 Problem

For what value of k does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.1}$$

represent a pair of straight lines?

2 Solution

Equation (1.1) can also be written as

$$\mathbf{x}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mathbf{x} + \begin{pmatrix} d & e \end{pmatrix} \mathbf{x} + f = 0 \tag{2.1}$$

$$\mathbf{V} = \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \tag{2.2}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.3}$$

$$f = -2 \tag{2.4}$$

Block Matrix

$$= \begin{pmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{pmatrix} \tag{2.5}$$

Determinant of the Block Matrix

$$\Delta = \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} \tag{2.6}$$

If the equation (1.1) represents a pair of straight lines then the Determinant is zero

$$\Delta = 0 \tag{2.7}$$

$$\implies \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} = 0 \tag{2.8}$$

$$\implies 6 \times (6 - 25/4) - k/2(-k - 5) + 2(5k/4 + 6) = 0 \tag{2.9}$$

$$\implies k^2 + 10k + 21 = 0 \tag{2.10}$$

$$\implies \boxed{k = -3} \tag{2.11}$$

$$\implies \boxed{k = -7} \tag{2.12}$$

Substituting k=-3 in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{2.13}$$

Equation (2.12) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \tag{2.14}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.15}$$

$$f = -2 \tag{2.16}$$

To find the separate equations of the straight lines we will use spectral decomposition.
Characteristic equation of $\mathbf{V}$ is given by:

$$\left|V - \lambda \mathbf{I}\right| = \begin{vmatrix} 6 - \lambda & -3/2 \\ -3/2 & -3 - \lambda \end{vmatrix} = 0 \tag{2.17}$$

$$\implies \lambda^2 - 3\lambda - 81/4 = 0 \tag{2.18}$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3 + 3\sqrt{10}}{2}, \lambda_2 = \frac{3 - 3\sqrt{10}}{2} \tag{2.19}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{2.20}$$

$$\implies (\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \tag{2.21}$$

For $\lambda_1 = \frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} \end{pmatrix} \tag{2.22}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V} - \lambda \mathbf{I})$

$$\begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \tag{2.23}$$

$$\xleftrightarrow{R_1 \rightarrow \frac{2}{9-3\sqrt{10}} R_1} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \tag{2.24}$$

$$\xleftrightarrow{R_1 \rightarrow 3/2 R_1 + R_2} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.25}$$

So we get,

$$x_1 + (3 + \sqrt{10})x_2 = 0 \tag{2.26}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \begin{pmatrix} -(3+\sqrt{10}) \\ 1 \end{pmatrix} \tag{2.27}$$

Similarly for $\lambda_2 = \frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2 = \begin{pmatrix} -(3-\sqrt{10}) \\ 1 \end{pmatrix} \tag{2.28}$$

We know that $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.29}$$

$$\implies \mathbf{D} = \begin{pmatrix} \frac{3+3\sqrt{10}}{2} & 0 \\ 0 & \frac{3-3\sqrt{10}}{2} \end{pmatrix} \tag{2.30}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.31}$$

$$\implies \mathbf{P} = \begin{pmatrix} -(3+\sqrt{10}) & -(3-\sqrt{10}) \\ 1 & 1 \end{pmatrix} \tag{2.32}$$