

Assignment 4

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Abstract—This document determines the value of k for which the given equation represents a pair of straight lines.

1 PROBLEM

For what value of k does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \mathbf{x} + (4 \quad 5) \mathbf{x} - 2 = 0 \quad (1.1)$$

represent a pair of straight lines?

2 SOLUTION

Equation (1.1) can also be written as

$$\mathbf{x}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mathbf{x} + (d \quad e) \mathbf{x} + f = 0 \quad (2.1)$$

$$\mathbf{V} = \begin{pmatrix} 6 & k/2 \\ k/2 & -3 \end{pmatrix} \quad (2.2)$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.3)$$

$$f = -2 \quad (2.4)$$

Block Matrix

$$= \begin{pmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{pmatrix} \quad (2.5)$$

Determinant of the Block Matrix

$$\Delta = \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} \quad (2.6)$$

If the equation (1.1) represents a pair of straight lines then the Determinant is zero

$$\Delta = 0 \quad (2.7)$$

$$\Rightarrow \begin{vmatrix} 6 & k/2 & 2 \\ k/2 & -3 & 5/2 \\ 2 & 5/2 & -2 \end{vmatrix} = 0 \quad (2.8)$$

$$\Rightarrow 6 \times (6 - 25/4) - k/2(-k - 5) + 2(5k/4 + 6) = 0 \quad (2.9)$$

$$\Rightarrow k^2 + 10k + 21 = 0 \quad (2.10)$$

$$\Rightarrow \boxed{k = -3} \quad (2.11)$$

$$\Rightarrow \boxed{k = -7} \quad (2.12)$$

Substituting $k=-3$ in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \mathbf{x} + (4 \quad 5) \mathbf{x} - 2 = 0 \quad (2.13)$$

Equation (2.12) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix} \quad (2.14)$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \quad (2.15)$$

$$f = -2 \quad (2.16)$$

To find the separate equations of the straight lines we will use spectral decomposition.

Characteristic equation of \mathbf{V} is given by:

$$|V - \lambda \mathbf{I}| = \begin{vmatrix} 6 - \lambda & -3/2 \\ -3/2 & -3 - \lambda \end{vmatrix} = 0 \quad (2.17)$$

$$\Rightarrow \lambda^2 - 3\lambda - 81/4 = 0 \quad (2.18)$$

The Eigen Values of \mathbf{V} are:

$$\lambda_1 = \frac{3 + 3\sqrt{10}}{2}, \lambda_2 = \frac{3 - 3\sqrt{10}}{2} \quad (2.19)$$

Let \mathbf{p}_1 and \mathbf{p}_2 be the Eigen vector corresponding to λ_1 and λ_2 respectively

Eigen vector \mathbf{p} is given as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.20)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.21)$$

For $\lambda_1 = \frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} \end{pmatrix} \quad (2.22)$$

To find \mathbf{p}_1 Use Augmented Matrix of $(\mathbf{V} - \lambda_1 \mathbf{I})$

$$\begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \quad (2.23)$$

$$\xleftrightarrow{R_1 \rightarrow \frac{2}{9-3\sqrt{10}} R_1} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix} \quad (2.24)$$

$$\xleftrightarrow{R_1 \rightarrow 3/2 R_1 + R_2} \begin{pmatrix} 1 & 3 + \sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.25)$$

So we get,

$$x_1 + (3 + \sqrt{10})x_2 = 0 \quad (2.26)$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{1}{20 + 6\sqrt{10}} \begin{pmatrix} -(3 + \sqrt{10}) \\ 1 \end{pmatrix} \quad (2.27)$$

Similarly for $\lambda_2 = \frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2 = \frac{1}{20 - 6\sqrt{10}} \begin{pmatrix} -(3 - \sqrt{10}) \\ 1 \end{pmatrix} \quad (2.28)$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where \mathbf{P} and \mathbf{V} are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.29)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{3+3\sqrt{10}}{2} & 0 \\ 0 & \frac{3-3\sqrt{10}}{2} \end{pmatrix} \quad (2.30)$$

Hence the rotation matrix \mathbf{P} is

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.31)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \frac{-(3+\sqrt{10})}{20+6\sqrt{10}} & \frac{-(3-\sqrt{10})}{20-6\sqrt{10}} \\ \frac{1}{20+6\sqrt{10}} & \frac{1}{20-6\sqrt{10}} \end{pmatrix} \quad (2.32)$$