Assignment 4

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***Abstract*—This document determines the value of k for which the given equation represents a pair of straight lines.**

1 Problem

For what value of k does the equation

$$\mathbf{x}^T\begin{pmatrix}6 & k/2\\ k/2 & -3\end{pmatrix}\mathbf{x}+\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-2=0 \tag{1.1}$$

represent a pair of straight lines?

2 Solution

(1.1) can also be written as

$$\mathbf{x}^T\begin{pmatrix}a & b\\ b & c\end{pmatrix}\mathbf{x}+\begin{pmatrix}d & e\end{pmatrix}\mathbf{x}+f=0 \tag{2.1}$$

$$\begin{aligned}\mathbf{V}&=\begin{pmatrix}6 & k/2\\ k/2 & -3\end{pmatrix}\\ \mathbf{u}&=\begin{pmatrix}2\\ 5/2\end{pmatrix}\\ f&=-2\end{aligned} \tag{2.2}$$

Block Matrix

$$=\begin{pmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{pmatrix} \tag{2.3}$$

Determinant of the Block Matrix

$$\Delta=\begin{vmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{vmatrix} \tag{2.4}$$

If the (1.1) represents a pair of straight lines
then the Determinant is zero

$$\begin{aligned}\Delta&=0\\ \implies \begin{vmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{vmatrix}&=0\\ \implies 6\times(6-25/4)-k/2(-k-5)+2(5k/4+6)&=0\\ \implies k^2+10k+21&=0\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\implies \boxed{k=-3}\\ &\implies \boxed{k=-7}\end{aligned} \tag{2.6}$$

Substituting k=-3 in (1.1)

$$\mathbf{x}^T\begin{pmatrix}6 & -3/2\\ -3/2 & -3\end{pmatrix}\mathbf{x}+\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-2=0 \tag{2.7}$$

(2.2) can be represented as

$$\begin{aligned}\mathbf{V}&=\begin{pmatrix}6 & -3/2\\ -3/2 & -3\end{pmatrix}\\ \mathbf{u}&=\begin{pmatrix}2\\ 5/2\end{pmatrix}\\ f&=-2\end{aligned} \tag{2.8}$$

To find the separate equations of the straight lines we will use spectral decomposition.
Characteristic equation of $\mathbf{V}$ is given by:

$$\begin{aligned}\left|V-\lambda\mathbf{I}\right|=\begin{vmatrix}6-\lambda & -3/2\\ -3/2 & -3-\lambda\end{vmatrix}&=0\\ \implies \lambda^2-3\lambda-81/4&=0\end{aligned} \tag{2.9}$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1=\frac{3+3\sqrt{10}}{2},\lambda_2=\frac{3-3\sqrt{10}}{2} \tag{2.10}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned}\mathbf{V}\mathbf{p}&=\lambda\mathbf{p}\\ \implies (\mathbf{V}-\lambda\mathbf{I})\mathbf{p}&=0\end{aligned} \tag{2.11}$$

For $\lambda_1=\frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V}-\lambda_1\mathbf{I})=\begin{pmatrix}\frac{9-3\sqrt{10}}{2} & -3/2\\ -3/2 & \frac{-9-3\sqrt{10}}{2}\end{pmatrix} \tag{2.12}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V}-\lambda\mathbf{I})$

$$\begin{aligned}&\begin{pmatrix}\frac{9-3\sqrt{10}}{2} & -3/2 & 0\\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0\end{pmatrix}\\ &\xleftrightarrow{R_1\to\frac{2}{9-3\sqrt{10}}R_1}\begin{pmatrix}1 & 3+\sqrt{10} & 0\\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0\end{pmatrix}\\ &\xleftrightarrow{R_1\to 3/2R_1+R_2}\begin{pmatrix}1 & 3+\sqrt{10} & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned} \tag{2.13}$$

So we get,

$$x_1+(3+\sqrt{10})x_2=0 \tag{2.14}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1=\frac{1}{\sqrt{20+6\sqrt{10}}}\begin{pmatrix}-(3+\sqrt{10})\\ 1\end{pmatrix} \tag{2.15}$$

Similarly for $\lambda_2=\frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2=\frac{1}{\sqrt{20-6\sqrt{10}}}\begin{pmatrix}-(3-\sqrt{10})\\ 1\end{pmatrix} \tag{2.16}$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$\implies \mathbf{D} = \begin{pmatrix} \frac{3+3\sqrt{10}}{2} & 0 \\ 0 & \frac{3-3\sqrt{10}}{2} \end{pmatrix} \tag{2.17}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}$$
$$\implies \mathbf{P} = \begin{pmatrix} \frac{-(3+\sqrt{10})}{\sqrt{20+6\sqrt{10}}} & \frac{-(3-\sqrt{10})}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \tag{2.18}$$

We know that

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{2.19}$$

where $\mathbf{c}$ is the point of intersection of the lines Let (α,β) be the point of intersection of the lines

$$\mathbf{Vc} = -\mathbf{u}$$
$$\begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.20}$$
$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1/9 \\ 8/9 \end{pmatrix}$$

Substituting values in (2.19)

$$\begin{pmatrix} \sqrt{\frac{3+3\sqrt{10}}{2}} & \pm\sqrt{\frac{3-3\sqrt{10}}{2}} \end{pmatrix} \times$$
$$\begin{pmatrix} -\frac{3+\sqrt{10}}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20+6\sqrt{10}}} \\ -\frac{3-\sqrt{10}}{\sqrt{20-6\sqrt{10}}} & -\frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \times \tag{2.21}$$
$$\begin{pmatrix} x + 1/9 \\ y - 8/9 \end{pmatrix} = 0$$

Simplifying (2.21) we get

$$3x - 3y + 3 = 0 \textit{ and } 2x + y - 2/3 = 0$$
$$\boxed{(3x - 3y + 3)(2x + y - 2/3) = 0} \tag{2.22}$$

Similarly substituting k=-7 in (1.1)

$$\mathbf{x}^T \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{2.23}$$

Equation (2.2) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}$$
$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.24}$$
$$f = -2$$

To find the separate equations of the straight lines we will use spectral decomposition.
Characteristic equation of $\mathbf{V}$ is given by:

$$\left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 6-\lambda & -7/2 \\ -7/2 & -3-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 3\lambda - 121/4 = 0 \tag{2.25}$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3+\sqrt{130}}{2}, \lambda_2 = \frac{3-3\sqrt{130}}{2} \tag{2.26}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\mathbf{Vp} = \lambda\mathbf{p}$$
$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.27}$$

For $\lambda_1 = \frac{3+\sqrt{130}}{2}$

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 \\ -7/2 & \frac{-9-\sqrt{130}}{2} \end{pmatrix} \tag{2.28}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V} - \lambda\mathbf{I})$

$$\begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix}$$
$$\xleftrightarrow{R_1 \to \frac{2}{9-\sqrt{130}}R_1} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix} \tag{2.29}$$
$$\xleftrightarrow{R_1 \to 7/2R_1 + R_2} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we get,

$$x_1 + (\frac{9+\sqrt{130}}{7})x_2 = 0 \tag{2.30}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{7}{\sqrt{260 + 18\sqrt{130}}}\begin{pmatrix} -\frac{9+\sqrt{130}}{7}) \\ 1 \end{pmatrix} \tag{2.31}$$

Similarly for $\lambda_2 = \frac{3-\sqrt{130}}{2}$

$$\mathbf{p}_2 = \frac{7}{\sqrt{260 - 18\sqrt{130}}}\begin{pmatrix} -\frac{9-\sqrt{130}}{7} \\ 1 \end{pmatrix} \tag{2.32}$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies \mathbf{D} = \begin{pmatrix} \frac{3+\sqrt{130}}{2} & 0 \\ 0 & \frac{3-\sqrt{130}}{2} \end{pmatrix} \tag{2.33}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.34}$$

$$\implies \mathbf{P} = \begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} \\ \frac{7}{\sqrt{260+18\sqrt{10}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \tag{2.35}$$

We know that

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{2.36}$$

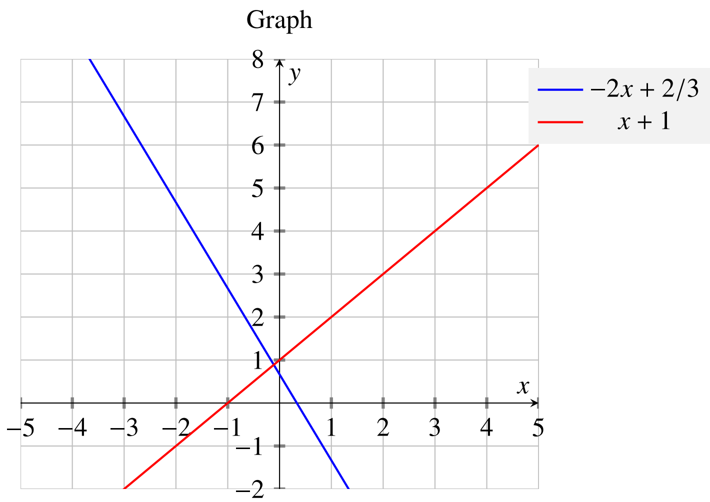

Fig. 1: This is a plot of pair of straight lines when k=-3

where **c** is the point of intersection of the lines Let (α, β) be the point of intersection of the lines

$$\mathbf{V}\mathbf{c} = -\mathbf{u}$$

$$\begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.37}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1/11 \\ 8/11 \end{pmatrix}$$

Substituting values in (2.36)

$$\begin{pmatrix} \sqrt{\frac{3+\sqrt{130}}{2}} & \pm\sqrt{\frac{3-\sqrt{130}}{2}} \end{pmatrix} \times \begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & \frac{7}{\sqrt{260+18\sqrt{130}}} \\ -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \times \begin{pmatrix} x - 1/11 \\ y - 8/11 \end{pmatrix} = 0 \tag{2.38}$$

Simplifying (2.38) we get

$$2x - 3y + 2 = 0 \; and \; 3x + y - 1 = 0$$
$$\boxed{(2x - 3y + 2)(3x + y - 1) = 0} \tag{2.39}$$

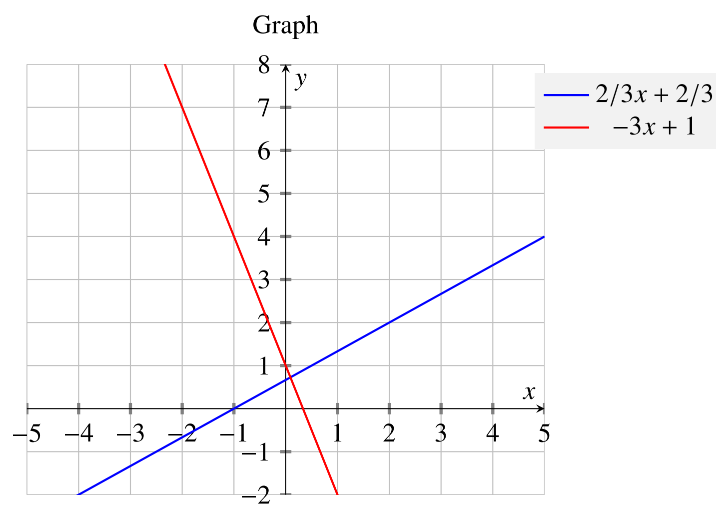

Fig. 2: This is a plot of pair of straight lines when k=-7