Assignment 7

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***Abstract*—This document explains the concept of Vector Space and its properties.**

Download all python codes from

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https://github.com/Shweta-SV/Assignment-7
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Download latex-tikz codes from

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https://github.com/Shweta-SV/Assignment-7
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1 Problem

Which of the following sets of functions from $\Re$ to $\Re$ is a vector space over $\Re$?

$$S_1 = \{f \mid \lim_{x\to 3} f(x) = 0\} \tag{1.0.1}$$

$$S_2 = \{g \mid \lim_{x\to 3} g(x) = 1\} \tag{1.0.2}$$

$$S_3 = \{h \mid \lim_{x\to 3} h(x)\ exists\} \tag{1.0.3}$$

is

1) Only S_1
2) Only S_2
3) S_1 and S_3 but not S_2
4) All the three are vector spaces

2 Theory

Let S be a set of functions. Let $f_1, f_2 \in S$ and $\alpha, \beta \in \Re$

For a set of functions to be considered as a vector space:

1) The linear combination of f_1 and f_2 should be in S.
   i.e. $\alpha f_1(x) + \beta f_2(x) \in S$
2) The $\mathbf{0}$ should belong to S
   i.e. $\mathbf{0} \in S$

3 Solution

Case1: Test for S_1

1) Let $f_1, f_2 \in S_1$ and $\alpha, \beta \in \Re$

$$\begin{aligned} \lim_{x\to 3} f_1(x) = 0 \\ \lim_{x\to 3} f_2(x) = 0 \end{aligned} \tag{3.0.1}$$

Then Using (3.0.1)

$$\begin{aligned} &\lim_{x\to 3}(\alpha f_1(x) + \beta f_2(x)) \\ &= \alpha\left(\lim_{x\to 3} f_1(x)\right) + \beta\left(\lim_{x\to 3} f_2(x)\right) \\ &= \alpha \times 0 + \beta \times 0 \\ &= 0 \\ &\therefore \alpha f_1(x) + \beta f_2(x) \in S_1 \end{aligned}$$

2) Let $f(x) = 0$
   then

$$\lim_{x\to 3} f(x) = 0$$
$$\therefore \mathbf{0} \in S_1$$

Hence, S_1 is a vector space.

Case2: Test for S_2

1) Let $g_1, g_2 \in S_2$ and $\alpha, \beta \in \Re$

$$\begin{aligned} \lim_{x\to 3} g_1(x) = 1 \\ \lim_{x\to 3} g_2(x) = 1 \end{aligned} \tag{3.0.2}$$

Then Using (3.0.2)

$$\begin{aligned} &\lim_{x\to 3}(\alpha g_1(x) + \beta g_2(x)) \\ &= \alpha\left(\lim_{x\to 3} g_1(x)\right) + \beta\left(\lim_{x\to 3} g_2(x)\right) \\ &= \alpha \times 1 + \beta \times 1 \\ &= \alpha + \beta \\ &\therefore \alpha g_1(x) + \beta g_2(x) \in S_1 \quad iff \quad \alpha + \beta = 1 \end{aligned}$$

2) Let $g(x) = 0$
   then

$$\lim_{x\to 3} g(x) = 1$$
$$\therefore \mathbf{0} \notin S_1$$

Hence, S_2 is not a vector space.

Case3: Test for S_3

1) Let $h_1, h_2 \in S_3$ and $\alpha, \beta \in \Re$

$$\begin{aligned} &\lim_{x\to 3} h_1(x) \ exists \\ &\lim_{x\to 3} h_2(x) \ exists \end{aligned} \tag{3.0.3}$$

   Then Using (3.0.3)

$$\lim_{x\to 3}(\alpha h_1(x) + \beta h_2(x)) \ exists$$
$$\therefore \alpha h_1(x) + \beta h_2(x) \in S_3$$

2) Let $h(x) = 0$
   then

$$\lim_{x\to 3^-} h(x) = 0 = \lim_{x\to 3^+} h(x)$$
$$\therefore \mathbf{0} \in S_1$$

Hence, S_3 is a vector space.

Therefore, Option (3) is correct.