

# Assignment 8

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**Abstract**—This document explains the concept of self-adjoint matrix and properties of eigen values and vectors.

Download all python codes from

<https://github.com/Shweta-SV/Assignment-8>

Download latex-tikz codes from

<https://github.com/Shweta-SV/Assignment-8>

## 1 PROBLEM

Let  $\mathbf{A}$  be an  $n \times n$  complex matrix. Assume that  $\mathbf{A}$  is a self-adjoint matrix and let  $\mathbf{B}$  denotes the inverse of  $\mathbf{A} + i\mathbf{I}$ . Then all eigen values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$  are:

- 1) purely imaginary
- 2) of modulus one
- 3) real
- 4) of modulus less than one

## 2 THEORY

- 1) If  $\mathbf{A}$  is a self-adjoint matrix, then it satisfies

$$\mathbf{A}^* = \mathbf{A} \quad (2.0.1)$$

where  $\mathbf{A}^*$  is the complex conjugate of  $\mathbf{A}$

- 2) For a self-adjoint(Hermitian) matrix the eigen values are real.
- 3) Let  $\mathbf{A}$  be an  $n \times n$  matrix,  $\lambda_A$  be its eigen values and  $\mathbf{X}$  be its eigen vector.

$$\mathbf{A}\mathbf{X} = \lambda_A\mathbf{X} \quad (2.0.2)$$

- 4) If  $\lambda_A$  be the eigen value of  $\mathbf{A}$ , then
  - a) Eigen value of  $\mathbf{A} + k\mathbf{I}$  is  $\lambda_A + k$
  - b) Eigen value of  $\mathbf{A}^p$  is  $\lambda_A^p$
  - c) Eigen value of  $\mathbf{A}^{-1}$  is  $1/\lambda_A$

## 3 SOLUTION

Since  $\mathbf{A}$  is an  $n \times n$  complex matrix and a self-adjoint matrix. Hence, eigen values of  $\mathbf{A}$  are real.

Let  $\lambda_A$  be the eigen value of  $\mathbf{A}$  and  $\mathbf{X}$  be its eigen vector.

$$\mathbf{A}\mathbf{X} = \lambda_A\mathbf{X} \quad (3.0.1)$$

The eigen value of  $\mathbf{B}$

$$\mathbf{B} = (\mathbf{A} + i\mathbf{I})^{-1}$$

Eigen value of  $\mathbf{A} + i\mathbf{I}$  is  $\lambda_A + i$

Eigen value of  $\mathbf{B}$  i.e.  $(\mathbf{A} + i\mathbf{I})^{-1}$  is  $\frac{1}{\lambda_A + i}$

Eigen value of  $\mathbf{A} - i\mathbf{I}$  is  $\lambda_A - i$

Now Using (3.0.1)

$$(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = \frac{1}{\lambda_A + i}\mathbf{X} \quad (3.0.2)$$

$$(\mathbf{A} - i\mathbf{I})\mathbf{X} = (\lambda_A - i)\mathbf{X} \quad (3.0.3)$$

Multiplying (3.0.2) by  $\mathbf{A} - i\mathbf{I}$

$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\mathbf{A} - i\mathbf{I})\frac{1}{\lambda_A + i}\mathbf{X} \quad (3.0.4)$$

Using (3.0.3) in (3.0.4)

$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\lambda_A - i)\frac{1}{\lambda_A + i}\mathbf{X}$$

$$(\mathbf{A} - i\mathbf{I})\mathbf{B}\mathbf{X} = \left(\frac{\lambda_A - i}{\lambda_A + i}\right)\mathbf{X} \quad (3.0.5)$$

From (3.0.5) the eigen values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$  are:

- 1)  $\frac{\lambda_A - i}{\lambda_A + i}$
- 2) not real
- 3) Magnitude:

$$\left|\frac{\lambda_A - i}{\lambda_A + i}\right| = \frac{\sqrt{\lambda_A^2 + 1}}{\sqrt{\lambda_A^2 + 1}} = 1 \quad (3.0.6)$$

Therefore, option (2) is correct.

What happens when the eigen values of  $\mathbf{A}$  are complex?

If  $\lambda_A$  is complex i.e.

$$\lambda_A = x + iy \quad (3.0.7)$$

from (3.0.5) Eigen values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$  are:

$$EV = \frac{\lambda_A - i}{\lambda_A + i} \quad (3.0.8)$$

Using (3.0.7) in (3.0.8) we get,

$$EV = \frac{x + i(y - 1)}{x + i(y + 1)} \quad (3.0.9)$$

Rationalizing (3.0.9) we get,

$$EV = \frac{x^2 - 2xi + y^2 - 1}{x^2 + (y + 1)^2} \quad (3.0.10)$$

From (3.0.10)

The eigen values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$  are complex.

They can be real only if the eigen values of  $\mathbf{A}$  are purely imaginary.

Verification of the result using a  $2 \times 2$  matrix.

Let

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix} \quad (3.0.11)$$

Characteristic equation of  $\mathbf{A}$ :

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= 0 \\ \Rightarrow \lambda^2 - \lambda - i &= 0 \end{aligned} \quad (3.0.12)$$

Eigen values of  $\mathbf{A}$ :

$$\begin{aligned} \lambda_1 &= -0.3 - 0.625i \\ \lambda_2 &= 1.3 + 0.625i \end{aligned} \quad (3.0.13)$$

Let  $\alpha$  be the eigen values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$

Using (3.0.10) we get

$$\begin{aligned} \alpha_1 &= -2.25 + 2.6i \\ \alpha_2 &= 0.25 - 0.6i \end{aligned} \quad (3.0.14)$$

Now let's verify (3.0.14)

$$(\mathbf{A} - i\mathbf{I})\mathbf{B} = \begin{pmatrix} -1 & 2 \\ -2i & -1 + 2i \end{pmatrix} \quad (3.0.15)$$

Characteristic equation of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ :

$$\begin{aligned} |\mathbf{A} - \alpha\mathbf{I}| &= 0 \\ \alpha^2 + (2 - 2i)\alpha + 1 + 2i &= 0 \end{aligned} \quad (3.0.16)$$

Eigen Values of  $(\mathbf{A} - i\mathbf{I})\mathbf{B}$  using (3.0.16)

$$\begin{aligned} \alpha_1 &= -2.25 + 2.6i \\ \alpha_2 &= 0.25 - 0.6i \end{aligned} \quad (3.0.17)$$

Since (3.0.14) and (3.0.17) are equal.

Hence the result is verified.

Eigen values of $\mathbf{A}$	Eigen Values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$
(1) If eigen values of $\mathbf{A}$ are real	(a) $\frac{\lambda_A - i}{\lambda_A + i}$ (b) not real (c) Magnitude = 1
(2) If eigen values of $\mathbf{A}$ are complex	(a) $\frac{x^2 - 2xi + y^2 - 1}{x^2 + (y + 1)^2}$ (b) complex
(3) If eigen values of $\mathbf{A}$ are purely imaginary	(a) $\frac{y^2 - 1}{(y + 1)^2}$ (b) real (c) Magnitude $\leq 1$