Assignment 8

Shweta Verma

***Abstract*—This document explains the concept of self-adjoint matrix and properties of eigen values and vectors.**

Download all python codes from

```
https://github.com/Shweta−SV/Assignment−8
```

Download latex-tikz codes from

```
https://github.com/Shweta−SV/Assignment−8
```

1 Problem

Let $\mathbf{A}$ be an $n \times n$ complex matrix. Assume that $\mathbf{A}$ is a self-adjoint matrix and let $\mathbf{B}$ denotes the inverse of $\mathbf{A} + i\mathbf{I}$. Then all eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are:

1) purely imaginary
2) of modulus one
3) real
4) of modulus less than one

2 Theory

1) If $\mathbf{A}$ is a self-adjoint matrix, then it satisfies
$$\mathbf{A}^* = \mathbf{A} \tag{2.0.1}$$
where $\mathbf{A}^*$ is the complex conjugate of $\mathbf{A}$
2) For a self-adjoint(Hermitian) matrix the eigen values are real.
3) Let $\mathbf{A}$ be an $n \times n$ matrix, λ_A be its eigen values and $\mathbf{X}$ be its eigen vector.
$$\mathbf{AX} = \lambda_A \mathbf{X} \tag{2.0.2}$$
4) If λ_A be the eigen value of $\mathbf{A}$, then
   a) Eigen value of $\mathbf{A} + k\mathbf{I}$ is $\lambda_A + k$
   b) Eigen value of $\mathbf{A}^p$ is λ_A^p
   c) Eigen value of $\mathbf{A}^{-1}$ is $1/\lambda_A$

3 Solution

Since $\mathbf{A}$ is an $n \times n$ complex matrix and a self-adjoint matrix. Hence, eigen values of $\mathbf{A}$ are real. Let λ_A be the eigen value of $\mathbf{A}$ and $\mathbf{X}$ be its eigen vector.
$$\mathbf{AX} = \lambda_A \mathbf{X} \tag{3.0.1}$$
The eigen value of $\mathbf{B}$
$$\mathbf{B} = (\mathbf{A} + i\mathbf{I})^{-1}$$
Eigen value of $\mathbf{A} + i\mathbf{I}$ is $\lambda_A + i$
Eigen value of $\mathbf{B}$ i.e. $(\mathbf{A} + i\mathbf{I})^{-1}$ is $\frac{1}{\lambda_A + i}$
Eigen value of $\mathbf{A} - i\mathbf{I}$ is $\lambda_A - i$
Now Using (3.0.1)
$$(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = \frac{1}{\lambda_A + i}\mathbf{X} \tag{3.0.2}$$
$$(\mathbf{A} - i\mathbf{I})\mathbf{X} = (\lambda_A - i)\mathbf{X} \tag{3.0.3}$$
Multiplying (3.0.2) by $\mathbf{A} - i\mathbf{I}$
$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\mathbf{A} - i\mathbf{I})\frac{1}{\lambda_A + i}\mathbf{X} \tag{3.0.4}$$
Using (3.0.3) in (3.0.4)
$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\lambda_A - i)\frac{1}{\lambda_A + i}\mathbf{X}$$
$$(\mathbf{A} - i\mathbf{I})\mathbf{BX} = \left(\frac{\lambda_A - i}{\lambda_A + i}\right)\mathbf{X} \tag{3.0.5}$$
From (3.0.5) the eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are:

1) $\frac{\lambda_A - i}{\lambda_A + i}$
2) not real
3) Magnitude:
$$\left|\frac{\lambda_A - i}{\lambda_A + i}\right| = \frac{\sqrt{\lambda_A^2 + 1}}{\sqrt{\lambda_A^2 + 1}} = 1 \tag{3.0.6}$$

Therefore, option (2) is correct.

What happens when the eigen values of $\mathbf{A}$ are complex?
If λ_A is complex i.e.
$$\lambda_A = x + iy \tag{3.0.7}$$

from (3.0.5) Eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ are:

$$EV = \frac{\lambda_A - i}{\lambda_A + i} \tag{3.0.8}$$

Using (3.0.7) in (3.0.8) we get,

$$EV = \frac{x + i(y-1)}{x + i(y+1)} \tag{3.0.9}$$

Rationalizing (3.0.9) we get,

$$EV = \frac{x^2 - 2xi + y^2 - 1}{x^2 + (y+1)^2} \tag{3.0.10}$$

From (3.0.10)
The eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ are complex.
They can be real only if the eigen values of $\mathbf{A}$ are purely imaginary.
Verification of the result using a 2×2 matrix.
Let

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix} \tag{3.0.11}$$

Characteristic equation of $\mathbf{A}$:

$$\begin{aligned} |\mathbf{A}-\lambda\mathbf{I}| &= 0 \\ \implies \lambda^2 - \lambda - i &= 0 \end{aligned} \tag{3.0.12}$$

Eigen values of $\mathbf{A}$:

$$\begin{aligned} \lambda_1 &= -0.3 - 0.625i \\ \lambda_2 &= 1.3 + 0.625i \end{aligned} \tag{3.0.13}$$

Let α be the eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$
Using (3.0.10) we get

$$\begin{aligned} \alpha_1 &= -2.25 + 2.6i \\ \alpha_2 &= 0.25 - 0.6i \end{aligned} \tag{3.0.14}$$

Now let's verify (3.0.14)

$$(\mathbf{A}-i\mathbf{I})\mathbf{B} = \begin{pmatrix} -1 & 2 \\ -2i & -1+2i \end{pmatrix} \tag{3.0.15}$$

Characteristic equation of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$:

$$\begin{aligned} |\mathbf{A}-\alpha\mathbf{I}| &= 0 \\ \alpha^2 + (2-2i)\alpha + 1 + 2i &= 0 \end{aligned} \tag{3.0.16}$$

Eigen Values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ using (3.0.16)

$$\begin{aligned} \alpha_1 &= -2.25 + 2.6i \\ \alpha_2 &= 0.25 - 0.6i \end{aligned} \tag{3.0.17}$$

Since (3.0.14) and (3.0.17) are equal.
Hence the result is verified.