

Assignment 8

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Abstract—This document explains the concept of self-adjoint matrix and properties of eigen values and vectors.

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<https://github.com/Shweta-SV/Assignment-8>

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<https://github.com/Shweta-SV/Assignment-8>

1 PROBLEM

Let \mathbf{A} be an $n \times n$ complex matrix. Assume that \mathbf{A} is a self-adjoint matrix and let \mathbf{B} denotes the inverse of $\mathbf{A} + i\mathbf{I}$. Then all eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are:

- 1) purely imaginary
- 2) of modulus one
- 3) real
- 4) of modulus less than one

2 THEORY

- 1) If \mathbf{A} is a self-adjoint matrix, then it satisfies

$$\mathbf{A}^* = \mathbf{A} \quad (2.0.1)$$

where \mathbf{A}^* is the complex conjugate of \mathbf{A}

- 2) For a self-adjoint(Hermitian) matrix the eigen values are real.
- 3) Let \mathbf{A} be an $n \times n$ matrix, λ_A be its eigen values and \mathbf{X} be its eigen vector.

$$\mathbf{A}\mathbf{X} = \lambda_A\mathbf{X} \quad (2.0.2)$$

- 4) If λ_A be the eigen value of \mathbf{A} , then
 - a) Eigen value of $\mathbf{A} + k\mathbf{I}$ is $\lambda_A + k$
 - b) Eigen value of \mathbf{A}^p is λ_A^p
 - c) Eigen value of \mathbf{A}^{-1} is $1/\lambda_A$

3 SOLUTION

Since \mathbf{A} is an $n \times n$ complex matrix and a self-adjoint matrix. Hence, eigen values of \mathbf{A} are real.

Let λ_A be the eigen value of \mathbf{A} and \mathbf{X} be its eigen vector.

$$\mathbf{A}\mathbf{X} = \lambda_A\mathbf{X} \quad (3.0.1)$$

The eigen value of \mathbf{B}

$$\mathbf{B} = (\mathbf{A} + i\mathbf{I})^{-1}$$

Eigen value of $\mathbf{A} + i\mathbf{I}$ is $\lambda_A + i$

Eigen value of \mathbf{B} i.e. $(\mathbf{A} + i\mathbf{I})^{-1}$ is $\frac{1}{\lambda_A + i}$

Eigen value of $\mathbf{A} - i\mathbf{I}$ is $\lambda_A - i$

Now Using (3.0.1)

$$(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = \frac{1}{\lambda_A + i}\mathbf{X} \quad (3.0.2)$$

$$(\mathbf{A} - i\mathbf{I})\mathbf{X} = (\lambda_A - i)\mathbf{X} \quad (3.0.3)$$

Multiplying (3.0.2) by $\mathbf{A} - i\mathbf{I}$

$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\mathbf{A} - i\mathbf{I})\frac{1}{\lambda_A + i}\mathbf{X} \quad (3.0.4)$$

Using (3.0.3) in (3.0.4)

$$(\mathbf{A} - i\mathbf{I})(\mathbf{A} + i\mathbf{I})^{-1}\mathbf{X} = (\lambda_A - i)\frac{1}{\lambda_A + i}\mathbf{X}$$

$$(\mathbf{A} - i\mathbf{I})\mathbf{B}\mathbf{X} = \left(\frac{\lambda_A - i}{\lambda_A + i}\right)\mathbf{X} \quad (3.0.5)$$

From (3.0.5) the eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are:

- 1) $\frac{\lambda_A - i}{\lambda_A + i}$
- 2) not real
- 3) Magnitude:

$$\left|\frac{\lambda_A - i}{\lambda_A + i}\right| = \frac{\sqrt{\lambda_A^2 + 1}}{\sqrt{\lambda_A^2 + 1}} = 1 \quad (3.0.6)$$

Therefore, option (2) is correct.

What happens when the eigen values of \mathbf{A} are complex?

If λ_A is complex i.e.

$$\lambda_A = x + iy \quad (3.0.7)$$

from (3.0.5) Eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are:

$$EV = \frac{\lambda_A - i}{\lambda_A + i} \quad (3.0.8)$$

Using (3.0.7) in (3.0.8) we get,

$$EV = \frac{x + i(y - 1)}{x + i(y + 1)} \quad (3.0.9)$$

Rationalizing (3.0.9) we get,

$$EV = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \quad (3.0.10)$$

From (3.0.10)

The eigen values of $(\mathbf{A} - i\mathbf{I})\mathbf{B}$ are real.