Assignment 9

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Abstract—**This document explains the concept of diagonalizability of a matrix.**

Download latex-tikz codes from

https://github.com/Shweta−SV/Assignment−9

1 Problem

Which of the following matrices is not diagonalizable over $\Re$?

1) $\begin{pmatrix}1 & 1 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}$
2) $\begin{pmatrix}1 & 1 & 0\\0 & 2 & 1\\0 & 0 & 3\end{pmatrix}$
3) $\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$
4) $\begin{pmatrix}1 & 0 & 1\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$

2 Theory

A matrix $\mathbf{A}$ is diagonalizable if it is similar to a diagonal matrix i.e.
if there exists an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{2.0.1}$$

1) If the Eigen values of a matrix are distinct then the matrix is diagonalizable.
2) If the Eigen values of a matrix are not distinct then : if Arithmetic multiplicity and geometric multiplicity are equal then the matrix is diagonalizable.
3) Arithmetic multiplicity (AM) = multiplicity of eigen values
4) Geometric multiplicity (GM) = no of variables - Rank($\mathbf{A} - \lambda\mathbf{I}$)

3 Solution

Option 1:

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned}\left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix}1-\lambda & 1 & 0\\0 & 2-\lambda & 0\\0 & 0 & 1-\lambda\end{vmatrix} = 0\\ \implies (1-\lambda)(1-\lambda)(2-\lambda) = 0\end{aligned} \tag{3.0.1}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2 \tag{3.0.2}$$

for $\lambda = 1$: AM = 2
for $\lambda = 1$: AM = 1
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned}\mathbf{A}\mathbf{p} = \lambda\mathbf{p}\\ \implies (\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = 0\end{aligned} \tag{3.0.3}$$

For $\lambda = 1$

$$\begin{aligned}(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{p} = \begin{pmatrix}0 & 1 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\mathbf{p}_1 = 0\\ \xleftrightarrow{R_2 \to R_2 - R_1}\begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\mathbf{p}_1 = 0\end{aligned} \tag{3.0.4}$$

From (3.0.4) Rank is
$R(\mathbf{A} - \lambda\mathbf{I}) = 1$
For $\lambda = 2$

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{p} = \begin{pmatrix}-1 & 1 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}\mathbf{p} = 0 \tag{3.0.5}$$

From (3.0.5) Rank is-
$R(\mathbf{A} - \lambda_1\mathbf{I}) = 2$ for $\lambda = 1$: GM = 2
for $\lambda = 1$: GM = 1
AM and GM are equal.
Hence we obtained that $\mathbf{A}$ is diagonalizable.

Option 2:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \\ \implies (1-\lambda)(2-\lambda)(3-\lambda) = 0 \end{aligned} \tag{3.0.6}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{3.0.7}$$

Since the Eigen values of the matrix are distinct Hence the matrix is diagonalizable.
Option 3:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \\ \implies (1-\lambda)(1-\lambda)(2-\lambda) = 0 \end{aligned} \tag{3.0.8}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2 \tag{3.0.9}$$

for $\lambda = 1$: AM $= 2$
for $\lambda = 1$: AM $= 1$
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned} \mathbf{A}\mathbf{p} = \lambda\mathbf{p} \\ \implies (\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = 0 \end{aligned} \tag{3.0.10}$$

For $\lambda = 1$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{p} = 0 \tag{3.0.11}$$

from (3.0.11) Rank is-
$R(\mathbf{A} - \lambda\mathbf{I}) = 2$
For $\lambda = 2$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{p}_3 = 0 \tag{3.0.12}$$

from (3.0.12) Rank is-
$R(\mathbf{A} - \lambda\mathbf{I}) = 2$

for $\lambda = 1$: GM $= 1$
for $\lambda = 1$: GM $= 1$
AM and GM are not equal.
Hence we obtained that $\mathbf{A}$ is not diagonalizable.
Option 4:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \\ \implies (1-\lambda)(2-\lambda)(3-\lambda) = 0 \end{aligned} \tag{3.0.13}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{3.0.14}$$

Since the Eigen values of the matrix are distinct Hence the matrix is diagonalizable.