Assignment 9

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***Abstract*—This document explains the concept of diagonalizability of a matrix.**

Download all python codes from

https://github.com/Shweta−SV/Assignment−9

Download latex-tikz codes from

https://github.com/Shweta−SV/Assignment−9

1 Problem

Which of the following matrices is not diagonalizable over $\Re$?

1) $\begin{pmatrix}1 & 1 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}$
2) $\begin{pmatrix}1 & 1 & 0\\0 & 2 & 1\\0 & 0 & 3\end{pmatrix}$
3) $\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$
4) $\begin{pmatrix}1 & 0 & 1\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$

2 Theory

A matrix $\mathbf{A}$ is diagonalizable if it is similar to a diagonal matrix i.e.
if there exists an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{2.0.1}$$

If the Eigen values of a matrix are distinct then the matrix is diagonalizable.

3 Solution

Option 1:

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\left|A - \lambda\mathbf{I}\right| = \begin{vmatrix}1-\lambda & 1 & 0\\0 & 2-\lambda & 0\\0 & 0 & 1-\lambda\end{vmatrix} = 0 \qquad \text{(3.0.1)}$$
$$\implies (1-\lambda)(1-\lambda)(2-\lambda) = 0$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2 \tag{3.0.2}$$

Let $\mathbf{p}_1$,$\mathbf{p}_2$ and $\mathbf{p}_3$ be the Eigen vector corresponding to λ_1,λ_2 and λ_3 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned}\mathbf{Ap} &= \lambda\mathbf{p}\\ \implies (\mathbf{A}-\lambda\mathbf{I})\mathbf{p} &= 0\end{aligned} \tag{3.0.3}$$

For $\lambda_1 = 1$

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{p} = \begin{pmatrix}0 & 1 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\mathbf{p}_1 = 0$$

$$\implies \mathbf{p}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \tag{3.0.4}$$

$$\implies \mathbf{p}_2 = \begin{pmatrix}1\\0\\2\end{pmatrix}$$

For $\lambda_3 = 2$

$$\begin{aligned}(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{p}_3 = \begin{pmatrix}-1 & 1 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}\mathbf{p}_1 = 0\\ \implies \mathbf{p}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}\end{aligned} \tag{3.0.5}$$

$$\begin{aligned}\mathbf{P} &= \begin{pmatrix}1 & 0 & 1\\1 & 0 & 2\\1 & 1 & 0\end{pmatrix}\\ \mathbf{D} &= \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}\end{aligned} \tag{3.0.6}$$

Here Using (3.0.6) in (2.0.1) we obtained that $\mathbf{A}$ is diagonalizable.
Option 2:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|A - \lambda\mathbf{I}\right| &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \\ \implies &(1-\lambda)(2-\lambda)(3-\lambda) = 0 \end{aligned} \tag{3.0.7}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{3.0.8}$$

Since the Eigen values of the matrix are distinct Hence the matrix is diagonalizable.
Option 3:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|A - \lambda\mathbf{I}\right| &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \\ \implies &(1-\lambda)(1-\lambda)(2-\lambda) = 0 \end{aligned} \tag{3.0.9}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2 \tag{3.0.10}$$

Let $\mathbf{p}_1$,$\mathbf{p}_2$ and $\mathbf{p}_3$ be the Eigen vector corresponding to λ_1,λ_2 and λ_3 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\begin{aligned} \mathbf{A}\mathbf{p} &= \lambda\mathbf{p} \\ \implies (\mathbf{A} - \lambda\mathbf{I})\mathbf{p} &= 0 \end{aligned} \tag{3.0.11}$$

For $\lambda_1 = 1$

$$\begin{aligned} (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{p} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{p}_1 = 0 \\ \implies \mathbf{p}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{3.0.12}$$

$$\implies \mathbf{p}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{p} &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{p}_3 = 0 \\ \implies \mathbf{p}_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{3.0.13}$$

$$\begin{aligned} \mathbf{P} &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{D} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned} \tag{3.0.14}$$

Here Using (3.0.14) in (2.0.1) we obtained that $\mathbf{A}$ is not diagonalizable.
Option 4:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic equation of $\mathbf{A}$ is given by:

$$\begin{aligned} \left|A - \lambda\mathbf{I}\right| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \\ \implies &(1-\lambda)(2-\lambda)(3-\lambda) = 0 \end{aligned} \tag{3.0.15}$$

The Eigen Values of $\mathbf{A}$ are:

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{3.0.16}$$

Since the Eigen values of the matrix are distinct Hence the matrix is diagonalizable.