Simulation of Jump Markov Linear Systems with Rao-Blackwellized Particle Filters and Applications to Financial Modeling and Trading

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***Abstract*—Jump Markov Linear Systems (JMLS) are a much more generalized representation of stochastic statespace models such as Hidden Markov Models and linear Gaussian state space models. By closely following the approach of Doucet et. al. [1], for a tracking application of a JMLS, a particle filter (PF) and Rao-Blackwellized Particle Filter (RBPF) were used. The performance of the tracking was also compared with an extended Kalman Filter (EKF). We give some practical applications of this method to: 1) Statistical arbitrage and pairs trading, 2) Modeling of macroeconomic indicators and signals, 3) Asset dynamics modeling via stochastic differential equations(SDE).**

***Index Terms*—JMLS, MCMC, RBPF, EKF, jump Markov linear system, financial modeling, trading, particle filter, Rao-Blackwellized particle filter, Markov chain Monte Carlo**

I. Introduction

A category of linear systems whose parameters evolve based on a discrete time Markov chain is called the Jump Markov Linear Systems(JMLS). This paper discusses the evaluation of the parameters of JMLS using the particle filter, extended Kalman filter, as well as the Rao-Blackwellized particle filter(RBPF) to sample some of the variables and marginalize out the remaining variables in order to obtain accurate results by reducing the variance. Further the paper discusses the following three applications to which JMLS could be applied:

Statistical arbitrage and pairs trading:
This application discusses about two time series which are cointegrated with order of integration of the linear combination of these series is less than the order of integration of either time series.

Macroeconomic factor modeling:
Macroeconomic indicators such as volatility, market indices, and inflation rates are important for economic policy as well as regime switches (such as periods of growth and downturn in the economy). This application discusses deducing various economic factors based on computations for economic signals, and trying to estimate the ”true state” of the economy. A few examples relate to predicting the ”true” volatility of the economy with the S & P VIX as a signal, credit ratings of various companies, and factor signals that are specific to companies.

Asset dynamics modeling via Stochastic differential equations:
This application discusses a shorter term domain for asset valuation and hedging - modeling the short term asset dynamics in the time domain (such as equity option and short-rates). Volume or time scales of traded assets within a short period of time can be modeled by a set of SDEs. This can be broken up into two SDEs, one representing the true, underlying asset state, and the signal (which may be indicated by prices or volume traded). A corresponding stochastic statespace model of the discretized evolution can be used as a model, upon which a filtering algorithm such as RBPF may be utilized to predict the asset dynamics.

II. Theory

Let us consider a general state space model with hidden Markov process r_t, continuous state process x_t and observed noisy variables y_t. r_t is a Markov process with with initial probability distribution $p(r_0)$ and transition probability as $p(r_t|r_{t-1})$. Then the jump system can be given by:

$$x_{t+1} = A(r_{t+1})x_t + \Gamma(r_{t+1})w_{t+1} + f(r_{t+1})u_{t+1} \quad (1)$$

$$y_t = C(r_{t+1})x_t + D(r_t)v_t + g(r_t)u_t \quad (2)$$

where u_k is the input to the system and w_k and v_k are Gaussian noise.

A. Related work

Recent decades have seen incredible developments in the online state estimation space because of its prolific use in the fields of engineering and computer science. JMLS has several applications in control systems, finance [4] [2] etc., which require online state estimation. One of the powerful algorithms used for state estimation in a JMLS is particle filter [1], [20] [19]. [1] derives recursive algorithms for optimal state estimation of JMLS. One of the major drawbacks [13] [14] of particle filtering technique is that the estimator becomes inefficient in terms of mean-square error(MSE) estimate in high dimensional spaces. Several works in the past have focused on improving efficiency of the estimator. [21] tries to overcome this problem by integrating dual tree recursions and fast multi-pole techniques with forward-backward smoothers. However this method assumes availability of large number(over a million) of particles. Several algorithms to enhance smoothing [24] and sampling strategies [23] [22] have been proposed and studied in an attempt to indirectly approach this problem. This paper utilizes Rao-Blackwellized particle filter [18] [5] to estimate the latent variable and compute the conditionally Gaussian part using Kalman filter. Further, the paper discusses the real world applications of JMLS in the field of finance [9]. For statistical arbitrage and pairs trading that are cointegrated, Kalman filters are presently used [6] [10]. The cointegration of securities, motivates towards considering one of the securities as the latent variable and other as the continuous time process.

III. Problem Formulation

The basic premise of the problem is formulated as follows:

The state (in this case, an object) moves according to dynamics coherent with a JMLS. However, this true state is not observed. A noisy signal of the state is measured (also coherently) according to dynamics observed by a JMLS. Based on this, state estimation is performed by using a few different methods: 1) Extended Kalman Filters, 2) Particle Filters, and 3) Rao-Blackwellized Particle Filters.

After this initial technique validation, three possible applications are explored: 1) estimate the time series of one series with respectd to another time series (where the two series are cointegrated), in order to capitalize on the differential, 2) model macroeconomic indicators by estimating the underlying (but unobserved state), one of which is the volatility index VIX, and 3) state estimation of a stochastic differential equation with two SDEs which are coupled together.

IV. Real World Applications

There are various applications of tracking and statistical parameter estimation techniques. These filtering algorithms can be done in an offline or online approach, i.e., batch or real-time. Besides the tracking applications outlined in [1], there are applications to tracking Poisson jump processes(Elliott et. al. [2]) and parameter and state estimation of time-varying stochastic statespace models. This paper poses three possible applications to financial modeling and trading:

1) Statistical arbitrage and pairs trading based on cointegration - Time series analysis is used for modeling various economic phenomena. There are many models for modeling complex time series data, one of which is the most general time series model being an $ARIMA(p,d,q)$ model, where an autoregressive and moving average time series are differenced by an integration factor d.

This can be represented as follows - given an $ARMA(p',q)$ model, the model is given by:

$$X_t-\alpha_1 X_{t-1}-\ldots-\alpha_{p'}X_{t-p'} = \varepsilon_t+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}$$

$$(1-\sum_{i=1}^{p'}\alpha_i L^i)X_t = (1+\sum_{i=1}^{q}\theta_i L^i)\varepsilon_t$$

where L is the lag operator.
Now, for any given time series X_t, an integration order d is given when the differenced time series, denoted by ϵ_t, is stationary, i.e.,

$$(1-L)^d X_t = \Delta^d X_t \sim \epsilon_t$$

Assuming the polynomial term for $AR(p')$ has a unit root (this fails the augmented Dickey-Fuller (ADF) test) of multiplicty d), then the polynomial can be re-written as:

$$(1-\sum_{i=1}^{p'}\alpha_i L^i) = (1-\sum_{i=1}^{p'-d}\phi_i L^i)(1-L)^d$$

Now, the above $ARMA(p',q)$ model that has unit root of multiplicity d can be written as:

$$(1-\sum_{i=1}^{p'}\alpha_i L^i)X_t$$

$$= (1-\sum_{i=1}^{p'-d}\phi_i L^i)(1-L)^d X_t = (1+\sum_{i=1}^{q}\theta_i L^i)\varepsilon_t$$

This is the $ARIMA(p,d,q)$ process. This can be thought of as an $ARMA(p+d,q)$ process with d unit roots.

A generalization of this is

$$(1-\sum_{i=1}^{p'-d}\phi_i L^i)(1-L)^d X_t = \delta + (1+\sum_{i=1}^{q}\theta_i L^i)\varepsilon_t$$

where δ is a drift term associated with this $ARIMA(p,d,q)$ model.

For a set of time series $[n]$, the collection of time series is cointegrated if $\exists$ order of integration d' that is smaller than d. Multicointegration explains this relationship for multiple orders of differencing. In order to look at the intuition, a simpler example can be used. Assuming $n = 2$, with two time series x_t and y_t that are cointegrated, then the linear combination of the series must be a stationary distribution u_t, i.e.,

$$x_t - \beta y_t \sim u_t$$

where the linear differenced series is stationary, one way to think about this is to look at the difference (or the gap) between two time series. This may happen as shown in Fig. 1 below:

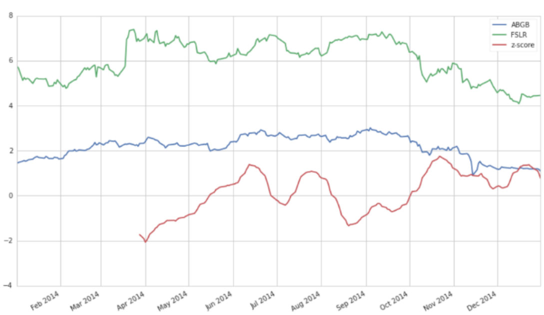


Fig. 1: ABGB and FSLR cointegration

The represented data are the ticker symbols ABGB and FSLR, which are Abengoa, S.A., and First Solar, respectively, as well as the moving average time series of the linear-differenced time series. The ”gap” time series process, which is the differential between ABGB and FSLR, can be seen to have a relatively stable mean, as well as a variance that seems stationary. However, the differenced time series shows the z-score of the moving average, which was used to create a trading strategy in Quantopian, a Python platform for financial modeling, trading strategy research and development, and backtesting.

Another example for cointegration is data from 1954-1994 Canadian interest rates in Fig. 2.

Afterwards, the differenced time series is represented in Fig. 3.

These figures show the cointegrated times series obtained from the cointegration analysis, as well as multi-cointegration (by differencing multiple times). Based on this data, similarly, the interest rate can be hedged (or speculated) by predicting this cointegrated time series terms based on the underlying, observable interest rate time series with a postulated model for the interest rate models with more complex frameworks such as the

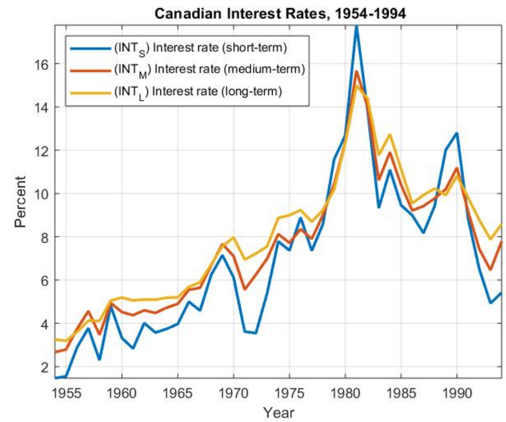


Fig. 2: Canadian Interest Rates

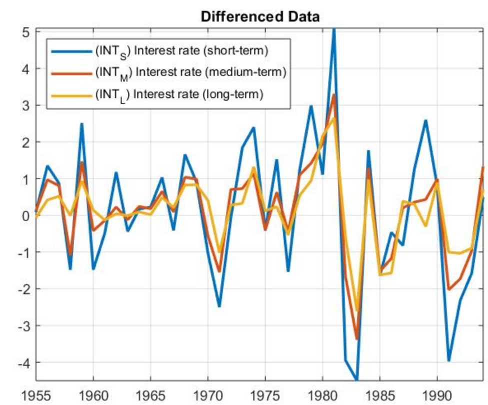
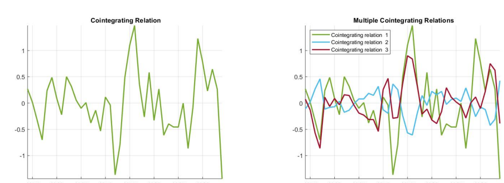


Fig. 3: Differenced Canadian Interest Rates

Heath-Jarrow-Morton (HJM) framework (Heath, Jarrow, Morton [25]).

The cointegrated and multicointegrated results are shown below in Fig. 4:



(a) Cointegration analysis

(b) Multicointegration analysis

Fig. 4

Cointegration viewed mathematically was as such, but there is also an intuitive way to view the cointegration relationships. Economic phenomena that have certain relationships that may be lagged can likely be viewed as being cointegrated. For example, the ticker symbols ABGB and FSLR are related since they are in the same verticals (ABGB uses renewable energy technology such as solar cells, whereas FSLR produces solar cells). Therefore, an economic production/consumer relationship along the consumer chain will have lagged indicators in the time series that signals cointegration.

These cointegration relationships may still be present during economic regime changes, but different external factors that may shift the production frontier or market demand curve in a production vertical may allow "jumps" in the statespace model, thus a Jump Markov Linear System can capture these state shifts that occur. One example of this may be when a material or production equipment can be produced at a cheaper cost, thus shifting the cointegration relationship. There may be still be a cointegration relationship, but the fundamental difference may push observed asset prices of one vs. the other to a new relationship. An RBPF may be able to capture these shifts and predict the new parameter estimates quickly, thus being able to model this phenomena much more efficiently.

For other factors such as risk-free interest rates, the relationship between various maturities should have cointegration relationships since risk-free interest rates should have the same yield, regardless of maturity. According to financial economic theory, this should be the case; however, this does not always happen and thus this leads to arbitrage and trading opportunity around U.S. Treasuries and LIBOR (London Inter-bank Offered Rate). This leads into the next part on macroeconomic factor modeling.

2) Macroeconomic factor and indicator modeling for policy - Macroeconomic factors such as inflation rates, risk-free interest rates, credit ratings of companies, true volatility vs. implied volatility can be tough to model. However, with the advent of big data, powerful computational methods and infrastructure, and better mathematical models, various such phenomena may be modeled using a framework that aligns with economic theory. One application of this is in estimating the factors in multi-factor models such as the Fama-French 3-Factor Models [28] or more complicated ones such as that proposed by Rickard and Torre [27]. Since economic indicators are often signals that are observed by a calculation methodology (credit rating algorithms by various companies such as S & P, Moody's and Fitch), the underlying, unobserved states can be estimated assuming a statespace model. With a Jump Markov Linear System, in addition to estimating the underlying state, the model may be used to predict regime changes such as growth economy, neutral (peak/trough) economy, downward economy, as well as various indicators that may signal this.

Another method is calculating volatility, which is in reality an unobserved metric. There are various stochastic volatility models to deal with this, and implied volatility [34] is used to calculate the volatility of an option chain, as shown in Fig.5 below, which was obtained on the NASDAQ website on May 9, 2019:

Option Chain for Apple Inc. (AAPL)

Calls	Last	Chg	Bid	Ask	Vol	Open Int	Root	Strike	Puts	Last	Chg	Bid	Ask	Vol	Open Int
May 03, 2019	29.13				0	0	AAPL	182.5	May 03, 2019	0.01				0	5136
May 03, 2019	26.67				0	0	AAPL	185	May 03, 2019	0.01				0	9278
May 03, 2019	24.13				0	0	AAPL	187.5	May 03, 2019	0.01				0	7341
May 03, 2019	21.73				0	0	AAPL	190	May 03, 2019	0.01				0	23395
May 03, 2019	18.98				0	0	AAPL	192.5	May 03, 2019	0.01				0	5793
May 03, 2019	16.71				0	0	AAPL	195	May 03, 2019	0.01				0	24888
May 03, 2019	14.17				0	0	AAPL	197.5	May 03, 2019	0.01				0	13103
May 03, 2019	11.75				0	0	AAPL	200	May 03, 2019	0.01				0	18446
May 03, 2019	9.30				0	0	AAPL	202.5	May 03, 2019	0.01				0	7016
May 03, 2019	6.75				0	0	AAPL	205	May 03, 2019	0.01				0	12958
May 03, 2019	4.26				0	0	AAPL	207.5	May 03, 2019	0.01				0	10688
May 03, 2019	1.80				0	0	AAPL	210	May 03, 2019	0.01				0	12924
May 03, 2019	0.02				0	13380	AAPL	212.5	May 03, 2019	0.73				0	2
May 03, 2019	0.01				0	19355	AAPL	215	May 03, 2019	3.24				0	0
May 03, 2019	0.01				0	10418	AAPL	217.5	May 03, 2019	6.04				0	0
May 03, 2019	0.01				0	14428	AAPL	220	May 03, 2019	8.32				0	0
May 10, 2019	17.50	-4.70	18.00	18.20	901	148	AAPL	182.5	May 10, 2019	0.05	0.02	0.04	0.05	1945	1881
May 10, 2019	15.25	-4.30	15.50	15.75	1377	254	AAPL	185	May 10, 2019	0.08	0.03	0.06	0.08	4317	3947

Fig. 5: Option Chain for AAPL on May 9, 2019 on NASDAQ

The VIX index is calculated by the Chicago Board Options Exchange, as described in [26], and shows how the calculation of option spreads between call and put options of different maturities and this is used as a basis for volatility trading (via options). As volatility is an indicator of how "risky" the economy is, predicting the true, unobserved volatility "state" and the change that may occur depending on other factors is a useful exercise.

As shown above, a Rao-Blackwellized Particle Filter may be used to estimate these continuous and discrete state changes, and then based on this, formulate a model for risk management and auditing, observe trading opportunities, or create valuation models for companies. This prediction can be used as the intermediate step, providing the link between positive and normative economics. As with descriptive statistics providing the data necessary to perform predictive analytics, which then aids in decision-making in the prescriptive analytics phase, this prediction of the unobserved, true state of the economy may aid in normative economics (which deals with economic policy).

3) Asset dynamics modeling via stochastic differential equations - The above two applications dealt with longer-term and medium-term dynamics. However, when looking at short-term asset models via stochastic differential equations (SDEs), which may be useful for modeling in high frequency domains. The stochastic differential equation may be modeled as a stochastic statespace model, then filtering can be used to deduce true, unobserved states as described in [7].

A stochastic differential equation formulation of an asset, in its most general form, is given below:

$$dS_t = \mu^s(\Theta, S_t, X_t)dt + \sigma^s(\Theta, S_t, X_t)dW_t^s$$

$$dX_t = \mu^x(\Theta, S_t, X_t)dt + \sigma^x(\Theta, S_t, X_t)dW_t^x$$

This is a model of asset dynamics of SDE as observed (y_t) vs. true state (x_t). The terms dW_t is a standard Brownian motion, whereas the σ represents the diffusion associated with the respective Brownian motion. the μ is the drift term across time dt. These stochastic differential equations are represented by the drift and diffusion terms in physical and engineering applications, while the financial analogs call these terms the momentum and volatility terms, respectively.

This is important in applications of any option pricing, hedging, trading, risk management, etc. In fact, the famous Black-Scholes option pricing formula is based on a geometric Brownian motion SDE in a simple sense [29], [30].

The general solution to the set of SDEs above can be written as follows:

$$S_t = S_0 + \int_0^t \mu^s(\Theta, S_{t'}, X_{t'})dt' + \int_0^t \sigma^s(\Theta, S_{t'}, X_{t'})dW_{t'}^s + \sum_{j=1}^{N_t^s} f^s(\xi_j^s, S_{\tau_j^s}, X_{\tau_j^s})$$

$$X_t = X_0 + \int_0^t \mu^x(\Theta, S_{t'}, X_{t'})dt' + \int_0^t \sigma^x(\Theta, S_{t'}, X_{t'})dW_{t'}^x + \sum_{j=1}^{N_t^x} f^x(\xi_j^x, S_{\tau_j^x}, X_{\tau_j^x})$$

This integral in reality is tough to compute analytically, thus other methods such as finite difference methods (FDMs) on the discretization of the model may be used. The discretization can be represented as a stochastic statespace model as with [32]. The procedure to then put everything together would be as follows:

1. Posit SDE model for underlying unobserved asset state and signal

2. Perform state estimation via filtering algorithm (i.e. RBPF)

3. Use FDM or other techniques to get SDE integral calculation

4. Verify model when data observed

This can be used to create a model of asset dynamics, with which hedging, speculation, valuation, and model validation, and much more can be performed. Example models include credit models such as vyne copulae, short-rate models including Vasicek, Hull-White, Cox-Ingersoll-Ross (CIR), and equity option pricing models. An example of an application with regime switches is given in [31].

V. Algorithms and Discussion

A. Extended Kalman Filter

The Extended Kalman (EKF) filter is a useful model for a nonlinear (but Gaussian) statespace model. The general nonlinear stochastic space model can be written as:

$$x_{k+1} = f(k, x_k) + w_k \tag{3}$$

$$y_k = h(k, x_k) + e_k \tag{4}$$

Assuming differentiability, the EKF estimate can be calculated for the iteratively as follows from [8]:

$$\hat{x}_{k+1|k} = f(k, \hat{x}_k) + K_k[y_k - h(k, \hat{x}_{k|k-1})] \tag{5}$$

Where the following equations are as such:

$$\hat{K}_k = \text{F(k, } {}_{k|k-1})P_k H^T(k, \hat{x}_{k|k-1})[H(k, \hat{x}_{k|k-1}) P_k H^T(k, \hat{x}_{t|t-1}) + R_{2,k}]^{-1}$$

$$P_{k+1} = F(k, \hat{x}_{k|k-1})P_k F^T(k, \hat{x}_{k|k-1}) + R_{1,k} - K_k[H(k, \hat{x}_{k|k-1})P_k H^T(k, \hat{x}_{t|t-1}) + R_{2,k}]K_k^T$$

The Jacobians for the observation and continuous state models were chosen as follows:

$$F(k, \hat{x}) = \frac{\partial}{\partial x} f(k, x)|_{x=\hat{x}}$$
$$H(k, \hat{x}) = \frac{\partial}{\partial x} h(k, x)|_{x=\hat{x}}$$

The Extended Kalman Filter uses a linearity approximation of the Taylor series expansion of the general function (hence the Jacobian calculation, which is of order 1). However, this can be further extended to higher orders as long as the calculation is mathematically and computationally feasible and reasonable. One thing to observe here is that this is only useful for errors that are low from a computational standpoint [8]. The Extended Kalman Filter may also be further improved in various other ways, such as the Implicit Extended Kalman Filter (for noninvertible functional forms), Iterated Extended Kalman Filter (which modifies the point about which the Taylor series expansion takes place), and much more [15].

Other methods of utilizing a Kalman Filter as the underlying core algorithm exist, such as the Unscented Kalman Filter and Spherical Kalman Filters. The Unscented Kalman Filter (UKF) is based on a combination of a Gaussian assumption to create a sampling distribution, then applying method of moments in order to estimate the various moments such as expectation, variance, and higher order moments [16]. However, both the Unscented Kalman Filter (UKF) and a more general form of the Extended Kalman Filter, the Second Order Extended Kalman Filter (SOEKF) - which performs better than the UKF in some cases - both suffer from a few issues. This is due to instability experienced when trying to fit nonlinear functions, which gives rise to numerical computation issues [17].

B. Rao-Blackwellized Particle Filter

1) Rao-Blackwellization: One of the major drawbacks of the particle filter is that sampling in high-dimensional spaces can be inefficient. Any Monte Carlo estimator is affected by variance due to the random simulation used in its construction. This inefficiency can be seen as increase in the mean square error of the state estimates affected by the variance. Hence, to obtain an accurate estimator it is desirable to keep the variance as small as possible.This inefficiency can be countered with a modification in the particle filter and this is called Rao-Blackwellization. The reduction in variance is explained in [33]

The key idea of Rao-Blackwell dimension reduction is to utilize the information that is inherent in the problem to analytically infer part of state parameters conditional upon other state components which can be estimated by MCMC methods. It states that for an unknown parameter Z and the data Y drawn from Z, if Z can be factorized into x, r where r is a sufficient statistic for Y then, Z can be estimated as,

$$\hat{Z}_{RB}(Y) = E[Z(Y)|r] \tag{6}$$

Given the equations of JMLS, Given N particles, $\{r^{(i)}_{0:t-1}, x^{(i)}_{0:t-1}\}$ at time t-1, approximately distributed by $p(r^{(i)}_{0:k-1}, x^{(i)}_{0:k-1}|y_{1:k-1})$, estimation of x_t and r_t can be written as,

$$p(r_{0:t}, x_{0:t}) = p(r_{0:t}|y_{0:t})p(x_{0:t}|y_{0:t}, r_{0:t}) \tag{7}$$

In the above equation, $r_{0:t}$ is the sufficient statistic and thus RBPF can be used to estimate $p(r_{0:t}|y_{0:t})$ and $p(x_{0:t}|y_{0:t}, r_{0:t})$ can be exactly estimated using Kalman filter.

2) Sequential importance sampling: For an ideal estimation of $p(r_{0:t}|y_{0:t})$, importance sampling method could be used. [18] explains the optimal choice of importance distribution for the sampling scheme. Here, the importance distribution is considered to be $q(r^{(i)}_t|r^{(i)}_{0:t-1}, y_{1:t})$ which minimizes the variance of the importance weights that are being calculated. When $q(r^{(i)}_t|r^{(i)}_{0:t-1}, y_{1:t})$ equals $p(r^{(i)}_t|r^{(i)}_{0:t-1}, y_{1:t})$, the normalized weights would equal N^{-1} for all i.

Algorithm for RBPF is given as follows.

Algorithm 1 Rao-Blackwellized Particle Filter

1: **procedure** RBPF()
2: For i = 1,2,..,N, Sample:

$$\hat{r}^{(i)}_t \sim q(r_t|r^{(i)}_{0:t-1})$$

Set:

$$(\hat{r}^{(i)}_{0:t}) \triangleq (\hat{r}^{(i)}_t, ^{(i)}_{0:t-1})$$

3: For i = 1,2,...,N, determine the importance weights:

$$w^{(i)}_t = \frac{p(\hat{r}^{(i)t}|y_{1:t})}{q(\hat{r}^{(i)}_t|r^{(i)}_{0:t-1}, y_{1:t})p(\hat{r}^{(i)}_{0:t-1}|y_{1:t-1})}$$

4: For i = 1,2,..,N, normalize the importance weights

$$\tilde{w}^{(i)}_t = w^{(i)}_t[\sum_{j=1}^{N} w^j_t]^{-1}$$

5: Perform selection: Multiply or suppress samples $(r^{(i)}_{0:k})$ with high or low respective importance weights to obtain N random samples $(\tilde{r}^{(i)}_0 : k)$ approximately distributed according to p$(^{(i)}_{0:k}|y_{1:k})$.

Degeneracy occurs because of the randomness present in the importance function over time and this causes variance of the importance weights to increase leading to one of the weights approaching 1 and remaining 0. The selection step is being performed in order to overcome this degeneracy. This step is performed using one of the several selection schemes(like residual sampling) once every few iterations of SIS.

VI. Experiments

Simulations were carried out to evaluate the performance of RBPF against particle filter and extended Kalman filter. Section VI-A discusses the problem of tracking a maneuvering target[1].

A. Tracking of a Maneuvering Target

This section discusses the simulation of JMLS for tracking a maneuvering target in noise as done in Doucet et. al. [1] using MC algorithms, further using EKF and RBPF and comparison between them. As given in [1], The state of the target at time t is denoted as $x_t \triangleq (l_{x,t}, s_{x,t}, l_{y,t}, s_{y,t})^T$, where $l_{x,t}(l_{y,t})$ and $s_{x,t}(s_{y,t})$ represent the position and velocity of the target in the x (resp. in the y) direction. It evolves according to a JMLS model of parameters

$$\mathbf{A} = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = 0.1 I_4$$

$$C = I_4, G = 0_{4xn_u}$$

and $D = \sqrt{3}dia(20, 1, 20, 1)$. The switching term is $F(r_t)u_t$, where r_t is a three-state Markov chain corresponding to the three possible maneuver commands:

straight;
left turn;
right turn.

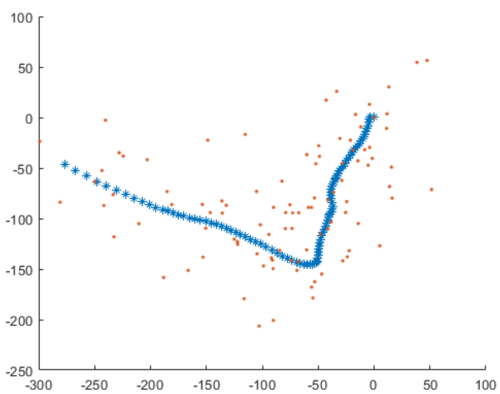
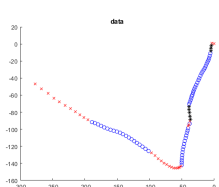
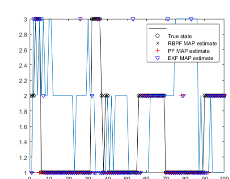


Fig. 6: Target maneuver with ground truth x_k(blue) and noisy observations y_k (dots)

It has the following transition probabilities: $p_{m,m} = 0.9$ and $p_{m,n} = 0.05$ for $m \neq n$. For any t $F(1)u_t = (0, 0, 0, 0)^T, F(2)u_t = (-1.225, -0.35, -1.225, 0.35)^T$, and $F(3)u_t = (1.225, 0.35, -1.225, -0.35)^T$ Sampling is done according to the optimal distribution.



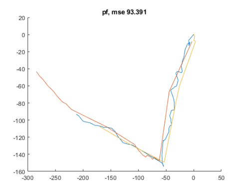
(a) Target maneuver jumping states



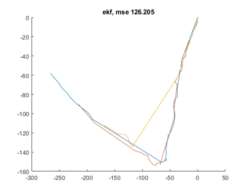
(b) Estimation of r_k by different algorithms

Fig. 7

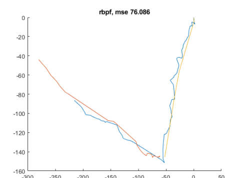
Fig. 6 shows the maneuvering target. The underlying state transitions have been realized in Fig. 7a. Simulations were carried out to determine the continuous and the discrete state processes using particle filter, Rao-Blackwellized particle filter and extended Kalman filter. The estimation of underlying r_k can be seen in the Fig. 7b and estimation of x_k can be observed in Fig. 8. Further, for a N = 500 particle filter, error rate between the algorithms were evaluated. It could be observed from Fig. 9 that out of all the three algorithms, RBPF performs the best.



(a) PF



(b) EKF



(c) RBPF

Fig. 8: Mean square error(MSE) of different algorithms

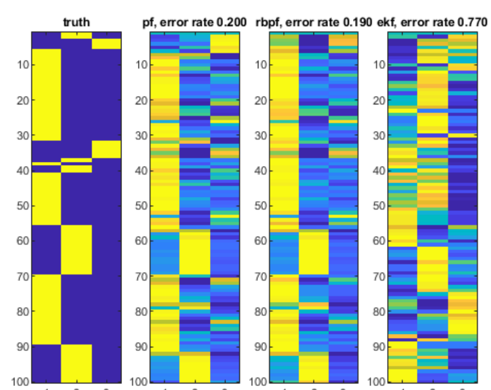


Fig. 9: Error rate of the algorithms

VII. Conclusion

RBPFs have been widely adopted in computer science and engineering fields because of its applications in conditionally Gaussian state-space models, generalized HMMs and a wide class of dynamic models.

When comparing the performance of the EKF in relation to the RBPF in a Jump Markov Linear System, the EKF performed quite poorly compared to the RBPF. The EKF is just a linear approximation to the Taylor series expansion, so although it may be able to predict approximately differentiable nonlinear functions, due to discontinuities arising from the jump process, the EKF was unable to predict well when compared to the RBPF. This demonstrates that RBPFs would outperform in a process with a separate jump process that enables switching of regimes. When compared to traditional particle filter, as expected, RBPF outperforms it by reducing the variance between the state estimates. This reduction in variance owes to marginalization carried out by RBPF that allows us to evaluate fewer parameters.

Based on this analysis, utilizing an RBPF may work quite well for modeling the uncertain nature of economic and financial systems, especially ones that may have drastic jumps that arise due to regime changes. Various policies related to fiscal and monetary policy would systemically affect the economy, and thus various asset prices and the relationships amongst them. In such a model, which may be modeled with a Jump Markov Linear System, and correspondingly an RBPF may be used.

Acknowledgment

The authors would like to thank Prof. Vikram Krishnamurthy and Buddhika Nettasinghe for the ECE 5412 course.

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We both (Shwetha and Hongsun) have contributed equally in all parts.