Hammond's Project and Simple Random Graphs

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1 Introduction

In this project, the authors seek to understand the behavior of Simple Random Bridge (SRBs), and more specfically how they behave under addition. An SRG of size N is a function $B : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ such that $B(0) = 0$ and $B(x+1)-B(x) = \pm 1$ for all $x \in \mathbb{Z}/N\mathbb{Z}$.

We say that a value $i \in \mathbb{Z}/N\mathbb{Z}$ is *down* in B if $B(i) < \min(B(i-1), B(i+1))$.

If B_1 and B_2 are size N bridges, then we say that the *minimal points* of B_1 under B_2 are those values $i \in \mathbb{Z}/N\mathbb{Z}$ such that $B_1(i)-B_2(i) = \min_j (B_1(j) - B_2(j))$. If 0 is not minimal then we define the *addition* of B_2 to B_1 as

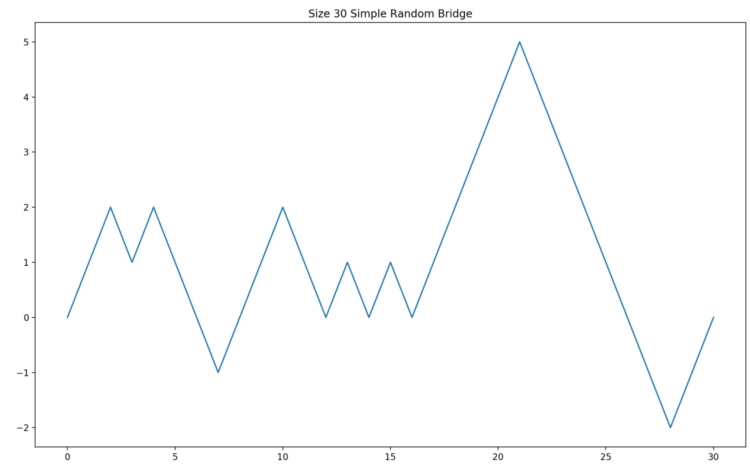
$$\tilde{B_1}(i) = \begin{cases} B_1(i) + 2 & i \text{ minimal under } B_2 \\ B_1(i) & \text{otherwise} \end{cases}$$

If 0 is minimal, then we define

$$\tilde{B_1}(i) = \begin{cases} B_1(i) - 2 & i \text{ not minimal under } B_2 \\ B_1(i) & \text{otherwise} \end{cases}$$

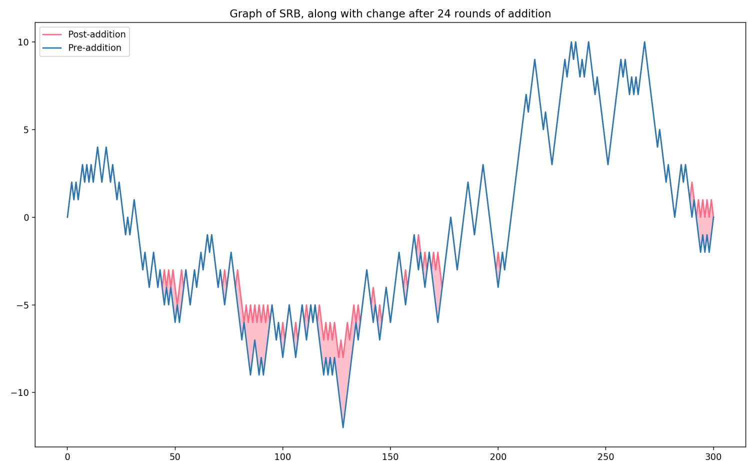
It is clear that in both cases $\tilde{B_1}$ is once again an SRB. We note the interpretation of addition as putting B_2 below B_1 (as graphs) and sliding B_2 up until it just touches B_1, at which point every down intersection point is flipped up. In the case that 0 is minimal, everything is then rescaled down so that $\tilde{B_1}(0) = 0$.

Figure 1: An Example of a SRB



The main interest of this paper is the result of repeated addition of a template bridge B_1 to random trial bridges, where in this context random means taken uniformly over the (finite) space of SRBs of size N.

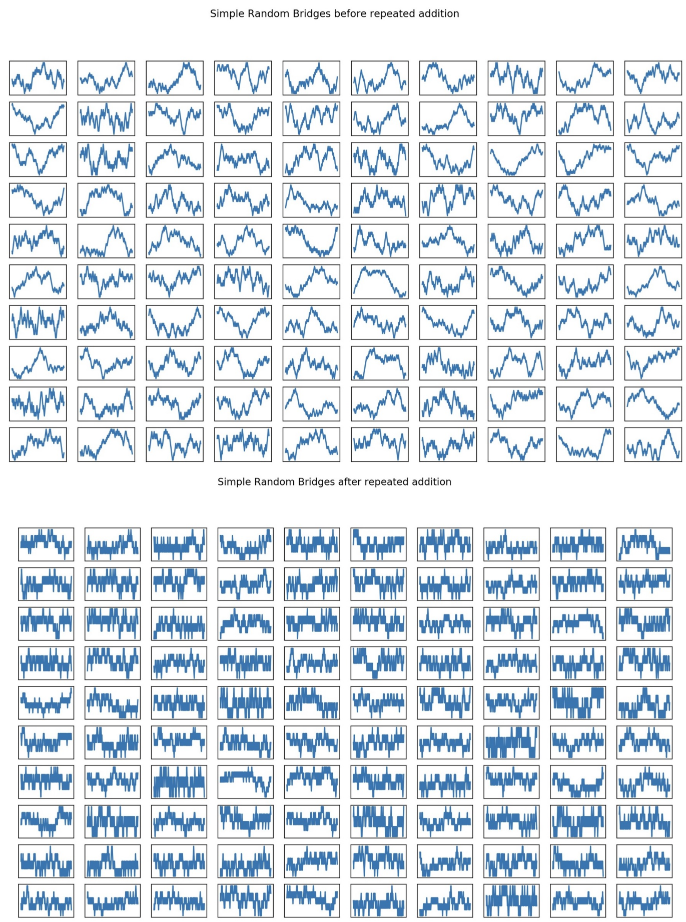
Figure 2: An SRB after repeated addition



2 Flattening effect of repeated addition

One natural phenominon to observe the probability that any given SRB appears after large rounds of repeated addition. A sampling of 100 random SRBs of size 176 before and after 10,000 rounds of repeated addition by random SRBs is shown in Figure 3.

Figure 3: An SRB after repeated addition

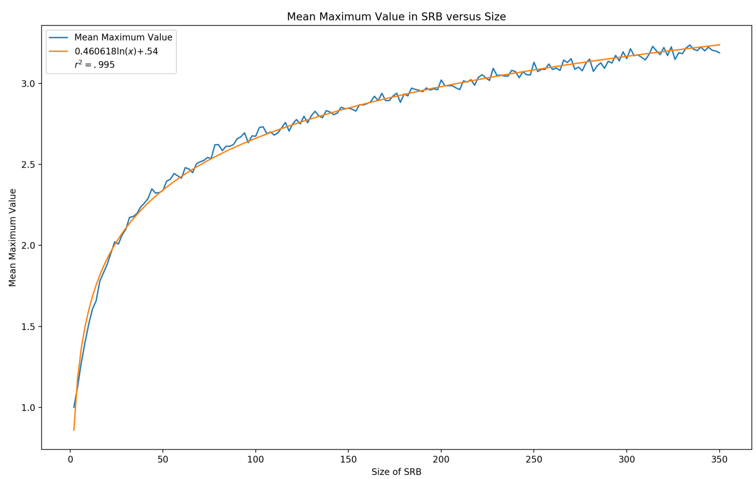


Note the the thick bars in the "after repeated addition" group. These show that the values of B are distributed across a very small image. This can be measured in the quantity $s(N) := \mathbb{E}_B [\max_i |B(i)|]$ where the expected value is taken over all bridges B of size N, weigthed according to how likely they are to appear after an arbitrarily large amount of flips from an arbitrary starting bridge. This example demonstrates the fact that, numerically, $s(176)$ seems to be about 2.8.

Figure 4 shows how the numerically approximated quanity $s(N)$ changes as N varies. Due to the extremely sharp fit of the superimposed logarithm, the following conjecture is clearly motivated:

Conjecture 1. *The value* $\lim_{N\to\infty} \frac{s(N)}{\ln(N)}$ *converges to a value* $0 < c < \infty$.

Figure 4: s(N) versus N



3 Probability of point specific flips under addition

One way that one might hope to understand the behavior of SRBs under addition would be to examine the probability that any individual down point is flipped. A specific SRB with its points colored according to the probability that they are flipped is shown in Figure 5. While sharp general results seem hard to come by, there are a few simple statements that one can make about these probabilities.

In the below proposition and throughout, the probability is taken uniformly as B is added with all test bridges B_0 of size N.

Proposition 1. *Let* B *be the SRB of size* N *defined by* $B(i) := (i \bmod(2))$, *which is well defined as there are no SRBs for which* N *is odd. It holds that* $\Pr[0 \text{ flips}] = \frac{2}{N+2}$.

Proof. By the definition of probability we have that

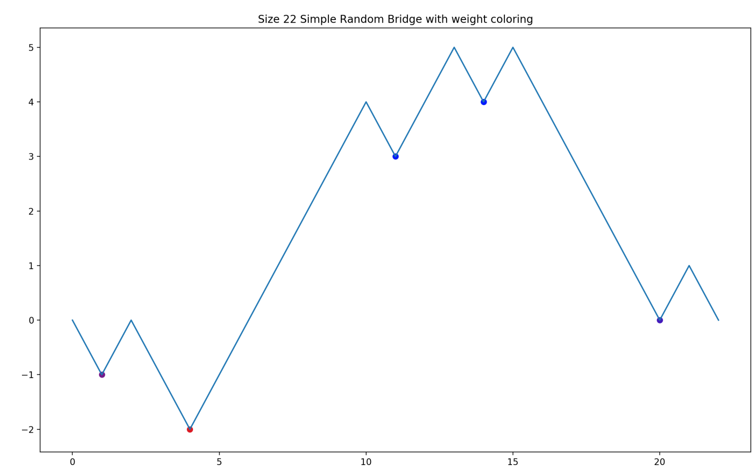
$$\Pr[0 \text{ flips}] = \frac{|\{B_0 \text{ flips } 0\}|}{|\{B_0 \text{ of size N}\}|}$$

Since $\sum_i (B(i+1) - B(i)) = 0$ we have that $B(i+1) - B(i) = 1$ and $B(i+1) - B(i) = -1$ exactly $N/2$ times, the placement of the $N/2$ $+1s$ in $B(i+1) - B(i)$ exactly determines the placement of $B(i+1) - B(i) = -1$s. Thus, the total amount of SRBs of size N is $\binom{N}{N/2}$.

Any bridge B_0 that never intersects B will flip 0 since then $\max(B_0(i) - B(i)) = 0 = B_0(0) - B(0)$. As discussed above any bridge can be determined by the distribution of $+1$s and -1s in $B_0(i+1) - B_0(i)$. B_0 will not intersect B if and only if the number of $-1s$ in any intial segment of $B_0(i)$s if greater or equal to the number of $+1$s.

These sorts of sequences of ± 1s are known as Dyck words, and it is well known that the amount of Dyck words of length N is exactly $\frac{2}{N+2}\binom{N}{N/2}$ [1] $\square$

Figure 5: An SRB with flip probability coloring



As a simple corollary, we arrive at a general lower bound for how likely a point is to be flipped, if it is (one of) the points the lowest in the image.

Corollary 1. *For any bridge B and index i, if $B(i) = \min_j B(j)$ then* $\Pr[i \text{ flips}] \geq \frac{2}{N+2}$.

Proof. If we define a new Bridge B' by $B'(j) := B(j+i) - B(i)$, then clearly every bridge with the Dyck word property as in the proof of Proposition 1 will once again not intersect B', since by assumtion of minimality we have that $B'(j) \geq 0$ uniformly. Thus,

$$|\{B_0 \text{ flips } i\}| \geq \frac{2}{N+2}\binom{N}{N/2}$$

and plugging this into

$$\Pr[i \text{ flips}] = \frac{|\{B_0 \text{ flips } i\}|}{|\{B_0 \text{ of size N}\}|}$$

we are done. $\square$

References

[1] Philippe Duchon. On the enumeration and generation of generalized dyck words. *Discrete Mathematics*, 225(1-3):121–135, 2000.