**Homework**

**Question 5:**

a) Solve the following question from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(b)

$$\begin{array}{l} p \to (q \land r) \\ \neg q \\ \hline \therefore \; \neg p \end{array}$$

| | | |
|---|---|---|
| 1 | $\neg q$ | Hypothesis |
| 2 | $\neg q \lor \neg r$ | Addition, 1 |
| 3 | $\neg (q \land r)$ | De Morgan's Law 2 |
| 4 | $p \to (q \land r)$ | Hypothesis |
| 5 | $\neg p$ | Modus Tollens 3, 4 |

(e)

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore \; r \end{array}$$

| | | |
|---|---|---|
| 1 | $p \lor q$ | Hypothesis |
| 2 | $\neg q$ | Hypothesis |
| 3 | $p$ | Disjunctive Syllogism 1, 2 |
| 4 | $\neg p \lor r$ | Hypothesis |
| 5 | $r$ | Disjunctive Syllogism 3, 4 |

2. Exercise 1.12.3, section c

(c) One of the rules of inference is Disjunctive syllogism:

$$\begin{array}{c} p \lor q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

| | | |
|---|---|---|
| 1 | $p \lor q$ | Hypothesis |
| 2 | $\neg p \to q$ | Conditional Identity, 1 |
| 3 | $\neg p$ | Hypothesis |
| 4 | $q$ | Modus Ponens, 2, 3 |

Disjunctive syllogism is valid.

3. Exercise 1.12.5, sections c, d

Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(c) I will buy a new car and a new house only if I get a job
I am not going to get a job.

---

∴ I will not buy a new car

j: I will get a job
h: I will get a new house
c: I will get a new car

The form of the argument is:

$$\begin{array}{l} (c \land h) \rightarrow j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

$$(((c \land h) \rightarrow j) \land \neg j) \rightarrow \neg c$$

| $c$ | $h$ | $j$ | $c \land h$ | $(c \land h) \rightarrow j$ | $\neg j$ | $\neg c$ | $(((c \land h) \rightarrow j) \land \neg j) \rightarrow \neg c$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | T |
| T | T | F | T | F | T | F | T |
| T | F | T | F | T | F | F | T |
| T | F | F | F | T | T | F | F |
| F | T | T | F | T | F | T | T |
| F | T | F | F | T | T | T | T |
| F | F | T | F | T | F | T | T |
| F | F | F | F | T | T | T | T |

In the hypothesis $(c \land h) \rightarrow j$, when c is True and h and j are False, the conclusion $\neg c$ is False.

Therefore, the argument is invalid.

(d) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.

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$\therefore$ I will not buy a new car.

j: I will get a job
h: I will get a new house
c: I will get a new car

The form of the argument is:

$$\begin{array}{l} (c \land h) \to j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

| | | |
|---|---|---|
| 1 | $(c \land h) \to j$ | Hypothesis |
| 2 | $\neg j$ | Hypothesis |
| 3 | $\neg (c \land h)$ | Modus Tollens 1, 2 |
| 4 | $\neg c \lor \neg h$ | De Morgan's Law, 3 |
| 5 | $\neg h \lor \neg c$ | Commutative Law, 5 |
| 6 | $h$ | Hypothesis |
| 7 | $\neg c$ | Disjunctive Syllogism, 5, 6 |

The argument is <u>valid</u>.

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a,b}.

(b)

$$\begin{array}{l} \exists x(P(x) \lor Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

| | P | Q |
|---|---|---|
| a | F | F |
| b | F | T |

When Q(b) is true, $\exists x(P(x) \lor Q(x))$ is true and when Q(a) is false, $\exists x \neg Q(x)$ is true.

There is no Q(a) nor Q(b) for which P(x) is true. Both of them are <u>false</u>.

Therefore, the argument is <u>invalid</u>.

2. Exercise 1.13.5, sections d, e

(d) Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.
The domain for each problem is the set of students in a class.

| | |
|---|---|
| Every student who missed class got a detention. | ← Hypothesis |
| Penelope is a student in the class. | ← Hypothesis |
| Penelope did not miss class. | ← Hypothesis |
| Penelope did not get a detention. | ← Conclusion |

M(x): x missed class
D(x): x got a detention
S(x): x is a student in the class

The form of the argument is:

$$
\begin{array}{ll}
\forall x\,(M(x) \rightarrow D(x)) & \leftarrow \text{Hypothesis} \\
S(\text{Penelope}) & \leftarrow \text{Hypothesis} \\
\neg M(\text{Penelope}) & \leftarrow \text{Hypothesis} \\
\hline
\therefore\ \neg D(\text{Penelope}) & \leftarrow \text{Conclusion}
\end{array}
$$

If we were to imply that M(Penelope) was True and that D(Penelope) was False, our first hypothesis $\forall x$ (M(Penelope) → D(Penelope)) will then result in False.

$$\forall x\,(M(\text{Penelope}) \rightarrow D(\text{Penelope}))$$

T → F ← Conditional statement (→) states that when p is true and q is false it is <u>false</u>

F

But our conclusion, ¬D(Penelope), will be True. Therefore, our argument will be <u>invalid</u>.

(e) Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

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Penelope did not get a detention.

M(x): x missed class
D(x): x got a detention
S(x): x is a student in the class
A(x): x got an A

The form of the argument is:

$\forall x\ (M(x) \lor D(x)) \rightarrow \neg A(x)$
$S(\text{Penelope})$
$A(\text{Penelope})$

---

$\therefore\ \neg D(\text{Penelope})$

| | | |
|---|---|---|
| 1 | $\forall x\ (M(x) \lor D(x)) \rightarrow \neg A(x)$ | Hypothesis |
| 2 | $S(\text{Penelope})$ | Hypothesis |
| 3 | $M(\text{Penelope}) \lor D(\text{Penelope}) \rightarrow \neg A(\text{Penelope})$ | Universal Instantiation,1& 2 |
| 4 | $(\neg((M(\text{Penelope}) \lor D(\text{Penelope}))) \lor \neg A(\text{Penelope})$ | Conditional Identity, 3 |
| 5 | $(\neg(M(\text{Penelope}) \land \neg D(\text{Penelope})) \lor \neg A(\text{Penelope})$ | De Morgan's Law, 4 |
| 6 | $\neg A(\text{Penelope}) \lor (\neg M(\text{Penelope}) \land \neg D(\text{Penelope}))$ | Commutative Identity, 5 |
| 7 | $A(\text{Penelope}) \rightarrow (\neg M(\text{Penelope}) \land \neg D(\text{Penelope}))$ | Conditional Identity, 6 |
| 8 | $A(\text{Penelope})$ | Hypothesis |
| 9 | $\neg M(\text{Penelope}) \land \neg D(\text{Penelope})$ | Modus Ponens, 7 & 8 |
| 10 | $\neg D(\text{Penelope}) \land \neg M(\text{Penelope})$ | Commutative Identity, 9 |
| 11 | $\neg D(\text{Penelope})$ | Simplification, 10 |

The argument is <u>valid</u>.

**Question 6:**

Solve Exercise 2.2.1, sections d, c from the Discrete Math zyBook:

Prove each of the following statements using a direct proof.

(d) The product of two odd integers is an odd integer.

*Solution*:

*Proof*. Direct proof. An odd integer can be expressed as 2k + 1 for some integer (in this case, k = integer). Since k = integer and 2k+1=odd integer (as per stated), let us choose two odd integers (we will pick e and f) to which each one of them will equal to their proper odd integer value (we will use 2k+1 and 2m+1). We will state the following:

e (1st odd integer) = 2k + 1
f (2nd odd integer) = 2m+ 1

Because we are trying to prove that the product of two odd integers is an odd integer, we will multiply both 1st and 2nd odd integers.

ef = (2k+1)(2m+1) ← multiplication of 1st and 2nd odd integer

ef = 4km + 2k + 2m +1 ← using the FOIL method (first, outer, inner and last) we calculate the products

ef = 2(2km+1+m) + 1 ← equivalent of 4km + 2k + 2m +1

Now, 2(2km+1+m) +1 is an odd number. We solve:

Let n = 2km + 1 + m

ef = 2(n) + 1

The multiplication of the two integers, e and f (ef) which were both odd integers, equals to 2n + 1 which stands for an odd integer. 2(some integer) + 1 = odd integer. Therefore, it is proven that the product of two odd integers is an odd integer. ■

(c) If $x$ is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

*Solution*:

*Proof*. Direct proof. Let us assume that for $x$(a real number) $x \leq 3$.
We will then solve for $12 - 7x + x^2 \geq 0$. We will first start stating the hypothesis $x \leq 3$. Then, we will subtract x from both sides of the inequality to get $(3 - x) \geq 0$. We solve:

$x \leq 3$ ← hypothesis

$x - x \leq 3 - x$ ← subtraction of $x$ from both sides of inequality

$0 \leq (3 - x)$ ← final result

or

$(3 - x) \geq 0$ ← the equivalent of $0 \leq (3 - x)$

Because $(3 - x) \geq 0$, $(4 - x)$ will be 1 more than $(3 - x)$. Therefore, we will plug such value, $(4 - x) \geq 0$, and then multiply it by $(3 - x)$. We solve:

$(3 - x)(4 - x) \geq 0$ ← multiplication of both $(3 - x)$ and $(4 - x)$

$12 - 3x - 4x + x^2 \geq 0$ ← using the FOIL method (first, outer, inner and last) we calculate the products

$12 - (3+4)x + x^2 \geq 0$

$12 - 7x + x^2 \geq 0$ ← final result of the multiplication

■

**Question 7:**

Solve Exercise 2.3.1, sections d, f, g, l from the Discrete Math zyBook:

(d) For every integer $n$, $n^2 - 2n + 7$ is even, then $n$ is odd.

*Solution*:

*Proof*. Proof by Contrapositive. We will begin the proof assuming that $n$ is an even integer. An even integer can be expressed as $2k$ for some integer $k$. We will then go ahead and state the following:

$$n = 2k \qquad \leftarrow \text{n equals 2k (even integer)}$$

Now that $n$ has become an even integer, we will go ahead and substitute all $n$'s in $n^2 - 2n + 7$ for its new value.

$$n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7 \qquad \leftarrow \text{ n is substituted for 2k}$$

$$n^2 - 2n + 7 = 2(2k^2 - 2k) + 7 \qquad \leftarrow \text{equivalent of } 4k^2 - 4k + 7$$

Since it was previously stated that $k$ was an integer, then $2(2k^2 - 2k)$ is an <u>even</u> integer (even integers are expressed as $2k$) but adding the 7, $2(2k^2 - 2k) + 7$, makes it an <u>odd</u> integer.
When $n$ is even, our equation $n^2 - 2n + 7$ is odd. Therefore, our contrapositive proof proves that when n is odd, our equation, $n^2 - 2n + 7$, will be odd as well. ■

(f) For every non-zero real number $x$, if $x$ is irrational, then $\frac{1}{x}$ is also irrational

If $\frac{1}{x}$ was to be rational and $x \neq 0$, then there exists integers a and b such that $\frac{1}{x} = \frac{p}{q}$

and p and q $\neq 0$.

*Solution*:

*Proof*. Proof by Contrapositive. A rational number equals an integer over an integer. Below, we will see that $\frac{1}{x}$ will equal $\frac{a}{b}$. We will cross multiply such values and then divide both sides by $a$ which will result in x = $\frac{b}{a}$.

$$\frac{1}{x} = \frac{a}{b} \qquad \leftarrow \text{cross multiplication of the values}$$

$$ax = b \qquad \leftarrow a \text{ times } x \text{ equals } b$$

$$x = \frac{b}{a} \qquad \leftarrow x \text{ equals } b \text{ over } a$$

Now, $a$ could have been a rational number if it was zero, <u>however</u>, our hypothesis states otherwise. It clearly states that "For every <u>non-zero</u> real number" as opposed to zero and therefore x is <u>irrational</u>. ∎

(g) For every pair of real numbers $x$ and $y$, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

*Solution*:

*Proof*. Proof by Contrapositive. Let us assume that $x > y$, then $x^3 + xy^2 > x^2y + y^3$. We will then use the factorization method on both sides which will give us a common value for both, which is $(x^2 + y^2)$.

$$x^3 + xy^2 > x^2y + y^3$$

$$x(x^2 + y^2) > y(x^2 + y^2) \qquad \leftarrow \text{the factorization method for both sides}$$

If we assume that $x^2 + y^2$ are exactly the same in both sides and it has been stated that $x$ is the greater value, our proof is True. ∎

(l) For every pair of real numbers $x$ and $y$, $x + y > 20$, then $x > 10$ or $y > 10$.

*Solution*:

*Proof*. Proof of contrapositive. Let us assume $x \leq 10$ and $y \leq 10$ then if we were to add both values together we would get $x + y \leq 20$:

$$x \leq 10 + y \leq 10$$

$x + y \leq 10 + 10$ ← addition of the variables to one side and addition of the numbers in the other

$x + y \leq 20$ ← final result

The statement is true. ∎

**Question 8:**

Solve Exercise 2.4.1, sections c, e from the Discrete Math zyBook:

Give a proof for each statement.

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

*Solution*:

*Proof*. Proof by contradiction. Let us start by assuming that these three real numbers are e, f, and g and these numbers are less than the average of such three real numbers. The average will be represented with the letter *a*. We solve in the following form:

e f g ← our three real numbers

Let us use the letter *a* for average:

$$a = \frac{e + f + g}{3}$$

← the average of three numbers ( in this case: e, f and g) is the division of such by 3 (because there are three numbers)

Now, our statement tell us that these numbers are <u>less</u> than the average, therefore:

$$e < a \qquad f < a \qquad g < a$$

If we added all three values, we would get:

$$e + f + g < 3\,a$$

And we know that our definition of the average $a$ equals to $\frac{e + f + g}{3}$ , therefore, $e + f + g = 3a$

But now, we also have that $3a = e + f + g < 3a$. Then we get, $3a < 3a$, which is a total contradiction. Our claim turns out to be true. ■

(e) There is no smallest integer.

Solution:

*Proof*. Proof by contradiction. Let us assume that there is a smallest integer and that smallest integer is m. That statement itself is letting us know that there should not be anything smallest than it.
So, if we were to subtract 1 from m, as a result we would get an integer.
m - 1 is an integer as well. But, the m - 1 < m statement contradicts the fact that m is the smallest integer.
Therefore, the theorem will be true. ■

**Question 9:**

Solve Exercise 2.5.1, section c, from the Discrete Math zyBook:

(c) If integers x and y have the same parity, then x + y is even.
The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

*Solution*:

*Proof*. Proof by cases. There are two scenarios. In the first one, we are assuming that both, x and y, are even. An integer, let us say e, will become 2e if we were to make it an even integer. And so we begin to plug in those values for x and y, to make them even.

Case 1: x and y are even integers.

$x = 2e$ ← an integer (e) is transformed to (2e) to become even
$y = 2f$ ← we follow the same step but add a different letter (f) for y

Now, let us add x plus y:

$x + y$
$2e + 2f$ ← x and y's new value as even integers
$2(e + f)$ ← final result

Since all integers that are multiplied by 2 are even, our scenario of x+y (or 2(e+f) ) is also proven to be even because it is multiplied by 2.

But now, let us test a different scenario in which x and y were both odd. An integer, let us say e (the same letter we used from the previous example), will become 2e+1 if we were to make it an odd integer. And so we begin to plug in those values for x and y, to make them odd.

Case 2: x and y are odd integers.

Now, let us add x plus y:

$x + y$
$2(e+1) + 2(f+1)$ ← x and y's new value as odd integers
$2(e+f+1)$ ← final result

If we were to follow the same rule, that all integers multiplied by 2 are even, then we can agree that x+y (or 2(e+f+1) is also proven to be even because it is also multiplied by 2. ■