Homework

Question 5:

a. Use mathematical induction to prove that for any positive integer n, 3 divide n^3+ 2n (leaving no remainder. Hint: you may want to use the formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Solution:

We need to prove that for any given positive integer n, the number 3 divides n^3 + 2n without a remainder. In order to make this possible using induction, we follow the following steps:

Base case: Because we have a positive integers, we use n=1 and then substituted into the formula to get: 1^3+ 2(1) = 3, which is divisible by 3. Therefore, the base case stands true.

We go ahead and assume that we have n=k then for some integer k and we assume that k^3 + 2k is divisible by three, therefore, we must prove that $(k + 1)^3 + 2(k + 1)$is divisible by 3.

We follow a simplification method and get the following:

$$= (k + 1)^3 + 2(k+1)$$

$$= (k+1)(k+1)(k+1) + 2k + 2$$

$$= (k^2 + 2k + 1)(k + 1) + 2k + 2$$

$$= k^3 + k^2 + 2k^2 + 2k + k + 1 + 2k + 2$$

$$= k^3 + 2k + 3k^2 + 3k + 3$$

Then, from using **induction**, we conclude that k^3 + 2k is divisible by three and, therefore, we can factor the equation as:

$$k^3 + 2k + 3(k^2 + k + 1)$$

And because we have acquired both parts, which are divisible by three, the whole equation is divisible by three.

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b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

<u>*Solution*</u>:

We begin by assuming that our **base case** is n=2, then the factors of n are: 1, 2 which evaluate to prime.
Next, we assume that for all $k \leq n$, k can either be prime or a product of prime factors.
Therefore, we have to show that n+1 is prime or a product of primes.
We have this scenario in which we assume that there are two numbers a and b such that $2 \leq a$ and $b \leq n$ and ab= n+1.
If n+1 is not a prime, then both (a and b), have to be prime or the product of some prime factors.
We know that, in that case, ab will also have to be the product of prime factors.
If numbers ab do not exist then n+1 must be prime itself, and therefore, the proof stands firm.
And it is because of **strong induction** that we have proven that n+1 is either a prime or a product of prime factors.

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Question 6:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 7.4.1, sections a-g

Define P(n) to be the assertion that: \sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}

a. Verify that P(3) is true.

Solution: P(3) is true because $1^2 + 2^2 + 3^2 = \frac{3(3+1)\,((3*2)+1))}{6} = 14$

b. Express P(k)

Solution: $P(k) = \sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6}$

c. Express P(k+1)

Solution: $P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

d. In an inductive proof that for every positive integer n,
\sum_{j=1}^n j ^ 2 = \frac{n(n+1)(2n+1)}{6},
what must be proven in the base case?

Solution: Since this applies for every positive integer, the base case is P(1) which evaluates to $P(1) = \frac{6}{6} = 1$ which is true.

e. In an inductive proof that for every positive integer n,
\sum_{j=1}^n j ^ 2 = \frac{n(n+1)(2n+1)}{6},
what must be proven in the inductive step?

Solution: We need to prove in the inductive step that: $P(k) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

f. What would be the inductive hypothesis in the inductive step from your previous answer?

Solution: The inductive hypothesis is: $P(k) = \sum_{j=1}^{k} j^2 = \frac{(k+1)(2k+1)}{6}$

g. Prove by induction that for any positive integer n,
\sum_{j=1}^n j ^ 2 = \frac{n(n+1)(2n+1)}{6}

Solution:

It has been proven the base case in (d). Also, we are aware that our hypothesis is located in (f). Now, we must prove that our inductive step from (e) works for any integer k.

Therefore:

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{2k^3 + 9k^2 + 13k + 6}{6} \quad \leftarrow \text{After simplification}$$

And we can also simplify our inductive step to the same, and since they are equal, we can prove the statement by induction.

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b) Exercise 7.4.3, section c

c. Prove that for n≥ 1, \sum_{j=1}^n\frac{1}{j^2}≤2-\frac 1 n

Solution:

Prove that for $n \geq 1$: $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Base case n = 1:

$$f(1) = \frac{1}{1^2} \leq 2 - \frac{1}{1} = 1 \leq 1$$ ← it is true for our base case

Inductive step: We are assuming the following:

$$\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$$

Therefore, we have to prove the following:

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

Next, with our initial assumption, we add: $\frac{1}{(k+1)^2}$ to each side.

Therefore:

$$\sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} \leq 2 - \frac{k+2}{(k+1)^2}$$

Next, the result of the proof step with the k+1 is:

$$2 - \frac{1}{k+1} = 2 - \frac{k+1}{(k+1)^2}$$

In the final step, we go ahead and simplify and the final result is:

$$-\frac{k+2}{(k+1)^2} \leq -\frac{k+1}{(k+1)^2}$$

It is indeed true because - (k+2) ≤ (-k+1) and, finally, we have proven our theorem. ■

c) Exercise 7.5.1, section a

Prove each of the following statements using mathematical induction.

a. Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$

<u>*Solution*</u>:

We must prove that for any positive integer n, 3^{2n}-1 is divisible by 4.
In order to prove this, we have strong induction.
We will start off with **base case** n=1, and therefore, our formula will result in$3^{2(1)}$- 1 = 8 which is divisible by 4. It is necessary to use **strong induction** for any integer j between 1 and k, 3^{2n}- 1 is divisible by 4, finally prove that $3^{2(k+1)}$- 1 is also evenly divisible by 4.

We have the following scenario:

$$= 3^{2(k+1)} - 1$$

$$= 3^{2k+2} - 1$$

$$= 3^{2k} + 3^{2} - 1$$

$$= 3^{2k} + 3^{2} - 1 + 8 - 8$$

$$= 9\,(9^{k} - 1) + 8$$

$$= 9\,(3^{2k} - 1) + 8$$

Because our assumption was that 3^{2k} - 1 is divisible by 4, 9 times that is also divisible by 4. And after adding 8 to it, it remains divisible by 4. Therefore, we have proven our statement.

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