Elliptic Fourier Descriptor

1 Description of a Closed Contour

Any continuous closed curve C in the 2D plane can be expressed as a function $f:\mathbb{R}\rightarrow\mathbb{R}^2$, with

$$f(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

with the continuous parameter t being varied over the range $[0,T]$ and $f(0)=f(T)$. Note that $x(t)$ and $y(t)$ are independent, real-valued functions, and t is typically the path length along the curve.

Sampling a closed curve C at K randomly spaced positions $t_0,t_1,...,t_p,...t_{K-1}$, with $\Delta t_p=t_p-t_{p-1}$, resulting a closed contour with a sequence of discrete 2D coordinates $V=(\boldsymbol{v}_0,\boldsymbol{v}_1,...,\boldsymbol{v}_p,...,\boldsymbol{v}_{K-1})$, with

$$\boldsymbol{v}_p=\begin{pmatrix}x(t_p)\\y(t_p)\end{pmatrix}$$

Since the contour is closed, thus

$$\boldsymbol{v}_p=\boldsymbol{v}_{p+kT}$$

for $0\leq p<K$ and any $k\in\mathbb{Z}$.

Furthermore, a closed contour is typically regarded as **piecewise linear** when processing it using computer algorithm, which mean the curve between $\boldsymbol{v}_p$ and $\boldsymbol{v}_{p-1}$ is regarded as a straight line.

2 Fourier Expansion of $x(t)$ and $y(t)$

$x(t)$ is a periodic function over the range $[0,T]$, the Fourier series expandsion for $x(t)$ of the complete closed contour is defined as

$$x(t)=A_0+\sum_{n=1}^{\infty}a_n cos\frac{2n\pi t}{T}+\sum_{n=1}^{\infty}b_n sin\frac{2n\pi t}{T}$$

where

$$\begin{aligned}A_0&=\frac{1}{T}\int_0^T x(t)dt\\a_n&=\frac{2}{T}\int_0^T x(t)cos\frac{2n\pi t}{T}dt\\b_n&=\frac{2}{T}\int_0^T x(t)sin\frac{2n\pi t}{T}dt\end{aligned}$$

Note that a closed curve is regarded as piecewise linear, thus

$$\begin{aligned}A_0&=\frac{1}{T}\int_0^T x(t)dt\\&=\frac{1}{T}\sum_{p=1}^{K}\int_{t_{p-1}}^{t_p}x(t)dt\\&=\frac{1}{T}\sum_{p=1}^{K}\int_{t_{p-1}}^{t_p}(\frac{x_p-x_{p-1}}{t_p-t_{p-1}}(t-t_{p-1})+x_{p-1})dt\\&=\frac{1}{T}\sum_{p=1}^{K}(\frac{x_p-x_{p-1}}{t_p-t_{p-1}}\frac{(t-t_{p-1})^2}{2}+x_{p-1}t)\Big|_{t_{p-1}}^{t_p}\\&=\frac{1}{T}\sum_{p=1}^{K}\frac{(x_p+x_{p-1})}{2}(t_p-t_{p-1})\\&=\frac{1}{T}\sum_{p=1}^{K}\frac{(x_p+x_{p-1})}{2}\Delta t_p\end{aligned}$$

The expression of A_0 looks quite different from the orginal paper (*Elliptic Fourier Features of a Closed Contour*), but the equation in the original paper can be simplified and they are the same.

and a_n can also be obtain

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T x(t) cos\frac{2n\pi t}{T}dt \\
&= \frac{2}{T}\sum_{p=1}^{K}\int_{t_{p-1}}^{t_p} x(t) cos\frac{2n\pi t}{T}dt \\
&= \frac{2}{T}\sum_{p=1}^{K}\int_{t_{p-1}}^{t_p} (\frac{x_p - x_{p-1}}{t_p - t_{p-1}}(t - t_{p-1}) + x_{p-1}) cos\frac{2n\pi t}{T}dt \\
&= \frac{1}{n\pi}\sum_{p=1}^{K}\int_{t_{p-1}}^{t_p} (\frac{x_p - x_{p-1}}{t_p - t_{p-1}}(t - t_{p-1}) + x_{p-1}) d(sin\frac{2n\pi t}{T}) \\
&= \frac{1}{n\pi}\sum_{p=1}^{K}\Big[(\frac{x_p - x_{p-1}}{t_p - t_{p-1}}(t - t_{p-1}) + x_{p-1}) sin\frac{2n\pi t}{T}\Big|_{t_{p-1}}^{t_p} - \frac{x_p - x_{p-1}}{t_p - t_{p-1}}\int_{t_{p-1}}^{t_p} sin\frac{2n\pi t}{T}dt\Big] \\
&= \frac{1}{n\pi}\sum_{p=1}^{K}(x_p sin\frac{2n\pi t_p}{T} - x_{p-1} sin\frac{2n\pi t_{p-1}}{T}) - \frac{1}{n\pi}\sum_{p=1}^{K}\frac{\Delta x_p}{\Delta t_p}\int_{t_{p-1}}^{t_p} sin\frac{2n\pi t}{T}dt \\
&= -\frac{1}{n\pi}\sum_{p=1}^{K}\frac{\Delta x_p}{\Delta t_p}\int_{t_{p-1}}^{t_p} sin\frac{2n\pi t}{T}dt \\
&= \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\frac{\Delta x_p}{\Delta t_p}\Big[cos\frac{2n\pi t_p}{T} - cos\frac{2n\pi t_{p-1}}{T}\Big]
\end{aligned}$$

Similarily b_n can also be calculated as

$$b_n = \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\frac{\Delta x_p}{\Delta t_p}\Big[sin\frac{2n\pi t_p}{T} - sin\frac{2n\pi t_{p-1}}{T}\Big]$$

The same way, the Fourier series expandsion for $y(t)$ of the complete closed contour is defined as

$$y(t) = C_0 + \sum_{n=1}^{\infty} c_n cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} d_n sin\frac{2n\pi t}{T}$$

where

$$C_0 = \frac{1}{T}\sum_{p=1}^{K}\frac{(y_p + y_{p-1})}{2}\Delta t_p$$

and

$$c_n = \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\frac{\Delta y_p}{\Delta t_p}\Big[cos\frac{2n\pi t_p}{T} - cos\frac{2n\pi t_{p-1}}{T}\Big]$$

$$d_n = \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\frac{\Delta y_p}{\Delta t_p}\Big[sin\frac{2n\pi t_p}{T} - sin\frac{2n\pi t_{p-1}}{T}\Big]$$

3 Truncated Fourier Approximation of a Closed Contour

The truncated Fourier approximation of a closed contour can be written as

$$\begin{aligned}
x(t) &= A_0 + \sum_{n=1}^{N} X_n(t) \\
y(t) &= C_0 + \sum_{n=1}^{N} Y_n(t)
\end{aligned}$$

where the components of X_n, Y_n $(1 \le n \le N)$ are

$$X_n(t) = a_n cos\frac{2n\pi t}{T} + b_n sin\frac{2n\pi t}{T}$$
$$Y_n(t) = c_n cos\frac{2n\pi t}{T} + d_n sin\frac{2n\pi t}{T}$$

It can also be expressed as matrix form

$$\begin{bmatrix} X_n(t) \\ Y_n(t) \end{bmatrix} = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} cos\frac{2n\pi t}{T} \\ sin\frac{2n\pi t}{T} \end{bmatrix}$$

4 Geometric Interpretation of Fourier Coefficients

4.1 A_0 and C_0 Correspond to Contour's Centroid

When the contour is regularly sampled, $\Delta t_p = t_p - t_{p-1} = \frac{T}{K}$, thus

$$\begin{aligned} A_0 &= \frac{1}{T}\sum_{p=1}^{K}\frac{(x_p + x_{p-1})}{2}\Delta t_p \\ &= \frac{1}{T}\sum_{p=1}^{K}(x_p + x_{p-1})\frac{T}{K} \\ &= \frac{1}{K}\sum_{p=0}^{K-1}x_p \\ &= \bar{x} \\ C_0 &= \frac{1}{T}\sum_{p=1}^{K}\frac{(y_p + y_{p-1})}{2}\Delta t_p \\ &= \frac{1}{T}\sum_{p=1}^{K}(y_p + y_{p-1})\frac{T}{K} \\ &= \frac{1}{K}\sum_{p=0}^{K-1}y_p \\ &= \bar{y} \end{aligned}$$

A_0 and C_0 is simply the average of x and y coordinate when the contour is regularly sampled.

4.2 a_n, b_n, c_n and d_n Correspond to Ellipse

The n coefficients contribute to

$$X_n(t) = a_n cos\frac{2n\pi t}{T} + b_n sin\frac{2n\pi t}{T}$$
$$Y_n(t) = c_n cos\frac{2n\pi t}{T} + d_n sin\frac{2n\pi t}{T}$$

By removing $cos\frac{2n\pi t}{T}$ and $sin\frac{2n\pi t}{T}$ utilizing the equation $cos\frac{2n\pi t}{T} + sin\frac{2n\pi t}{T} = 1$, it is easy to get that

$$(c_n^2 + d_n^2)X_n(t)^2 + (a_n^2 + b_n^2)Y_n(t)^2 - 2(a_n c_n + b_n d_n)X_n(t)Y_n(t) - (a_n d_n - b_n c_n)^2 = 1$$

It is obvious that the contour constructed by $X_n(t)$ and $Y_n(t)$ is **an ellipse with its center coinciding with the origin of the coordinate system**.

5 Invariance of Start Point, Translation, Rotation and Scale

5.1 Invariance of Start Point

A start point displaced λ units in the direction of rotation around the contour from the original start point is written as

$$X_n^*(t^*) = a_n^* cos\frac{2n\pi t^*}{T} + b_n^* sin\frac{2n\pi t^*}{T}$$
$$Y_n^*(t^*) = c_n^* cos\frac{2n\pi t^*}{T} + d_n^* sin\frac{2n\pi t^*}{T}$$

It can also be written as

$$
\begin{aligned}
X_n(t^*+\lambda) &= a_n cos\frac{2n\pi(t^*+\lambda)}{T} + b_n sin\frac{2n\pi(t^*+\lambda)}{T} \\
&= (a_n cos\frac{2n\pi\lambda}{T} + b_n sin\frac{2n\pi\lambda}{T})cos\frac{2n\pi t^*}{T} + (-a_n sin\frac{2n\pi\lambda}{T} + b_n cos\frac{2n\pi\lambda}{T})sin\frac{2n\pi t^*}{T} \\
Y_n(t^*+\lambda) &= c_n cos\frac{2n\pi(t^*+\lambda)}{T} + d_n sin\frac{2n\pi(t^*+\lambda)}{T} \\
&= (c_n cos\frac{2n\pi\lambda}{T} + d_n sin\frac{2n\pi\lambda}{T})cos\frac{2n\pi t^*}{T} + (-c_n sin\frac{2n\pi\lambda}{T} + d_n cos\frac{2n\pi\lambda}{T})sin\frac{2n\pi t^*}{T}
\end{aligned}
$$

Thus

$$
\begin{aligned}
a_n^* &= a_n cos\frac{2n\pi\lambda}{T} + b_n sin\frac{2n\pi\lambda}{T} \\
b_n^* &= -a_n sin\frac{2n\pi\lambda}{T} + b_n cos\frac{2n\pi\lambda}{T} \\
c_n^* &= c_n cos\frac{2n\pi\lambda}{T} + d_n sin\frac{2n\pi\lambda}{T} \\
d_n^* &= -c_n sin\frac{2n\pi\lambda}{T} + d_n cos\frac{2n\pi\lambda}{T}
\end{aligned}
$$

which can be written as matrix form as

$$
\begin{bmatrix} a_n^* & c_n^* \\ b_n^* & d_n^* \end{bmatrix} = \begin{bmatrix} cos\frac{2n\pi\lambda}{T} & sin\frac{2n\pi\lambda}{T} \\ -sin\frac{2n\pi\lambda}{T} & cos\frac{2n\pi\lambda}{T} \end{bmatrix} \begin{bmatrix} a_n & c_n \\ b_n & d_n \end{bmatrix}
$$

5.1.1 The start point invariance method in original paper

The point on the first semimajor axis of the first Fourier coefficients ellipse is selected as the start point. The first Fourier coefficients ellipse is written as

$$
\begin{aligned}
X_1 &= a_1 cos\theta + b_1 sin\theta \\
Y_1 &= c_1 cos\theta + d_1 sin\theta
\end{aligned}
$$

The point on the first semimajor axis is equivalent to find the maximum of $\sqrt{X_1^2+Y_1^2}$, thus

$$
\begin{aligned}
\frac{d\sqrt{X_1^2+Y_1^2}}{d\theta} &= \frac{1}{2\sqrt{X_1^2+Y_1^2}}(2X_1\frac{dX_1}{d\theta} + 2Y_1\frac{dY_1}{d\theta}) \\
&= \frac{1}{\sqrt{X_1^2+Y_1^2}}((a_1 cos\theta + b_1 sin\theta)(-a_1 sin\theta + b_1 cos\theta) + (c_1 cos\theta + d_1 sin\theta)(-c_1 sin\theta + d_1 cos\theta))
\end{aligned}
$$

Let $\frac{d\sqrt{X_1^2+Y_1^2}}{d\theta} = 0$, we get

$$
tan2\theta = \frac{2tan\theta}{1-tan^2\theta} = \frac{2(a_1b_1+c_1d_1)}{a_1^2-b_1^2+c_1^2-d_1^2}
$$

which yeilds

$$
\theta_1 = \frac{1}{2}arctan\Big[\frac{2(a_1b_1+c_1d_1)}{a_1^2-b_1^2+c_1^2-d_1^2}\Big]
$$

where $0 \le \theta_1 < \pi$.

5.1.2 The start point invariance method in the code

Ellipse has two semimajor axis, sometimes it is difficult to determine the semimajor axis and the normalized coefficients have -1 multiplier differences
Thus, the point having the maimum distance from the centroid (A_0, C_0) is selected as the start point.

The Fourier coefficients that are correct for the start point is now

$$
\begin{bmatrix} a_n^* & c_n^* \\ b_n^* & d_n^* \end{bmatrix} = \begin{bmatrix} cos(n\theta_1) & sin(n\theta_1) \\ -sin(n\theta_1) & cos(n\theta_1) \end{bmatrix} \begin{bmatrix} a_n & c_n \\ b_n & d_n \end{bmatrix}
$$

5.2 Invariance of Trannslation

A_0 and C_0 is simply the average of x and y coordinate when the contour is regularly sampled. To make Fourier coefficients invariant against translation, it is thus sufficient to zero A_0 and C_0, thereby shifting the shape's center to the origin of the coordinate system. Alternatively, it is achieved by simply ignoring coefficient A_0 and C_0.

5.3 Invariance of Rotation

The start point variance removed X_n^* and Y_n^* are expressed in matrix form

$$\begin{bmatrix} X_n^*(t^*) \\ Y_n^*(t^*) \end{bmatrix} = \begin{bmatrix} a_n^* & b_n^* \\ c_n^* & d_n^* \end{bmatrix} \begin{bmatrix} cos\frac{2n\pi t^*}{T} \\ sin\frac{2n\pi t^*}{T} \end{bmatrix}$$

The effect of anti-clockwise axial rotation on the Fourier coefficients is readily apparent

$$\begin{bmatrix} X_n^{**}(t^*) \\ Y_n^{**}(t^*) \end{bmatrix} = \begin{bmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{bmatrix} \begin{bmatrix} X_n^*(t^*) \\ Y_n^*(t^*) \end{bmatrix} = \begin{bmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{bmatrix} \begin{bmatrix} a_n^* & b_n^* \\ c_n^* & d_n^* \end{bmatrix} \begin{bmatrix} cos\frac{2n\pi t^*}{T} \\ sin\frac{2n\pi t^*}{T} \end{bmatrix}$$

Note that

$$\begin{bmatrix} X_n^{**}(t^*) \\ Y_n^{**}(t^*) \end{bmatrix} = \begin{bmatrix} a_n^{**} & b_n^{**} \\ c_n^{**} & d_n^{**} \end{bmatrix} \begin{bmatrix} cos\frac{2n\pi t^*}{T} \\ sin\frac{2n\pi t^*}{T} \end{bmatrix}$$

Therefore, it is easy to get

$$\begin{bmatrix} a_n^{**} & b_n^{**} \\ c_n^{**} & d_n^{**} \end{bmatrix} = \begin{bmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{bmatrix} \begin{bmatrix} a_n^* & b_n^* \\ c_n^* & d_n^* \end{bmatrix} = \begin{bmatrix} cos\psi & -sin\psi \\ sin\psi & cos\psi \end{bmatrix} \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} cos(n\theta_1) & -sin(n\theta_1) \\ sin(n\theta_1) & cos(n\theta_1) \end{bmatrix}$$

5.3.1 The rotation invariance method in original paper

The semimajor axis of the first Fourier coefficients ellipse is selected to aligned with horizontal axis of the coordiante system. Thus, the axial rotation angle is obvious

$$\begin{aligned} \psi_1 &= arctan\left[\frac{Y_1^*(0)}{X_1^*(0)}\right] \\ &= arctan\frac{c_1^*}{a_1^*} \end{aligned}$$

where $0 \leq \psi_1 < 2\pi$. We need to rotate $-\psi$ to obtain rotation invariance.

5.3.2 The rotation invariance method in the code

The point having the maimum distance from the centroid (A_0, C_0) is selected is selected to aligned with horizontal axis of the coordiante system.Thus, the axial rotation angle is obvious

$$\begin{aligned} \psi_1 &= arctan\left[\frac{y(0)}{x(0)}\right] \\ &= arctan\frac{\sum_{i=1}^{N} c_i^*}{\sum_{i=1}^{N} a_i^*} \end{aligned}$$

Thus, it is easy to get

$$\begin{bmatrix} a_n^{**} & b_n^{**} \\ c_n^{**} & d_n^{**} \end{bmatrix} = \begin{bmatrix} cos(-\psi_1) & -sin(-\psi_1) \\ sin(-\psi_1) & cos(-\psi_1) \end{bmatrix} \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} cos(n\theta_1) & -sin(n\theta_1) \\ sin(n\theta_1) & cos(n\theta_1) \end{bmatrix}$$

5.4 Invariance of Scale

The independence of scale is achieved by dividing each of the coefficients by the magnitude of the simimajor axis.

The magnitude of the semimajor axis is

$$\sqrt{X_1^*(0)^2 + Y_1^*(0)^2} = \sqrt{a_1^{*2} + b_1^{*2}}$$

and

$$s_1 = \frac{1}{\sqrt{a_1^{*2} + b_1^{*2}}}$$

The Fourier coefficients corrected by scale is illustrated as

$$\begin{bmatrix} a_n^{***} & b_n^{***} \\ c_n^{***} & d_n^{***} \end{bmatrix} = s_1 \begin{bmatrix} a_n^{**} & b_n^{**} \\ c_n^{**} & d_n^{**} \end{bmatrix}$$

5.5 Invariance of Start Point, Rotation and Scale

The invariance of start point, rotation and scale can be expressed as

$$\begin{bmatrix} a_n^{***} & b_n^{***} \\ c_n^{***} & d_n^{***} \end{bmatrix} = s_1 \begin{bmatrix} cos(-\psi_1) & -sin(-\psi_1) \\ sin(-\psi_1) & cos(-\psi_1) \end{bmatrix} \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} cos(n\theta_1) & -sin(n\theta_1) \\ sin(n\theta_1) & cos(n\theta_1) \end{bmatrix}$$

6 References

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2. Burger, W., Burge, M.J. (2013). Fourier Shape Descriptors. In: Principles of Digital Image Processing. Undergraduate Topics in Computer Science. Springer, London. https://doi.org/10.1007/978-1-84882-919-0_6