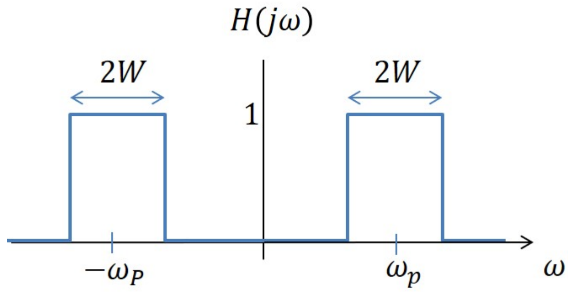
Objectives

- Apply Fourier Transforms and Properties
- Improve your familiarity with sampling

General instructions for this homework

- Please reach out to members of the teaching team if you find yourselves stuck. We are eager to help.

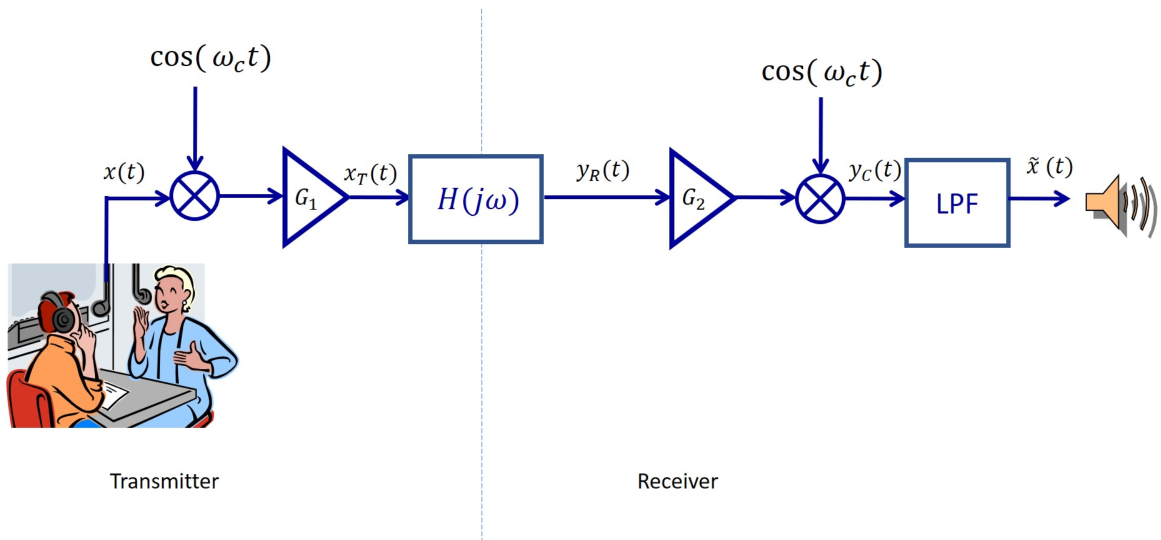
1. Consider the following frequency response of a bandpass filter with bandwidth $2W$ and center frequency ω_p.



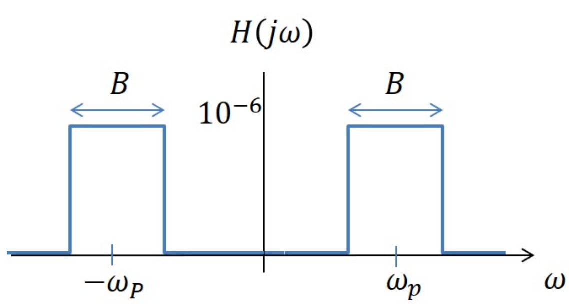
   (a) Why is this called a bandpass filter?

   (b) What is the impulse response of this filter, $h(t)$? Note that you can solve this using Fourier Transform properties and known Fourier transform pairs, or you can solve it using the inverse-Fourier transform integral.

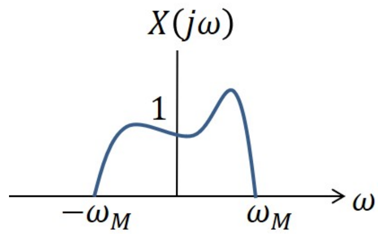
2. Consider the following figure which shows a simple model for an Amplitude-Modulation (AM) radio system.



   The signal from a microphone, $x(t)$ is multiplied by $\cos(\omega_c t)$ and scaled by G_1. The resulting signal $x_T(t)$ is transmitted using an antenna over the air and received as $y_R(t)$ at the receiver, and the frequency response of the system relating $x_T(t)$ and $y_R(t)$ is $H(j\omega)$, which is shown below (not to scale).



   $y_R(t)$ is scaled by G_2, multiplied by $\cos(\omega_c t)$, and lowpass filtered to produce an estimate of $x(t)$, which we call $\tilde{x}(t)$. Suppose that $x(t)$ has the following Fourier transform, and let $B > 2\omega_M$.



(a) Please sketch $X_T(j\omega)$.

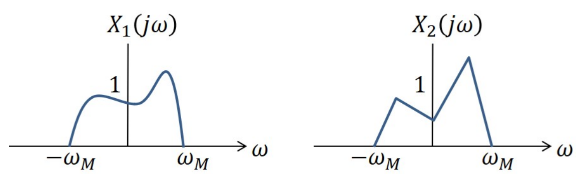
(b) Please find a value of ω_c such that the signal $x_T(t)$ and $y_R(t)$ are identical up to a scale factor.

(c) Please sketch $Y_c(j\omega)$ assuming the value of ω_c from the previous part.

(d) Please sketch the frequency response of the low-pass-filter marked LPF in system diagram, and provide values of G_1 and G_2 such that $\tilde{x}(t) = x(t)$.

This is a simplified, but surprisingly accurate model of AM radio transmission. The band-pass characteristic of the system (or called channel in the wireless comms. community) is due to the antennas at the transmitter and receiver which are tuned to radiate and absorb electromagnetic (EM) waves in fixed a range of frequencies. Additionally, regulatory constraints (FCC will only allow stations to operate in certain frequency ranges), and EM propagation properties further cause the channel to be effectively band-pass.

3. Consider two signals $x_1(t)$ and $x_2(t)$ with $X_1(j\omega)$ and $X_2(j\omega)$ given by



Suppose that

$$y(t) = x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t)$$

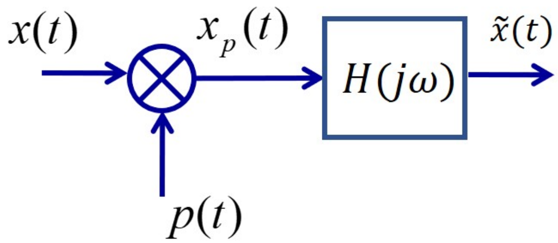
(a) Please sketch $Y(j\omega)$ assuming the signals due to $x_1(t)$ and $x_2(t)$ do not overlap in $Y(j\omega)$. Please find the relationships between ω_1, ω_2 and ω_M for this to be possible. You may find the multiplication-convolution Fourier transform property to be useful here.

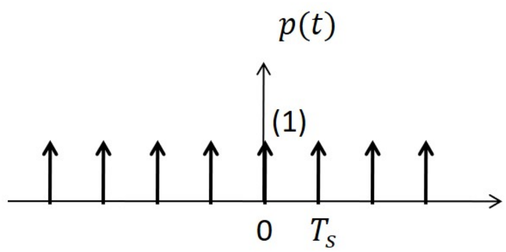
(b) Define $y_1(t) = y(t)\cos(\omega_1 t)$. Please sketch $Y_1(j\omega)$, and describe in words how you can recover $x_1(t)$ from $y_1(t)$.

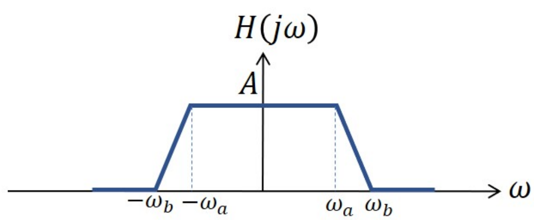
(c) Define $y_2(t) = y(t)\cos(\omega_2 t)$. Please sketch $Y_2(j\omega)$, and describe in words how you can recover $x_2(t)$ from $y_2(t)$.

This problem is intended to show you how AM radio systems with multiple stations work. Each station gets its own frequency (there are two stations with frequencies ω_1 and ω_2 in this example).

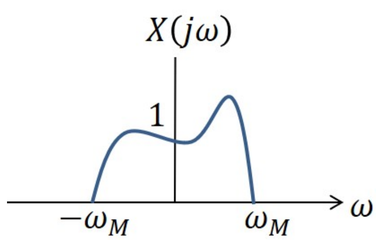
4. Consider the following system which samples and reconstructs a continuous-time signal $x(t)$, with $p(t)$ and $H(j\omega)$ as shown. The reconstructed signal is $\tilde{x}(t)$







Suppose that $X(j\omega)$ is given by the following:

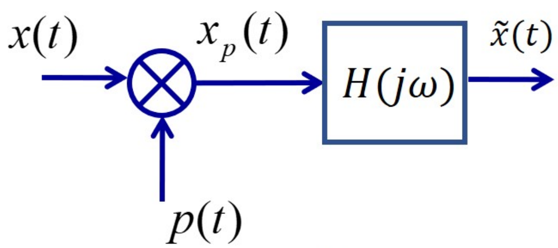


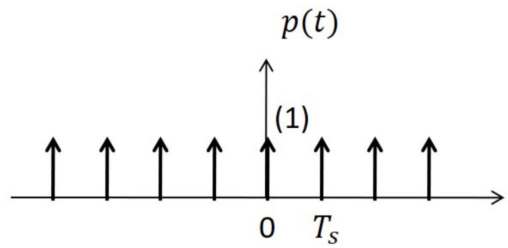
(a) Please sketch $X_p(j\omega)$

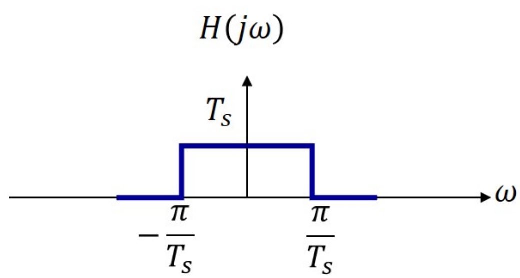
(b) Please find the relationships between T_s, ω_M, ω_a, ω_b and A such that $\tilde{x}(t) = x(t)$.

Note that the $H(j\omega)$ given above is still an idealized model for a low-pass filter, but it is much closer to a real low-pass filter compared to a rectangle in the frequency domain.

5. In class, we saw and heard an example of a cosine at 440 Hz looking and sounding identical to a cosine at 2560 Hz when the sampling frequency was 3000 Hz. Here, we are going to try to analyze that case in the frequency domain. Consider a system which samples a continuous-time (CT) signal $x(t)$ and restores it to a CT signal $\tilde{x}(t)$ using an ideal low-pass filter as follows:







Suppose that $x_1(t) = \cos(\omega_1 t)$ and $x_2(t) = \cos(\omega_2 t)$. To simplify notation, assume that the angular sampling frequency is $\omega_s = \frac{2\pi}{T_s}$.

(a) Assume that $\omega_1 = 2\pi \times 440$ rad/s, $\omega_2 = 2\pi \times 2560$ rad/s and $\omega_s = 2\pi \times 3000$ rad/s.

i. Let $x(t) = x_1(t)$. Sketch $X_p(j\omega)$ and find $\tilde{x}(t)$.

ii. Let $x(t) = x_2(t)$. Sketch $X_p(j\omega)$ and find $\tilde{x}(t)$.

(b) (Optional) Repeat the previous part without assuming specific values for ω_1, ω_2 and ω_s, but instead assume the following

- $2\omega_1 < \omega_s$
- $2\omega_2 > \omega_s$ and $\omega_2 < \omega_s$.

You will see that $\tilde{x}(t)$ is identical for both $x(t) = x_1(t)$ and $x(t) = x_2(t)$ if $\omega_s - \omega_2 = \omega_1$.