Applied Regression

Problem H.4

a) Linear model $y = \underset{n\times(p+1)}{X}\beta + \varepsilon$

$$\hat{\mu} = X\hat{\beta} = Hy$$

$$H = X(X'X)^{-1}X' \quad , \quad \hat{\beta} = (X'X)^{-1}X'y$$

$$E(\varepsilon) = 0 \quad , \quad V(\varepsilon) = \sigma^2 I_n$$
$$E(y) = X\beta \quad , \quad V(y) = \sigma^2 I_n$$

Mean vector

$$\begin{aligned}
E(\hat{\mu}) &= E(X\hat{\beta}) \\
&= X\,E(\hat{\beta}) \\
&= X\,E((X'X)^{-1}X'y) \\
&= X\,E((X'X)^{-1}X'(X\beta+\varepsilon)) \\
&= X\left(E((X'X)^{-1}X'X\beta) + (X'X)^{-1}X\,E(\varepsilon)\right) \\
&= X\,E(\beta) + 0 \\
&= X\beta
\end{aligned}$$

$\therefore \boxed{E(\hat{\mu}) = \mu}$

$$\begin{aligned}
V(\hat{\mu}) &= V(X\hat{\beta}) \\
&= X\,V(\hat{\beta})\,X' \\
&= X\,V((X'X)^{-1}X'y)\,X' \\
&= X\left[(X'X)^{-1}X'\,V(y)\,((X'X)^{-1}X')'\right]X' \quad \because (AB)' = B'A' \\
&= X\left[(X'X)^{-1}X'\sigma^2 I \cdot (X')'((X'X)^{-1})'\right]X' \\
&= X\left[(X'X)^{-1}X' \cdot X\,(X'X)^{-1} \cdot \sigma^2\right]X' \\
&= X\left[(X'X)^{-1} \cdot \sigma^2\right]X' \\
&= X(X'X)^{-1}X' \cdot \sigma^2
\end{aligned}$$

$\therefore \boxed{V(\hat{\mu}) = H\sigma^2}$

b) $Var(\hat{\mu}) = \sigma^2 \hat{h}$

$$\frac{1}{n}\sum_{i=1}^{n} Var(\hat{\mu}_i) = \frac{\sigma^2}{n}\sum_{i=1}^{n} h_{ii}$$

The trace of square matrix is equal to sum of its diagonal elements
As. $tr(AB) = tr(BA)$
Applying this to hat matrix

$$\frac{1}{n}\sum_{i=1}^{n} Var(\hat{\mu}_i) = \frac{\sigma^2}{n} tr(H)$$

$$= \frac{\sigma^2}{n} tr(X(X'X)^{-1}X')$$

$$= \frac{\sigma^2}{n} tr(XX'(X'X)^{-1}) \quad \ldots [\because tr(ABC) = tr(ACB) = tr(BAC)]$$

$$= \frac{\sigma^2}{n} tr(I_{(p+1)\times(p+1)}) \quad \ldots \left[\underset{(p+1)\times n}{X'}\underset{n\times(p+1)}{X} = I_{(p+1)\times(p+1)}\right]$$

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} Var(\hat{\mu}_i) = \frac{\sigma^2(p+1)}{n}}$$

c) An idempotent matrix is a matrix which when multiplied by itself, yields itself.
i.e. $M.M = M$

$$H = X(X'X)^{-1}X'$$

$$\therefore H.H = (X(X'X)^{-1}X')(X(X'X)^{-1}X')$$

$$= X(X'X)^{-1}\underbrace{(X'X)(X'X)^{-1}}_{I}X'$$

$$= X(X'X)^{-1}.I.X'$$

$$= X(X'X)^{-1}X'$$

$$\boxed{H.H = H}$$

A symmetric matrix is a square matrix that is equal to its transpose.
i.e. $M = M'$

$$H' = (X(X'X)^{-1}X')'$$

$$= (X')'[(X'X)^{-1}]'X' \quad \ldots [(ABC)' = C'B'A']$$

$$= X[(X'X)']^{-1}X'$$

$\therefore \quad H' = X\,(X'(X')')^{-1}\,X'$

$\quad\quad = X\,(X'X)^{-1}X'$

$\therefore \boxed{H' = H}$

Show: $\quad 0 \leq h_{ii} \leq 1$

i) $\quad h_{ii} \geq 0$

Let $a_i \in R^n$ is a vector with all components equal to 0, except for i^{th} term which is 1

$\therefore \underset{\substack{n\times 1 \\ \text{vector}}}{a_i} = (0\ 0\ \underset{\uparrow}{1}\ 0\ 0 \ldots)'$

Quadratic form $\quad \underset{1\times n}{a_i'}\ \underset{n\times n}{H}\ \underset{n\times 1}{a_i} = h_{ii} \geq 0$

Let $H w = \lambda w \longrightarrow ①$

where $\lambda \rightarrow$ eigen value

$\quad\quad w \rightarrow$ eigen vector

$H.Hw = \lambda Hw$

$H^2 w = \lambda(\lambda w) = \lambda^2 w \rightarrow ②$

H is idempotent

$H^2 w = Hw = \lambda w \rightarrow ③$

$\lambda^2 w = \lambda w$

$\lambda(\lambda-1)w = 0$

$\lambda = 0$ or 1

eigen values are non-negative, so is H is positive semidefinite and ~~h_{ii}~~ p.s.d. has non-negative diagonal elements

$\therefore \boxed{h_{ii} \geq 0}$

ii) $\quad h_{ii} \leq 1$

$h_{ii} = a_i' H. a_i$

$\quad = a_i'(I - M)a_i \quad \because$ Projection matrix $\quad M = I - H$

$\quad = a_i'.Ia_i - \underbrace{a_i' M a_i}_{\geq 0} \quad\quad M$ is p.s.d. as $(I-H)$ is p.s.d.

$\quad = 1 -$

As we subtract something +ve from 1,

$\therefore \boxed{h_{ii} \leq 1}$

d) $h_{ii} = x_i'(X'X)^{-1}x_i$

$$X'X = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+\cdots 1 & x_1+x_2+\cdots+x_n \\ x_1+x_2\cdots+x_n & x_1^2+x_2^2+\cdots+x_n^2 \end{pmatrix}$$

$$= \begin{pmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{pmatrix}$$

$$= \begin{pmatrix} n & n\bar{x} \\ n\bar{x} & \Sigma x_i^2 \end{pmatrix}$$

Determinant

$$|X'X| = n\Sigma x_i^2 - (n\bar{x})^2$$
$$= n(\Sigma x_i^2 - n\bar{x}^2)$$
$$= nS_{xx}$$

$$(X'X)^{-1} = \frac{X'X}{|X'X|} = \frac{1}{nS_{xx}} \begin{pmatrix} \Sigma x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix}$$

$\therefore$ $h_{ii} = x_i'(X'X)^{-1}x_i$

$$= (1 \; x_i) \frac{1}{nS_{xx}} \begin{pmatrix} \Sigma x_j^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix} \begin{pmatrix} 1 \\ x_i \end{pmatrix}$$

$$= \frac{1}{nS_{xx}} \left(\sum_{j=1}^{n} x_j^2 - 2n\bar{x}x_i + nx_i^2 \right)$$

$$= \frac{1}{nS_{xx}} \left(\sum_{j=1}^{n} x_j^2 - \underbrace{n\bar{x}^2 + [n\bar{x}^2}_{\text{Adding \& substracting } n\bar{x}^2} - 2n\bar{x}x_i + nx_i^2] \right)$$

$$= \frac{1}{nS_{xx}} \left(\sum_{j=1}^{n} x_j^2 - n\bar{x}^2 \right) + \frac{1}{nS_{xx}} [n\bar{x}^2 - 2n\bar{x}x_i + nx_i^2]$$

$$= \frac{1}{nS_{xx}} \cdot S_{xx} + \frac{1}{nS_{xx}} n[x_i - \bar{x}]^2$$

$$h_{ii} = \frac{1}{n} + \frac{1}{S_{xx}}[x_i - \bar{x}]^2$$

The 2nd term is square, so will be positive.

$\therefore$ $\boxed{h_{ii} \geq \frac{1}{n}}$

e) Refer html code file