Machine Learning

Homework 7 solution - Shyam Arumugaswamy

1.

$f_o(x) = -(x1 + x2)$

$f_1(x) = x1^2 + x2^2 -1 <= 0$

Lagrangian is given as

$L(x1,x2,\alpha) = - x1- x2 + \alpha(x12+x22-1)$

KTT :
$fi(x^*) <= 0$
$\alpha i.xi = 0$
$\alpha i >= 0$
$\nabla_x L(x^*, \alpha^*) = 0$

$\frac{\partial(L(x1,x2,\alpha)}{\partial x1} = 0$

$-1 + 2 \alpha x1 = 0$

$x1 = ½\alpha$ ---→1

$\frac{\partial(L(x1,x2,\alpha)}{\partial x2} = 0$

$-1 + 2 \alpha x2 = 0$

$x2 = ½\alpha$ ---→2

From complementary slackness KTT/

$\alpha i.xi = 0$

so,

$\alpha(x12+x22-1) = 0$ ---→3

Use values of x1, x2 in eq 3

α(1/4 + ¼ - 1) = 0

$\alpha = \pm\frac{1}{\sqrt{2}}$ ---→ 4

subsitutiong value of α in eq 1 & 2,

we get

x1 = $\pm\frac{1}{2\sqrt{2}}$, x2 = $\pm\frac{1}{2\sqrt{2}}$

now, substituting values of x1, x2 in fo(x), fo(x) is minimum at value of $-\frac{1}{\sqrt{2}}$

at (x1,x2) = ($\frac{1}{2\sqrt{2}}$, $\frac{1}{2\sqrt{2}}$)

2.

Similarity:

SVM and perceptron algorithm find a hyperplane which separates the two classes of data

Difference:

Major practical difference is that perceptron can be trained online but SVM cannot. SVM maximizes the region around a hyperplane so that no data point lies in the region. Perceptron finds the first hyperplane separating the two classes.

3.

we know that Slater's constraint qualification guarantees that the duality gap is zero if the objective function is convex and the constraints are convex

First we show that the objective function

f_0(w, b, ꞩ) = 1/2wTw + C$\sum_{i=1}^{m}$ꞩi

with C >= 0 is convex. We prove that the parabola f(x) = x^2 is convex by showing that for x, y ϵ R the inequality holds.

f(y) >= f(x) + $(y - x)^T \nabla$ f(x)

We have

f(x) + (y - x) $\frac{df}{dx}$ = x^2 + (y - x) 2x = $-x^2$ + 2xy

= $- x^2 + 2xy - y^2 + y^2 = -(x - y)^2 + y^2 <= y^2$ = f(y)

The linear function $f(x) = x$ is convex because

$f((1 - t)x + ty) = (1 - t)x + ty = (1 - t)f(x) + tf(y)$:

Now, the objective function $f0(w, b, \xi)$ is convex because it is a sum of (scaled) convex functions.

The constraints

$fi(x) = yi(w^T xi + b) - 1 + \xi I >= 0 \quad i = 1 \ldots m$

are convex because we can write them in the form

$fi(x) = a^T x + c$

with $a = y_I w$ and $c = y_I b - 1 + \xi i$

The constraints

$gi(\xi i) = \xi i \quad i = 1; \ldots m >= 0$

are convex because we can also write them in the form

$gi(\xi i) = a \xi i + c$

with $a = 1$ and $c = 0$.

Thus Slater's constraint qualification applies and the duality gap is zero for the SVM

Programming assignment 7: SVM

In [21]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

from sklearn.datasets import make_blobs

from cvxopt import matrix, solvers
```

Your task

In this sheet we will implement a simple binary SVM classifier.

We will use **CVXOPT** http://cvxopt.org/ (http://cvxopt.org/) - a Python library for convex optimization. If you use `Anaconda`, you can install it using

```
conda install cvxopt
```

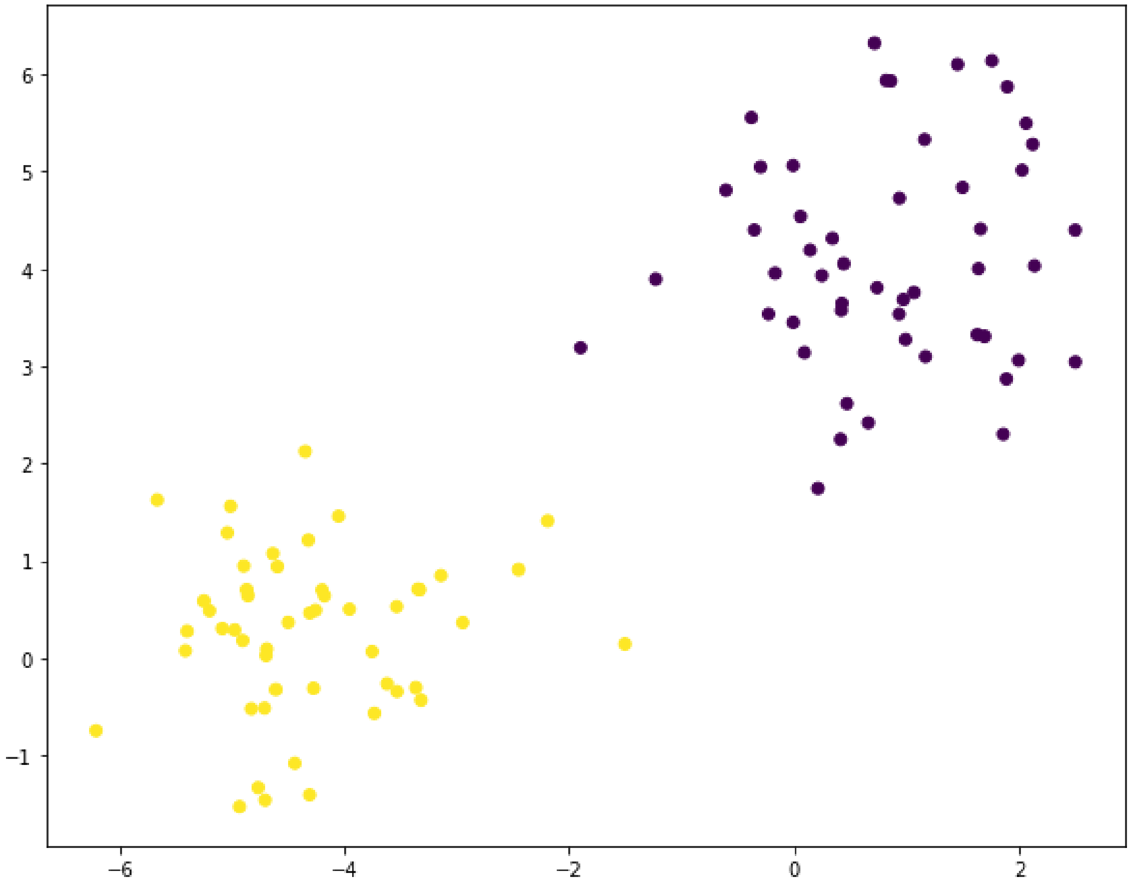
As usual, your task is to fill out the missing code, run the notebook, convert it to PDF and attach it you your HW solution.

Generate and visualize the data

In [22]:

```
N = 100  # number of samples
D = 2  # number of dimensions
C = 2  # number of classes
seed = 3  # for reproducible experiments

X, y = make_blobs(n_samples=N, n_features=D, centers=2, random_state=seed)
y[y == 0] = -1  # it is more convenient to have {-1, 1} as class labels (instead of {0,
 1})
y = y.astype(np.float)
plt.figure(figsize=[10, 8])
plt.scatter(X[:, 0], X[:, 1], c=y)
plt.show()
```



Task 1: Solving the SVM dual problem

Remember, that the SVM dual problem can be formulated as a Quadratic programming (QP) problem. We will solve it using a QP solver from the `CVXOPT` library.

The general form of a QP is

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T \mathbf{P} \mathbf{x} - \mathbf{q}^T \mathbf{x}$$

$$\text{subject to} \quad \mathbf{G}\mathbf{x} \preceq \mathbf{h}$$

$$\text{and} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\preceq$ denotes "elementwise less than or equal to".

Your task is to formulate the SVM dual problems as a QP and solve it using `CVXOPT`, i.e. specify the matrices $\mathbf{P}, \mathbf{G}, \mathbf{A}$ and vectors $\mathbf{q}, \mathbf{h}, \mathbf{b}$.

In [23]:

```
def solve_dual_svm(X, y):
    """Solve the dual formulation of the SVM problem.

    Parameters
    ----------
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).

    Returns
    -------
    alphas : array, shape [N]
        Solution of the dual problem.
    """
    # TODO
    # These variables have to be of type cvxopt.matrix

    N = X.shape[0]
    DIM = X.shape[1]

    K = y[:, None] * X
    K = np.dot(K, K.T)
    P = matrix(K)
    q = matrix(-np.ones((N, 1)))
    G = matrix(-np.eye(N))
    h = matrix(np.zeros(N))
    A = matrix(y.reshape(1, -1))
    b = matrix(np.zeros(1))
    solvers.options['show_progress'] = False
    solution = solvers.qp(P, q, G, h, A, b)
    alphas = np.array(solution['x'])
    return alphas
```

Task 2: Recovering the weights and the bias

In [24]:

```
def compute_weights_and_bias(alpha, X, y):
    """Recover the weights w and the bias b using the dual solution alpha.

    Parameters
    ----------
    alpha : array, shape [N]
        Solution of the dual problem.
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).

    Returns
    -------
    w : array, shape [D]
        Weight vector.
    b : float
        Bias term.
    """



    # get weights
    w = np.sum(alpha * y[:, None] * X, axis = 0)

    # get bias
    cond = (alpha > 1e-4).reshape(-1)
    b = y[cond] - np.dot(X[cond], w)

    return w, b[0]
```

Visualize the result (nothing to do here)

In [25]:

```
def plot_data_with_hyperplane_and_support_vectors(X, y, alpha, w, b):
    """Plot the data as a scatter plot together with the separating hyperplane.

    Parameters
    ----------
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).
    alpha : array, shape [N]
        Solution of the dual problem.
    w : array, shape [D]
        Weight vector.
    b : float
        Bias term.
    """
    plt.figure(figsize=[10, 8])
    # Plot the hyperplane
    slope = -w[0] / w[1]
    intercept = -b / w[1]
    x = np.linspace(X[:, 0].min(), X[:, 0].max())
    plt.plot(x, x * slope + intercept, 'k-', label='decision boundary')
    # Plot all the datapoints
    plt.scatter(X[:, 0], X[:, 1], c=y)
    # Mark the support vectors
    support_vecs = (alpha > 1e-4).reshape(-1)
    plt.scatter(X[support_vecs, 0], X[support_vecs, 1], c=y[support_vecs], s=250, marke
r='*', label='support vectors')
    plt.xlabel('$x_1$')
    plt.ylabel('$x_2$')
    plt.legend(loc='upper left')
```

The reference solution is

```
w = array([[-0.69192638],
           [-1.00973312]])

b = 0.907667782
```

Indices of the support vectors are

```
[38, 47, 92]
```

In [26]:

```
alpha = solve_dual_svm(X, y)
w, b = compute_weights_and_bias(alpha, X, y)
plot_data_with_hyperplane_and_support_vectors(X, y, alpha, w, b)
plt.show()
```