# PART-I SET THEORY

## ❖ INTRODUCTION

- ✓ The theory of set was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on 'problems on trigonometric series'.
- ✓ The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relation and function. The study of geometry, sequences, probability, etc. requires knowledge of sets. In this unit, we discuss basic definitions and operation involving sets.

## ❖ SET

- ✓ A set is collection of well-defined objects.
- ✓ Each of the objects in the set is called an element of the set.
- ✓ Elements of a set are usually denoted by lower case letter (a, b, c ...). While sets are denoted by capital letters (A, B, C ...).
- ✓ The symbol '$\in$ (is belongs to)' indicates the membership in a set. While the symbol '$\notin$ (is not belongs to)' is used to indicate that an element is not in the set.
- ✓ For example, if $A = \{1, 2, 3, a, b\}$ then we can write that $a \in A$, $1 \in A$ but $4 \notin A$.

## ❖ REPRESENTATION OF SETS

- ✓ Listing method (or Tabular form): In listing method, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all natural numbers less than 6 is described in listing method as {1, 2, 3, 4, 5}.
- ✓ Property method (or Set-builder form): In property method, all the elements of a set possess a single common property which is not possessed any element outside the set. For example, the set of all natural number between 4 & 9 described in property method as

$$\{x\text{: x is a natural number } \& \ 4 < x < 9\}$$

## ❖ SOME DEFINITIONS

- ✓ Empty set (null set) does not contain any element and It is denoted as { } or **ϕ**.
- ✓ A set which contain at least one element is called Non-empty set.
- ✓ A set which contain exactly one element is called Singleton set.
- ✓ A set which contain finite number of elements is called finite set.
- ✓ A set which contain infinite number of elements is called infinite set.
- ✓ Two set A and B are said to be equal if they have exactly the same elements and we write A=B. Otherwise, the sets are said to be unequal and we write A≠B.

- ✓ In any discussion in set theory, there always happens to be a set that contains all set under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

❖ SUBSET

- ✓ If every element of set A is an element of B then we can say that A is a subset of B($A \subset B$).

$$\text{i.e. } A \subset B \text{ if } \forall a \in A \Longrightarrow a \in B$$

- ✓ For example, if $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$ then $A \subset B$.

Remark: if A is a subset of B then B is a superset of A.

- ✓ Let A be any non-empty set.
  1. ϕ & A are improper subset of A and remaining all are called proper subset of A.
  2. If set A has n elements then there are total $2^n$ subsets of A.

❖ OPERATIONS ON SETS

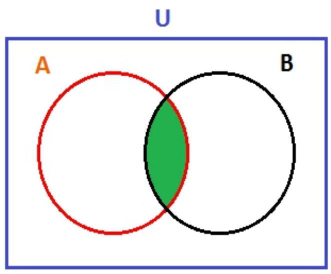
- ✓ Let U is a universal set and A & B are any two sets. Then
  1. The Union of A & B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. Symbolically, we write A U B and usually read as 'A union B'.
  $$A \cup B = \{x: x \in A \text{ or } x \in B\}$$
  2. The Intersection of A & B is the set of all elements which are common to both A and B. Symbolically, we write $A \cap B$ and usually read as 'A intersection B'.
  $$A \cap B = \{x: x \in A \text{ and } x \in B\}$$
  3. The Difference of the sets A & B in this order is the set of elements which belong to A but not to B. Symbolically, we write $A - B$ and usually read as 'A minus B'.
  $$A - B = \{x: x \in A \text{ and } x \notin B\}$$
  4. The Symmetric difference of sets A & B is the set of elements which belong to A U B but not to $A \cap B$. Symbolically, we write $A\Delta B$ and usually read as 'A delta B'.
  $$A\Delta B = \{x: x \in A \cup B \text{ and } x \notin A \cap B\}$$
  5. The complement of A is the set of all elements which belong to universal se U but not to A. Symbolically, we write A'.
  $$A' = \{x: x \in U \text{ and } x \notin A\}$$

Example: Let $\mathbf{U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}}$, $\mathbf{A = \{1, 2, 3, 4, 5\}}$ & $\mathbf{B = \{4, 5, 6, 7, 8\}}$. Find $\mathbf{A \cup B}$, $\mathbf{A \cap B}$, $\mathbf{A - B}$, $\mathbf{A\Delta B}$, $\mathbf{A'}$, $\mathbf{B'}$.

- ✓ Solution: $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A \cap B = \{4, 5\}$, $A - B = \{1, 2, 3\}$, $A' = \{6, 7, 8, 9, 10\}$
  $B' = \{1, 2, 3, 9, 10\}$, $A\Delta B = (A \cup B) - (A \cap B) = \{1, 2, 3, 6, 7, 8\}$

## ❖ VENN DIAGRAM

- ✓ The relationship between sets can be represented by means of diagrams which are known as Venn diagrams. Venn diagrams are named after the English logician, John Venn (1834-1883).
- ✓ This diagrams consist of rectangle and closed curves usually circles. The universal set is represented by rectangle and its subsets by circles.



Figure(1)

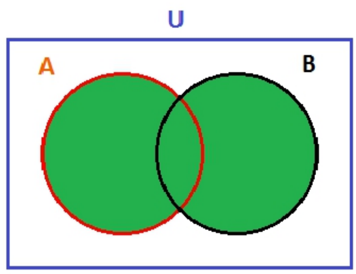


Figure (2)

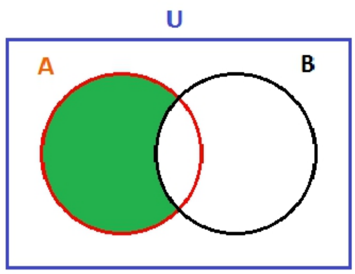
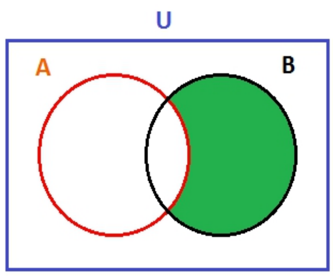


Figure (3)



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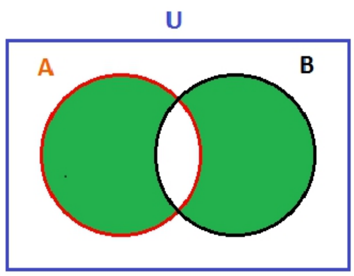

Figure (5)

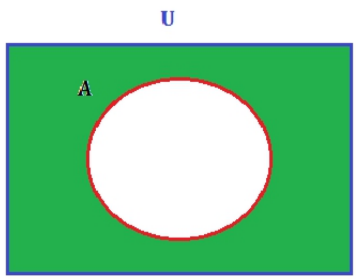

Figure (6)

- ✓ The shaded area in figure (1) is $A \cap B$ and the shaded area in figure (2) is $A \cup B$.
- ✓ The shaded area in figure (3) is $A - B$ and the shaded area in figure (4) is $B - A$.
- ✓ The shaded area in figure (5) is $A \Delta B$ and the shaded area in figure (6) is $A'$.

## ❖ METHOD-1: BASIC EXAMPLES ON SET THEORY

| | | | |
|---|---|---|---|
| H | 1 | Give the another description of the following sets:<br>(a) $A = \{x: x \text{ is an integer and } 5 \leq x \leq 12\}$<br>(b) $B = \{2, 4, 6, 8, \dots\}$<br>(c) $C = \{x: x \in \mathbb{N}, 4 < x^2 < 40\}$<br>(d) $D = \{x: x^3 - x = 0, x \in \mathbb{Z}\}$<br>Answer: $\{\mathbf{5, 6, 7, 8, 9, 10, 11, 12}\}$, $\{\mathbf{x: x \text{ is an even number}} \,\&\, \mathbf{x} \in \mathbb{N}\}$, $\{\mathbf{3, 4, 5, 6}\}$, $\{\mathbf{-1, 0, 1}\}$ | |

| | | | |
|---|---|---|---|
| C | 2 | Give the another description of the following sets:<br>(a) $A = \{x: x \in \mathbb{N}, x \text{ is a multiple of } 7 \ \& \ x \le 49\}$<br>(b) $B = \{1, 8, 27, 64, 125, 216\}$<br>(c) $C = \{x: x \in \mathbb{R}, \sin 2x = 0\}$<br>(d) $D = \{x: x \text{ is a divisor of } 36, x \in \mathbb{N}\}$<br>Answer: $\{\mathbf{7, 14, 21, 28, 35, 42, 49}\}$, $\{\mathbf{y: y = x^3, x} \in \mathbb{N} \textbf{ and } \mathbf{x \le 6}\}$, $\{\mathbf{y: y = k\pi, k} \in \mathbb{Z}\}$,<br>$\{\mathbf{1, 2, 3, 4, 6, 9, 12, 18, 36}\}$ | |
| C | 3 | Give the another description of the following sets:<br>(a) $A = \{(x, y): x, y \in \mathbb{N}, \lvert x - y \rvert \text{ is even number } \& \ x, y \le 6\}$<br>(b) $B = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$<br>(c) $C = \{(x, y): x, y \in \mathbb{N}, x \text{ divides } y \ \& \ y \le 9\}$ | |
| H | 4 | Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$, $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Find $A \cup B, A \cap B$, $A - B, B - A, A \Delta B, A', B'$. | |
| T | 5 | Let $A = \{4k + 1: k \in \mathbb{Z}\}$, $B = \{6k - 1: k \in \mathbb{Z}\}$. Find $A \cap B$.<br>Answer: $\mathbf{A \cap B = \{12k + 5: k} \in \mathbb{Z}\}$ | |
| C | 6 | If $A = \{x: x \in \mathbb{R}, x^2 - 3x - 4 = 0\}$, $B = \{x: x \in \mathbb{Z}, x^2 = x\}$, then find $A \cup B, A \cap B, A - B, A\Delta B$. | |
| C | 7 | Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$.<br>Verify (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (b) $(A \cup B)' = A' \cap B'$, (c) $A - B = A \cap B'$, (d) $A\Delta B = B\Delta A$, (e) $A - C = A - (A \cap C)$. | |
| H | 8 | Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 6\}$, $C = \{1, 2, 3\}$.<br>Verify (a) $(A - B) \cup B = A \cup B$, (b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$, (c) $A - (B - C) = (A - B) \cup (A \cap C)$, (d) $A\Delta A = \phi$, (e) $A\Delta\phi = A$. | |
| C | 9 | Draw the Venn diagram for the following:<br>$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$, $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$ | |
| H | 10 | Draw the Venn diagram for the following:<br>$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$ | |
| H | 11 | Prove the following statement using Venn diagram.<br>(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, (b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ | |
| C | 12 | Prove the following statement using Venn diagram.<br>(a) $(A \cup B)' = A' \cap B'$, (b) $A - (B \cup C) = (A - B) \cap (A - C)$ | |

## ❖ CARDINAL NUMBER OF SET

- ✓ The number of elements in a finite set A is called the cardinal number of set A and is denoted by $\mathbf{n(A)}$ **or** $|\mathbf{A}|$.
- ✓ For example, if $A = \{1, 2, 3, 5, 8\}$ then $n(A) = 5$.

## ❖ IMPORTANT RESULT OF CARDINAL NUMBER OF SETS:

- ✓ Let A, B, C are finite sets in a finite universal set U. Then
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (The Inclusion-Exclusion Principle)
  - $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
  - $n(A - B) = n(A) - n(A \cap B)$
  - $n(A \Delta B) = n(A) + n(B) - 2\, n(A \cap B)$
  - $n(A') = n(U) - n(A)$
  - $n(A' \cap B') = n(U) - n(A \cup B)$
  - $n(A' \cup B') = n(U) - n(A \cap B)$

## ❖ METHOD-2: EXAMPLES ON CARDINAL NUMBER OF SETS

| | | | |
|---|---|---|---|
| C | 1 | Let A and B be sets such that $n(A) = 50, n(B) = 50, n(A \cup B) = 75$. Find $n(A \cap B)$.<br>Answer: 25 | |
| H | 2 | Let A and B be sets such that $n(A) = 12, n(A \cup B) = 36, n(A \cap B) = 8$. Find $n(B)$.<br>Answer: 32 | |
| C | 3 | A & B are the subsets of universal set U. Let $n(A) = 20, n(B) = 30, n(U) = 80, n(A \cap B) = 10$. Find $n(A' \cap B')$.<br>Answer: 40 | |
| C | 4 | Among 50 patients admitted to a hospital, 25 are diagnosed with pneumonia, 30 with bronchitis, and 10 with both pneumonia and bronchitis. Determine:<br>(a) The number of patients diagnosed with pneumonia or bronchitis (or both).<br>(b) The number of patients not diagnosed with pneumonia or bronchitis.<br>Answer: 45, 5 | |
| T | 5 | A survey in 1986 asked households whether they had a VCR, a CD player or cable TV. 40 had a VCR. 60 had a CD player and 50 had cable TV. 25 owned VCR and CD player. 30 owned a CD player and had cable TV. 35 owned a VCR and had cable TV. 10 households had all three. How many households had at least one of the three?<br>Answer: 70 | |

| | | | |
|---|---|---|---|
| H | 6 | Among 18 students in a room, 7 study mathematics, 10 study science and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?<br>Answer: 2 | |

## ❖ POWER SET

- ✓ The set or family of all the subsets of a given set A is said to be the power set of A and is denoted by $P(A)$. Symbolically $P(A) = \{X : X \subseteq A\}$.
- ✓ It means, if $X \in P(A) \Longrightarrow X \subseteq A$. Further, $\phi \in P(A)$ & $A \in P(A)$.
- ✓ For example, if $A = \{a, b, c\}$ then $P(A) = \{\ \phi, A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\ \}$

Remark: If A has n elements, then its power set $\mathbf{P(A)}$ has $\mathbf{2^n}$ elements.

## ❖ CARTESIAN PRODUCT

- ✓ If A and B are any two non-empty sets, Cartesian product of A & B is defined and denoted as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- ✓ For example, if $A = \{a, b\}$ and $B = \{1, 2\}$ then Cartesian product of A and B is written as

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

- ✓ If A & B are empty sets, then $A \times B = \phi$.
- ✓ In general, $A \times B \neq B \times A$.
- ✓ If A has m elements and B has n elements, then $A \times B$ has $\mathbf{m} \times \mathbf{n}$ elements.
- ✓ Two dimensional Cartesian plane is set of all ordered pair resulting from the product $R \times R$, where R is a set of all real numbers.

## ❖ METHOD-3: EXAMPLES ON POWER SETS AND CARTESIAN PRODUCT

| | | | |
|---|---|---|---|
| H | 1 | Give the power sets of $\{X, Y, Z\}$ and $\{1, 2, 3\}$. | |
| C | 2 | Give the power sets of following:<br>(a) $A = \{x : x \text{ is multiple of } 4, x \in \mathbb{N} \text{ and } x \leq 16\}$<br>(b) $B = \{x : x \text{ is a prime number and } x < 8\}$ | |
| H | 3 | Let $A = \{a, b, c, d\}$ be a set. How many elements in $P(A)$? How many proper and improper subsets of A?<br>Answer: 16, 14, 2 | |
| H | 4 | Let $A = \{a, b, c\}$ and $B = \{1, 2\}$ be the sets, then write $A \times B$.<br>Answer: $\mathbf{A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}}$ | |

| | | | |
|---|---|---|---|
| T | 5 | If $A \times A = B \times B$, then prove that $A = B$. | |
| H | 6 | Let $A = \{\alpha, \beta\}$ and $B = \{1, 2, 3\}$ be the sets, then write $B \times B, A \times A$ & $(A \times B) \cap (B \times A)$. | |

❖ BASIC SET IDENTITIES

✓ Properties of the operation of Union:

- $A \cup B = B \cup A$ (Commutative law)
- $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
- $A \cup \phi = A$ (Identity element)
- $A \cup A = A$ (Idempotent property)
- $U \cup A = U$ (Law of U)

✓ Properties of the operation of intersection:

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap \phi = \phi$
- $A \cap A = A$
- $U \cap A = A$

✓ Properties of the operation of difference, complement and symmetric difference of a set:

- $A - B \neq B - A$
- $(A')' = A$
- $U' = \phi$
- $\phi' = U$
- $A\Delta B = B\Delta A$
- $A\Delta(B\Delta C) = (A\Delta B)\Delta C$
- $A\Delta A = \phi$
- $A\Delta\phi = A$

✓ Some important results:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive law)
- $(A \cup B) \cap A = A$
- $(A \cap B) \cup A = A$
- $(A \cup B) \cap B = B$
- $(A \cap B) \cup B = B$
- $A \cup A' = U$

- $A \cap A' = \phi$
- $(A \cup B)' = A' \cap B'$ (De Morgan's law)
- $(A \cap B)' = A' \cup B'$ (De Morgan's law)
- $A - (B \cup C) = (A - B) \cap (A - C)$ (De Morgan's law)
- $A - (B \cap C) = (A - B) \cup (A - C)$ (De Morgan's law)
- $A - B = A \cap B'$
- $B - A = B \cap A'$

✓ Some results on Cartesian product:

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$

❖ METHOD-4: EXAMPLES ON BASIC SET IDENTITIES

| | | | |
|---|---|---|---|
| H | 1 | If $A = \{1, 3, 5, 7\}, B = \{2, 4, 7, 8\}, C = \{5, 6, 8, 9\}$, then check the following identities:<br>(a) $(A \cup B) \cup C = A \cup (B \cup C)$<br>(b) $A\Delta B = B\Delta A$<br>(c) $A - C \neq C - A$ | |
| C | 2 | Let $A = \{1, 2, 3, 6, 7\}, B = \{2, 3, 4, 8, 9\}, C = \{3, 4, 5, 10, 11\}$. Check the following identities:<br>(a) $(A \cap B) \cap C = A \cap (B \cap C)$<br>(b) $A\Delta(B\Delta C) = (A\Delta B)\Delta C$<br>(c) $A\Delta A = \emptyset$ | |
| H | 3 | If A={x: x is a divisor of 24}, B={x: x is a divisor of 18}, C={x: x is a divisor of 6}, then check the following identities:<br>(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>(b) $A - (B \cup C) = (A - B) \cap (A - C)$<br>(c) $A - (B \cap C) = (A - B) \cup (A - C)$ | |
| C | 4 | Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$, $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Then check the following identities:<br>(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>(b) $(A \cup B)' = A' \cap B'$<br>(c) $(A \cap B)' = A' \cup B'$ | |
| T | 5 | State and prove De Morgan laws. | |

| | | | |
|---|---|---|---|
| C | 6 | Do as direct:<br>(a) If $A \subset B$ & $C \subset D$, then prove that $(A \cup C) \subset (B \cup D)$<br>(b) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>(c) Prove that $A - B = A \cap B'$ | |
| H | 7 | Do as direct:<br>(a) If $A \subset B$ & $C \subset D$, then prove that $(A \cap C) \subset (B \cap D)$<br>(b) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>(c) Prove that $B - A = B \cap A'$ | |
| H | 8 | Prove the following identities using Venn diagram:<br>(a) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$<br>(b) $A - B = A - (A \cap B)$ | |
| C | 9 | Prove the following identities using Venn diagram:<br>(a) $A \cap (B - C) = (A \cap B) - (A \cap C)$<br>(b) $(A \cap B) \cup (A - B) = A$ | |
| T | 10 | If $A \cap B = \emptyset$, $A \cup B = U$, then prove that $A' = B$. | |

## PART-II FUNCTION

### ❖ INTRODUCTION

- ✓ Function deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair.
- ✓ Practically in every day of our lives, we pair the members of two sets of numbers. For example,
  1. Each hour of the day is paired with the local temperature reading by T.V. Station's weatherman,
  2. A teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson.
- ✓ Finally, we shall learn about special relations called functions.
- ✓ The function is a special relation from one set to another set, in which every element of first set is in relation (uniquely) with the elements of another set.
- ✓ The relation, from A to B, shown in figure (1) and (2) are functions. But a relation, from A to B, shown in figure (3) is not a function because $4 \in A$ is not in relation to any element of B. Also a relation, from A to B, shown in figure (4) is not a function because $2 \in A$ is in relation with two elements of B. Now, we will define a function.

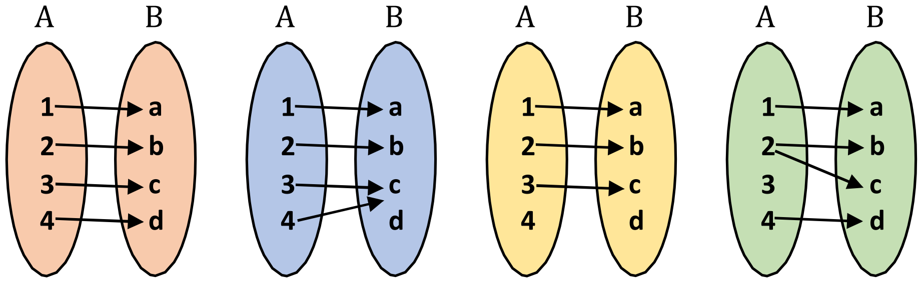


Figure (1) Figure (2) Figure (3) Figure (4)

## ❖ DEFINITION: (FUNCTION)

- ✓ Let A and B be two non-empty set. Suppose that to each elements of a set A we assign a unique element of set B; the collection of such assignments is called a function from A to B.
- ✓ Functions are ordinarily denoted by symbols. For example, let f denote a function from A to B. Then we write $\mathbf{f: A \rightarrow B}$.
- ✓ The set A is called domain of the function. (i. e. $D_f = A$)
- ✓ The set B is called co-domain of the function.
- ✓ Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$, then
  - $f = \{(1, a), (2, b), (3, c)\}$ is function from A to B.(OR $f: A \rightarrow B, f(1) = a, f(2) = b, f(3) = c$).
  - $f = \{(1, a), (2, b), (3, b)\}$ is function from A to B.(OR $f: A \rightarrow B, f(1) = a, f(2) = b, f(3) = b$).
  - $f = \{(1, a), (2, a), (3, a)\}$ is function from A to B.(OR $f: A \rightarrow B, f(1) = a, f(2) = a, f(3) = a$),

Note: If $\mathbf{n(A) = m}$ & $\mathbf{n(B) = n}$, then we can create $\mathbf{n^m}$ different functions from A to B.

- ✓ $f: N \rightarrow N, f(x) = 7x$ i.e. f(1)=7, f(2)=14, f(3)=21,.... etc

Note:

- Here 7 is an image of 1, 14 is an image of 2......etc.
- The set of images of all elements of domain is called Range of given function. In above example (2) range of f is {7, 14, 21, 28...} and it is denoted by $R_f$.

- ✓ $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + 1$.
- ✓ $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sin x$.
- ✓ $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = e^x$.
- ✓ $f: \mathbb{R}^+ \rightarrow \mathbb{R}, f(x) = \log x$.

Note: Any function from $\mathbb{R}$ to $\mathbb{R}$ is called real function.

- ✓ A program written in a high-level language is mapped into a machine language by a compiler. Similarly, the output from a computer is a function of its input.

## ❖ SOME STANDARD FUNCTIONS:

- ✓ Identity function:
  - Let A be a nonempty set. The function $f: A \rightarrow A$ define by $f(x) = x, \forall x \in A$ is called identity function on A. identity function on A is denoted by $\mathbf{I_A}$.

- This function maps any element of A onto itself. For this function, the range is entire codomain.

✓ Constant function:

- A function whose range is singleton set is called a constant function.
- Thus, the function $f: A \to B, f(x) = c, \forall x \in A$ where c is fixed elements of B, is a constant function, i.e. $\forall x \in A, f(x) = c$.

✓ Modulus function:

- The function $f: \mathbb{R} \to \mathbb{R}, f(x) = |x|$ is called modulus function or absolute value function where

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- The range of this function is $\mathbb{R}^+ \cup \{0\}$.

✓ Peano's successor function:

- The function $f: \mathbb{N} \to \mathbb{N}, f(n) = n + 1, \forall n \in \mathbb{N}$ is called Peano's successor function.
- Obviously $f(1) = 2, f(2) = 3, f(3) = 4 \dots \dots$ then, the range of this function is $\mathbb{N} - \{1\}$.

✓ Greatest integer function (Floor function):

- The function $f: \mathbb{R} \to \mathbb{R}, f(x) = [x]$ or $x], \forall x \in \mathbb{R}$ is called greatest integer function, where

$$[x] \text{ or } x] = \text{greatest intrger not exceeding } x$$

- $[1.2] = 1, [2] = 2, [-1.7] = -2 \dots \dots$Etc.
- The range of this function is $\mathbb{Z}$.

✓ Ceiling function:

- The function $f: \mathbb{R} \to \mathbb{R}, f(x) = ]x], \forall x \in \mathbb{R}$ is called ceiling function, where

$$]x] = \text{least integer not less than } x$$

- $[1.2] = 2, [2] = 2, [-1.7] = -1 \dots \dots$Etc.
- The range of this function is $\mathbb{Z}$.

❖ ALGEBRA OF REAL FUNCTIONS:

✓ Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ are two function with $A \cap B \neq \phi$, then

- The addition of functions is defined as

$$(f + g): (A \cap B) \to \mathbb{R}, (f + g)(x) = f(x) + g(x), \forall x \in A \cap B$$

- The subtraction of functions is defined as

$$(f - g): (A \cap B) \to \mathbb{R}, (f - g)(x) = f(x) - g(x), \forall x \in A \cap B$$

- The multiplication of functions is defined as

$$(f \cdot g): (A \cap B) \to \mathbb{R}, (f \cdot g)(x) = f(x) \cdot g(x), \forall x \in A \cap B$$

- The division (quotient) of functions is defined as

$$\left(\frac{f}{g}\right) : (A \cap B) - \{x : g(x) = 0\} \to \mathbb{R}, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in (A \cap B) - \{x : g(x) = 0\}$$

❖ METHOD-5: BASIC EXAMPLES ON FUNCTIONS

| | | | |
|---|---|---|---|
| H | 1 | Define: Peano's successor function, identity function, modulus function | |
| H | 2 | Define: Integer part function, Ceiling function | |
| C | 3 | Which of the following relations are function? If yes, determine domain and range.<br>(a) $\{(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)\}$<br>(b) $\{(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)\}$<br>(c) $\{(1,3), (1,5), (2,5)\}$ | |
| H | 4 | Which of the following relations are function? If yes, determine domain and range.<br>(a) $\{(1,1), (3,1), (5,1), (7,1), (9,1)\}$<br>(b) $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$<br>(c) $\{(2,3), (2,5), (3,5)\}$ | |
| H | 5 | Let $f(x) = x^2$ and $f(x) = 2x + 1$ be the real functions, then find the following:<br>$(f + g)(x), (f - g)(x), (fg)(x), (f/g)(x)$. | |
| C | 6 | Let $f(x) = \sqrt{x}$ and $f(x) = x$ be the real functions, then find the following:<br>$(f + g)(x), (f - g)(x), (fg)(x), (f/g)(x)$. | |
| C | 7 | Find the domain and range of following real functions:<br>(a) $f(x) = \lvert x - 1\rvert$, (b) $f(x) = \sqrt{9 - x^2}$ | |
| H | 8 | Find the domain and range of following real functions:<br>(a) $f(x) = \lvert x - 2\rvert$, (b) $f(x) = \sqrt{4 - x^2}$ | |
| C | 9 | Draw the graph of following real function:<br>(a) $f(x) = \lvert x - 1\rvert$ (b) $f(x) = x^2$, (c) $f(x) = \sin x$ | |
| H | 10 | Draw the graph of following real function:<br>(a) $f(x) = \log x$ (b) $f(x) = x^2 - 1$, (c) $f(x) = 3x + 1$ | |
| H | 11 | If $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2x + 1$, then find $f(1), f(2), f(-2), f(0)$.<br>Answer: 0, 1, 9, 1 | |
| C | 12 | If $f: \mathbb{R} \to \mathbb{R}, f(x) = 2\lvert x\rvert - \lvert -x\rvert$, then find $f(3), f(1/2), f(-3)$.<br>Answer: 3, 1/2, 3 | |

## ❖ INJECTIVE FUNCTION (ONE TO ONE FUNCTION)

- ✓ Definition: A function $f: A \to B$ is called one to one (injective) function if distinct elements of A are mapped into distinct elements of B. In other words, f is one to one if

$$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \text{ OR } f(x_1) = f(x_2) \Longrightarrow x_1 = x_2, \forall x_1, x_2 \in A$$

NOTE:

- ✓ If given function is not a one to one function then it is called many to one function.
- ✓ A function, shown in figure (1), is a one to one function. But a function, shown in figure (2) is not a one to one function.

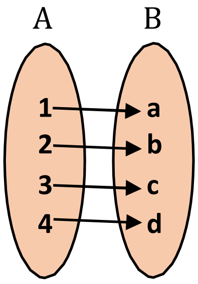

Figure (1)

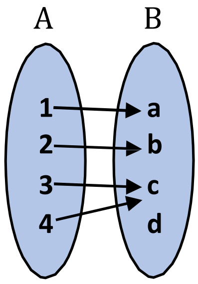

Figure (2)

Example:

- ✓ Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5, 6\}$ then
  1. $f: A \to B, f = \{(1, 2), (2, 4), (3, 6)\}$ is one to one function.
  2. $f: A \to B, f = \{(1, 2), (2, 2), (3, 6)\}$ is not one to one function. $[f(1) = 2 \ \& \ f(2) = 2]$
- ✓ $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$ is one to one function.
- ✓ $f: \mathbb{R} \to [-1, 1], f(x) = \sin x$ is not one to one function. $[f(0) = \sin 0 = 0 \ \& \ f(\pi) = \sin \pi = 0]$

## ❖ SURJECTIVE FUNCTION (ONTO FUNCTION)

- ✓ Definition: A function $f: A \to B$ is called onto (Surjective) if $R_f = B$ (Range of f = Codomain of f).
- ✓ If given function is not onto function then it is called into function.
- ✓ A function, shown in figure (1), is an onto function. But a function, shown in figure (2) is not an onto function.

Examples:

- ✓ Let $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$ then
  - (i) $f: A \to B, f = \{(1, 1), (2, 4), (3, 9)\}$ is onto function.
  - (ii) $f: A \to B, f = \{(1, 4), (2, 9), (3, 4)\}$ is not onto function. $[R_f = \{4, 9\} \neq B]$
- ✓ $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$ is onto function.
- ✓ $f: \mathbb{N} \to \mathbb{N}, f(n) = n + 2$ is not onto function. $[R_f = \{3, 4, 5, 6, \dots\} \neq \mathbb{N}(\text{codomain})]$

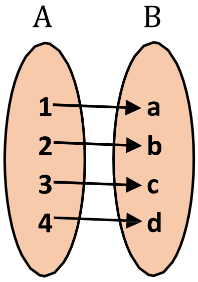

Figure (1)

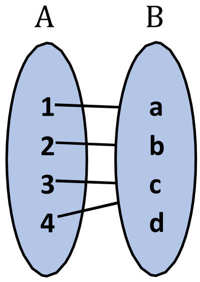

Figure (2)

## ❖ BIJECTIVE FUNCTION

- ✓ A function is said to be Bijective function if it is one to one and onto both.

## ❖ COMPOSITION OF TWO FUNCTION

- ✓ Consider functions $f: A \rightarrow B$ and $g: C \rightarrow D$. Then we may define a new $gof: A \rightarrow D$, called the composition of f & g, as follows:

$$(gof)(x) = g(f(x)), x \in A; \text{where } B \subseteq C$$

- ✓ That is, we find the image of x under f and then find the image of f(x) under g.

Note: Similarly we can define **fog**.

Example

- ✓ Let $A = \{1, 2, 3\}$, $B = \{p, q\}$, $C = \{a, b\}$. Also let $f: A \rightarrow B$ be $f = \{(1, p), (2, p), (3, q)\}$ and $g: B \rightarrow C$ be $g = \{(p, b), (q, b)\}$. Then

$$fog = \{(1, b), (2, b), (3, b)\}$$

- ✓ Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x + 2$ & $g(x) = x - 2$, then

$$(gof)(x) = g(f(x)) = g(x + 2) = (x + 2) - 2 = x$$
$$(fog)(x) = f(g(x)) = f(x - 2) = (x - 2) + 2 = x$$

## ❖ PROPERTIES OF COMPOSITION OF FUNCTIONS

- ✓ Let $f: A \rightarrow B$, then $foI_A = f = I_B of$
- ✓ Let the function $f: A \rightarrow B$ and $g: B \rightarrow C$ satisfy $gof = I_A$ & $fog = I_B$ then g is unique.
- ✓ In general, $fog \neq gof$.
- ✓ Let the function $f: A \rightarrow B$ and $g: B \rightarrow C$, then
  - If f and g are one to one function, gof is also one to one function.
  - If f and g are onto function, gof is also onto function.
- ✓ In usual notation, $(gof)^{-1}(x) = (f^{-1}og^{-1})(x)$.

## ❖ INVERSE OF FUNCTION

- ✓ Let $f: A \rightarrow B$ be a function and if there exist a function $g: B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$, we say g is the inverse function of f & g denote by $f^{-1}$.

Note: If $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ has the inverse $\mathbf{g}: \mathbf{B} \rightarrow \mathbf{A}$ if and only if $\mathbf{f}: \mathbf{A} \rightarrow \mathbf{B}$ is one to one and onto both.

Examples

- ✓ Let $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$ and $f: A \to B, f = \{(1,1), (2,4), (3,9)\}$ then

$$f^{-1}: B \to A, f^{-1} = \{(1,1), (4,2), (9,3)\}$$

- ✓ Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x + 2$.
  - For the inverse of f, let $y = x + 2 \Longrightarrow x = y - 2 \Longrightarrow f^{-1}(x) = x - 2$
  - Verification:

$$(fof^{-1})(x) = g(f(x)) = g(x+2) = (x+2) - 2 = x = I_{\mathbb{R}(\text{domain of f})}$$

$$(f^{-1}of)(x) = f(g(x)) = f(x-2) = (x-2) + 2 = x = I_{\mathbb{R}(\text{codomain of f})}$$

❖ METHOD-6: EXAMPLES ON INVERSE OF FUNCTION AND COMPOSITION OF FUNCTIONS

| | | | |
|---|---|---|---|
| H | 1 | Define: Injective function, Surjective function, inverse function, Bijective function. | |
| C | 2 | Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.<br>Answer: $\mathbf{gof} = \{(\mathbf{2}, \mathbf{7}), (\mathbf{3}, \mathbf{7}), (\mathbf{4}, \mathbf{11}), (\mathbf{5}, \mathbf{11})\}$ | |
| T | 3 | Find gof and fog, if $f : \mathbb{R} \to \mathbb{R}$ $g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$. | |
| H | 4 | Let $X = \{1, 2, 3\}$ and f, g, h, s be functions from X to X given by<br>$f = \{(1,2), (2,3), (3,1)\}, g = \{(1,2), (2,1), (3,3)\}, h = \{(1,1), (2,2), (3,1)\}$ and<br>$s = \{(1,1), (2,2), (3,3)\}$. Then find fog, gof, $(gof)^{-1}$, fohog, sog, gos, sos, fos. | |
| C | 5 | Let $f(x) = x + 2, g(x) = x - 2, h(x) = 3x$ for $x \in \mathbb{R}$.<br>Find gof, fog, $(fog)^{-1}$, fof, gog, foh, hog, hof, fohog. | |
| H | 6 | Let $f: \mathbb{R} \to \mathbb{R}, f(x) = -x^2$ and $g: \mathbb{R}^+ \to \mathbb{R}^+, g(x) = \sqrt{x}$. Find fog and gof, if possible.<br>Answer: $\mathbf{fog(x) = -x}$ & $\mathbf{gof(x)}$ **is not define** | |
| H | 7 | Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = x + 4$. Find fog and gof.<br>Answer: $\mathbf{fog(x) = x^2 + 8x + 14}$ & $\mathbf{gof(x) = x^2 + 2}$ | |
| C | 8 | Are the following functions one to one and onto? If yes, find its inverse?<br>(a) $f: \mathbb{R} \to \mathbb{R}, f(x) = ax + b$<br>(b) $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 2$<br>(c) $f: \mathbb{R} - \{-\frac{3}{2}\} \to \mathbb{R} - \{\frac{3}{2}\}, f(x) = \frac{3x+2}{2x+3}$ | |

| | | | |
|---|---|---|---|
| H | 9 | Are the following functions one to one and onto? If yes, find its inverse?<br>a) $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$<br>(b) $f: \mathbb{R}^+ \to \mathbb{R}^+, f(x) = x^2$<br>(c) $f: \mathbb{R} - \{-1\} \to \mathbb{R} - \{-1\}, f(x) = \frac{1-x}{1+x}$ | |

## PART-III COUNTING

### ❖ INTRODUCTION

- ✓ Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9.The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition.
- ✓ Somehow, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time.
- ✓ But, this method will be tedious, because the number of possible sequences may be large. Here, in this section, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements.
- ✓ In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of counting techniques.

### ❖ FUNDAMENTAL PRINCIPLES OF COUNTING

1. Fundamental principle of multiplication: Let there are two parts A and B of a certain event. The part A can be done in m different ways. If corresponding to each way of doing the part A of the event, the part B can be done in n ways, then there are $m \times n$ ways to complete the event.

Example: How many two digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

- ✓ Solution: Here, the unit's place can be filled by 2 and 4(two different ways) and the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by multiplication principle, the required number of two digits even numbers is $\mathbf{2 \times 5 = 10}$.

2. Fundamental principle of addition: suppose that A and B two disjoint events (i.e. they never occur together). Further suppose that A occur in m ways and B in n ways. Then A and B can occur in $m + n$ ways.

❖ PIGEONHOLE PRINCIPLE

- ✓ Statement: If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.
- ✓ Proof: we use a proof by contraposition.
- ✓ Suppose none of the m containers has more than one item. Then the total number of items would be at most m. it is contradicts with our hypothesis, there are n items and $n > m$ (i.e. we have at least $n = m + 1$ items), hence we proved.
- ✓ This principle is also called Dirichlet's box principle or Dirichlet's drawer principle.
- ✓ A function f from a set with n+1 elements to a set with n elements is not one to one.
- ✓ In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

❖ GENERALIZED PIGEONHOLE PRINCIPLE

- ✓ If the N objects are placed into K boxes, then there is at least one box containing at least $]\frac{N}{K}]$ objects.
- ✓ Question: Given a group of 100 people, at minimum, how many people were born in the same month?
- ✓ Answer: $]\frac{N}{K}] = ]\frac{100}{12}] = ]8.33] = 9$

Note: The principle just proves the existence of overlaps; it says nothing of the number of overlaps (which falls under the subject of probability distribution).

❖ FACTORIAL

- ✓ The continued product of first n natural numbers is called 'n factorial' and is denoted by n!.

$$\text{i.e. } n! = 1 \times 2 \times 3 \times \dots \dots \times (n-1) \times n$$

- ✓ For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

Remark:

1. $n! = n \times (n-1)!$ (Verify!)
2. $0! = 1$ (Why?)

❖ PERMUTATION

- ✓ Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.
- ✓ e.g. six arrangements can be made with three distinct objects a, b, c taking two at a time are

$$ab, ba, bc, cb, ac, ca$$

- ✓ Each of these arrangements is called a permutation.

Note: It should be noted that in permutations the order of arrangement is taken into account, when the order is changed, a different permutation is obtained.

❖ A NOTATION

- ✓ The number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$.

❖ DERIVATION OF THE FORMULA FOR $\mathbf{P(n, r)}$

- ✓ Suppose that we are given 'n' distinct objects and wish to arrange 'r' of these objects in a line where $1 \le r \le n$.
- ✓ Since there are 'n' ways of choosing the 1st object, after this is done 'n-1' ways of choosing the 2nd object and finally n-r+1 ways of choosing the $r^{th}$ object.
- ✓ It follows by the fundamental principle of counting that the number of different arrangement (or PERMUTATIONS) is given by
  - $\mathbf{P(n, r) = n\,(n-1)\,(n-2)\ldots\ldots\ldots(n-r+1)}$

$$= \frac{\mathbf{n\,(n-1)\,(n-2)\ldots\ldots\ldots(n-r+1)(n-r)(n-r-1)\ldots\ldots\ldots 3, 2, 1}}{\mathbf{(n-r)(n-r-1)\ldots\ldots\ldots 3, 2, 1}}$$

$$= \frac{\mathbf{n!}}{\mathbf{(n-r)!}}$$

❖ RESULTS:

- ✓ $P(n, n) = n!$
- ✓ $P(n, 0) = 1$
- ✓ Suppose that a set consists of 'n' objects of which $n_1$ are of one type, $n_2$ are of second type, …, $n_k$ are of $k^{th}$ type. Here $n = n_1 + n_2 + \cdots + n_k$. Then the number of different permutations of the objects is

$$\frac{n!}{n_1!\ n_2!\ \ldots\ n_k!}$$

Example: A number of different permutations of the letters of the word JISSISSITTI is

$$\frac{11!}{1!\ \ 4!\ \ 4!\ \ 2!} = 34650$$

- ✓ If 'r' objects are to be arranged out of 'n' objects and if repetition of an object is allowed then the total number of permutations is $n^r$.

Example: Different numbers of three digits can be formed from the digits 1, 2, 3, 4, 5 is

$$5^3 = 125$$

- ✓ The number of ways to arrange $n$ distinct objects along a fixed (i.e., cannot be picked up out of the plane and turned over) circle is $(n-1)!$

❖ GENERATING PERMUTATION

- ✓ Any set with n elements can be placed in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements.

- ✓ Many different algorithms have been developed to generate the n! Permutations of this set. We will describe one of these that is based on the lexicographic (or dictionary) ordering of the set of permutations of $\{1, 2, 3, \dots, n\}$.

❖ PROCEDURE TO OBTAIN NEXT PERMUTATION (LEXICOGRAPHIC ORDERING)

- ✓ An algorithm for generating the permutations of $\{1, 2, \dots, n\}$ can be based on a procedure that constructs the next permutation in lexicographic order following a given permutation $a_1a_2 \cdots a_n$. We will show how this can be done. First, suppose that $a_{n-1} < a_n$. Interchange $a_{n-1}$ and $a_n$ to obtain a larger permutation. No other permutation is both larger than the original per•mutation and smaller than the permutation obtained by interchanging $a_{n-1}$ & $a_n$. For instance, the next larger permutation after 234156 is 234165.
- ✓ On the other hand, if $a_{n-1} > a_n$, then a larger permutation cannot be obtained by interchanging these last two terms in the permutation. Look at the last three integers in the permutation. If $a_{n-2} < a_{n-1}$, then the last three integers in the permutation can be rearranged to obtain the next largest permutation. Put the smaller of the two integers $a_{n-1}$ & $a_n$ that is greater than $a_{n-2}$ in position $n - 2$. Then, place the remaining integer and $a_{n-2}$ into the last two positions in increasing order. For instance, the next larger permutation after 234165 is 234516. And so on.

Example: What is the next permutation in Lexicographic order after 362541?

- ✓ Answer: 364125.

❖ METHOD-7: EXAMPLES ON PERMUTATION

| | | | |
|---|---|---|---|
| H | 1 | Explain the pigeonhole principle. | |
| C | 2 | Evaluate: ${}^5P_2$, ${}^7P_3$, ${}^9P_4$<br>Answer: 20, 210, 3024 | |
| H | 3 | Find n, if (a) ${}^nP_5 = 42\ {}^nP_3$, (b) $3P(n, 4) = 5P(n - 1, 4)$.<br>**Answer: 10, 10** | |
| C | 4 | Find n if (a) ${}^nP_5 = 42\ {}^nP_3$ $(n > 4)$, (b) ${}^nP_4 = 42\ {}^nP_2$, (c) $2P(n, 2)+50=P(2n, 2)$.<br>Answer: 9, 9, 5 | |
| T | 5 | Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.<br>Answer: 3, 8 | |

| | | | |
|---|---|---|---|
| C | 6 | Suppose repetition are not permitted.<br>(a) How many three digit numbers can be formed from the six digits 2, 3, 5, 6, 7, 9?<br>(b) How many of these numbers are less than 400?<br>(c) How many are even?<br>**Answer: 120, 40, 40** | |
| C | 7 | find the number of ways that a party of seven persons can arrange themselves:<br>(a) in a row of seven chairs<br>(b) around a circular table<br>Answer: 7!, 6! | |
| H | 8 | Find the number of distinct permutation that can be formed from all the letters of:<br>(a) RADAR (B) UNUSUAL<br>**Answer: 30, 840** | |
| C | 9 | Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements:<br>(a) Do the words start with P,<br>(b) Do all the vowels always occur together,<br>(c) Do the vowels never occur together,<br>(d) Do the words begin with I and end in P?<br>Answer: 138600, 16800, 1646400, 12600 | |
| H | 10 | In a certain programming language, the variable should be length three and should be made up of two letters followed by a digit or of length two made up of a letter followed by a digit. How many variables can be formed? What if letters are not to be repeated?<br>**Answer: 6760, 7020** | |
| H | 11 | 6 boys and 6 girls are to be seated in a raw, how many ways can they be seated if<br>(a) All boys are to be seated together and all girls are to be seated together.<br>(b) No two girls should be seated together.<br>(c) Boys occupy extreme position.<br>Answer: 1036800, 6220800, p(6, 2) 10! | |
| C | 12 | How many ways can the letters in the word MISSISSIPPI can be arranged? What if P's are to be separated?<br>**Answer**: $\frac{11!}{4!\ 4!\ 2!}, \frac{11!}{4!\ 4!\ 2!} - \frac{10!}{4!\ 4!}$ | |

| | | | |
|---|---|---|---|
| H | 13 | If repetitions are not allowed, how many four digit numbers can be formed from 1, 2, 3, 7, 8, 5? How many of these numbers are less than 5000? How many of these are even? How many of these are odd? How many of these containing 3 and 5?<br>Answer: 360, 180, 120, 240, 144 | |
| C | 14 | A word that reads the same when read in forward or backward is called as palindrome. How many seven letter palindromes can be form English alphabets?<br>Answer: $\mathbf{26^4}$ | |
| T | 15 | Generate the permutations of the integers 1, 2, 3 in lexicographic order.<br>Answer: 123, 132, 213, 231, 312, 321 | |

❖ COMBINATION

- ✓ In a permutation we are interested in the order of arrangement of the objects. For example, ABC is a different permutation from BCA. In many problems, however, we are interested only in selecting or choosing objects without regard to order. Such selections are called combination.
- ✓ The total number of combination (selections) of 'r' objects selected from 'n' objects is denoted and defined by

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

❖ RESULTS:

- ✓ $\binom{n}{0} = \binom{n}{n} = 1$
- ✓ $\binom{n}{1} = n$
- ✓ $\binom{n}{r} = \frac{1}{r!}\ {}^{n}P_{r}$
- ✓ $\binom{n}{r} = \frac{n}{r} \times \binom{n-1}{r-1}$
- ✓ The number of r-combinations with repetition allowed (multi sets of size r) that can be selected from a set of n elements is $\binom{n+r-1}{r}$.

Examples:

1. The number of ways in which 3 card can be chosen from 8 cards is

$$\binom{8}{3} = \frac{8!}{3!\,(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

2. A club has 10 male and 8 female members. A committee composed of 3 men and 4 women is formed. In how many ways this be done?

$$\binom{10}{3} \times \binom{8}{4} = 120 \times 70 = 8400$$

3. Out of 6 boys and 4 girls in how many ways a committee of five members can be formed in which there are at most 2 girls are excluded?

$$\binom{4}{2}\times\binom{6}{3}+\binom{4}{1}\times\binom{6}{4}+\binom{4}{0}\times\binom{6}{5}=\mathbf{120+60+6=186}$$

## ❖ GENERATING COMBINATION

- ✓ An algorithm for generating the r-combinations of the set $\{1, 2, 3, \dots, n\}$ will be given. An r-combination can be represented by a sequence containing the elements in the subset in increasing order. The r-combinations can be listed using lexicographic order on these sequences. The next combinations after $a_1a_2 \dots a_r$ can be obtained in the following way:
- ✓ First, locate the last element $a_i$ in a sequence such that $a_i \neq n - r + i$. Then, replace $a_i$ with $a_i + 1$ and $a_j$ with $a_i + j - i + 1$, for $j = i + 1, i + 2, \dots$, r. It is left for the reader to show that this produces the next larger combination in lexicographic order.

Example: Find the next larger 4-combination of the set $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}$ after $\{\mathbf{1},\mathbf{2},\mathbf{5},\mathbf{6}\}$.

- ✓ Answer: {1, 3, 4, 5}.

## ❖ BINOMIAL COEFFICIENT

- ✓ $(a+b)^1 = a + b = \binom{1}{0} a^1b^0 + \binom{1}{1} a^0b^1$
- ✓ $(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0} a^2b^0 + \binom{2}{1} a^1b^1 + \binom{2}{2} a^0b^2$
- ✓ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0} a^3b^0 + \binom{3}{1} a^2b^1 + \binom{3}{2} a^1b^2 + \binom{3}{3} a^0b^3$

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- ✓ $(a+b)^n = \binom{n}{0} a^nb^0 + \binom{n}{1} a^{n-1}b^1 + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n}a^0b^n = \sum_{k=0}^{n}\binom{n}{k} a^nb^k$
- ✓ Above expression is called binomial theorem where n is positive integer and expression ${}^{n}C_k$ is often called binomial coefficient.

## ❖ PROPERTIES OF BINOMIAL COEFFICIENT

- ✓ $\binom{n}{k} = \binom{n}{n-k}$
- ✓ $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$
- ✓ $If\ \binom{n}{x} = \binom{n}{y}\ then\ either\ x = y\ or\ x + y = n$
- ✓ $\sum_{k=0}^{n}\binom{n}{k} = 2^n\ and\ \sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

## ❖ METHOD-8: EXAMPLES ON COMBINATION

| | | | |
|---|---|---|---|
| C | 1 | Show that ${}^{14}C_4$, ${}^{14}C_5$, ${}^{14}C_6$ are in AP. | |
| C | 2 | Prove that (a) ${}^{n}C_r = {}^{n}C_{n-r}$ (b) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ | |

| | | | |
|---|---|---|---|
| H | 3 | Find the value of n if $C(n,5) = C(n,13)$ and hence find $C(n,2)$.<br>Answer: 18 | |
| H | 4 | Find the value of n if $\binom{2n}{3} = 11\binom{n}{3}$ and hence find $\binom{n}{2}$.<br>**Answer: 6, 15** | |
| T | 5 | Find the value of r if (a) $\binom{8}{r} = 28$ (b) $\binom{12}{r} = \binom{12}{r+2}$.<br>Answer: 2 or 6, 5 | |
| H | 6 | If $C(n-1,4), C(n-1,5), C(n-1,6)$ are in A.P., find n.<br>Answer: 15, 8 | |
| H | 7 | Show that $C(2n,2) = 2C(n,2) + n^2$. | |
| C | 8 | In how many ways can a committee consisting of three men and two women be chosen from seven men and five women?<br>Answer: 350 | |
| C | 9 | A bag contains 6 white, 5 red marbles. Find the number of ways four marbles can be drawn from the bag if (a) they can be any color; (b) two must be white and two red; (c) they must all be of the same color.<br>Answer: 330, 150, 20 | |
| C | 10 | Out of 12 employee, group of four trainees is to be sent for “Software testing and QA” training of one month. (a) In how many ways can the four employees be selected? (b) What if there are two employees who refuse to go together? (c) What if there are two employees who want to go together? (d) What if there are two employees who want to go together and there are two employees who refuse to go together?<br>Answer: 495, 450, 255, 226 | |
| H | 11 | There are 2 white, 3 red, 4 green marbles. Three marbles are drawn. In how many ways can this be done so that at least one red marble is selected?<br>Answer: 64 | |

| | | | |
|---|---|---|---|
| H | 12 | 3 cards are chosen from a pack of 52 cards. In how many<br>(1) ways can this be done?<br>(2) ways can you select three cards so that all of them are face cards?<br>(3) of the selections, all cards are of the same color?<br>(4) of them all cards are of the same suit?<br>Answer: 22100, 220, 5200, 1144 | |
| C | 13 | A reception committee consisting of 6 students for the annual function of a college is to be formed from 8 boys and 5 girls. In how many ways can we do it if the committee is to contain (1) exactly four girls, (2) at most two girls, (3) at least three girls?<br>Answer: 140, 1008, 708 | |

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