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## SYLLABUS

## GTU PAPERS



## UNIT-1 » VECTOR CALCULUS

## ❖ SCALAR POTENTIAL FUNCTION OF VECTOR FUNCTION:

✓ The gradient of a scalar function  $f(x, y, z)$  is denoted by  $\text{grad } f$  or  $\nabla f$  and is defined as:

$$\text{grad } f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

✓ For a vector function  $\vec{F} = \nabla f$  or  $\vec{F} = \text{grad}(f)$  then  $f$  is called scalar potential of  $\vec{F}$

## ❖ PROCEDURE TO FIND SCALAR POTENTIAL FUNCTION OF A VECTOR FUNCTION:

✓ If  $\vec{F} = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$  or  $f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k}$  is given.

(1). Find  $A = \int f_1(x, y, z) dx$ ,  $B = \int f_2(x, y, z) dy$ ,  $C = \int f_3(x, y, z) dz$ .

(2). Write  $f$  as  $f = A + B + C$  without repeating the terms.

## METHOD – 1: EXAMPLES OF SCALAR POTENTIAL FUNCTION

H	1	Find the scalar potential function $f$ for $\vec{B} = (2x, 4y, 8z)$ . <b>Answer:</b> $x^2 + 2y^2 + 4z^2 + c$ .	
H	2	Find the scalar potential function $f$ for $\vec{A} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$ <b>Answer:</b> $xy^2 - \frac{z^3}{3} + c$ .	
T	3	Find the scalar potential function for $\vec{C} = \left( \frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2} \right)$ . <b>Answer:</b> $\frac{xy}{z} + c$ .	
C	4	A vector field is given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ . Find the scalar potential. <b>Answer:</b> $\frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c$ .	

## ❖ DIVERGENCE OF A VECTOR FUNCTION:

- ✓ The divergence of the vector function  $\vec{F} = (f_1, f_2, f_3) = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$  is denoted by  $\text{div } \vec{F}$  and is defined as:

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

- ✓ The divergence of a vector function is a scalar.
- ✓ A vector function  $\vec{F}$  is said to be solenoidal or incompressible or solenoidal field if  $\text{div } \vec{F} = 0$ .
- ✓ Result:

$$\text{div grad } f = \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

## METHOD – 2: EXAMPLES OF DIVERGENCE OF A VECTOR FUNCTION

H	1	Find divergence of the vector function $\vec{V} = 3x^2 \hat{i} + 5xy^2 \hat{j} + xyz^3 \hat{k}$ at the point (1, 2, 3). <b>Answer: 80.</b>	
C	2	If $\vec{A} = x^2z \hat{i} - 2y^3z^2 \hat{j} + xy^2z \hat{k}$ , find $\nabla \cdot \vec{A}$ at the point (1, -1, 1). <b>Answer: -3.</b>	
H	3	If $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$ , find $\text{div } \vec{F}$ <b>Answer: 6(x + y + z).</b>	
C	4	If $\phi = xyz - 2y^2z + x^2z^2$ , find $\text{div}(\text{grad } \phi)$ at the point (2, 4, 1). <b>Answer: 6.</b>	
C	5	Let $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$ , $r =  \vec{r} $ , and $\vec{a}$ is a constant vector. Find the value of $\text{div} \left( \frac{\vec{a} \times \vec{r}}{r^n} \right)$ . <b>Answer: 0.</b>	W 2019 (7)
C	6	Determine whether the vector field $\vec{U} = y^2 \hat{i} + 2xy \hat{j} - z^2 \hat{k}$ is solenoidal at a point (1, 2, 1). <b>Answer: Solenoidal.</b>	

T	7	Prove that $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$ is solenoidal at any point.	
T	8	Find the value of 'm' so that the vector $\vec{V} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + mz)\hat{k}$ is solenoidal. <b>Answer: m = -2.</b>	

### ❖ CURL OF VECTOR FUNCTION:

- ✓ The curl of the vector function  $\vec{F} = (f_1, f_2, f_3) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$  is denoted by  $\text{curl } \vec{F}$  and is defined as:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

- ✓ The curl of a vector function is a vector.
- ✓  $\text{Curl}(\text{grad } f) = \vec{0}$  i.e.  $\nabla \times (\nabla f) = \vec{0}$
- ✓  $\text{Div}(\text{curl } \vec{F}) = 0$  i.e.  $\nabla \cdot (\nabla \times \vec{F}) = 0$ .

### ❖ CONSERVATIVE FIELDS:

- ✓ Let  $F$  be a vector field defined on an open region  $D$  in space and suppose that for any two points  $A$  and  $B$  in  $D$  the line integral  $\int_C \vec{F} \cdot d\vec{r}$  is path independent in  $D$  then the field  $F$  is called conservative field on  $D$ .
- ✓ A vector function  $\vec{F}$  is said to be irrotational / irrotational field / conservative field if  $\text{curl } \vec{F} = \vec{0}$

### ❖ COMPONENT TEST FOR CONSERVATIVE FIELDS:

- ✓ Let  $\vec{F} = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$  be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then  $\vec{F}$  is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

- ✓ For a conservative function  $\vec{F}$  there exists a scalar function  $f$  such that  $\vec{F} = \nabla f$ , where  $\vec{F}$  is called conservative vector field.

### METHOD – 3: EXAMPLES OF CURL OF VECTOR FUNCTION

H	1	Obtain $\text{curl } \vec{F}$ at the point $(2, 0, 3)$ if $\vec{F} = ze^{2xy} \hat{i} + 2xy \cos y \hat{j} + (x + 2y) \hat{k}$ <b>Answer:</b> $(2, 0, -12)$ .	
C	2	Find $\text{curl } \vec{F}$ if $\vec{F} = (y^2 \cos x + z^3) \hat{i} + (2y \sin x - 4) \hat{j} + 3xz^2 \hat{k}$ whether $\vec{F}$ is irrotational? <b>Answer:</b> Yes.	
T	3	If $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$ , find $\text{curl } \vec{F}$ <b>Answer:</b> $\vec{0}$	
H	4	Calculate the curl of the vector $xyz \hat{i} + 3x^2y \hat{j} + (z^2x - y^2z) \hat{k}$ <b>Answer:</b> $(-2yz) \hat{i} + (xy - z^2) \hat{j} + (6xy - xz) \hat{k}$	W 2019 (3)
C	5	Find the value of 'a' if the vector $(ax^2y + yz) \hat{i} + (xy^2 - xz^2) \hat{j} + (2xyz - 2x^2y^2) \hat{k}$ has zero divergence. Find the curl of this vector when it has zero divergence. <b>Answer:</b> $a = -2, (4xz - 4x^2y, 4xy^2 - 2yz + y, 2x^2 + y^2 - z^2 - z)$ .	
H	6	Show that the vector field $\vec{F} = (y \sin z - \sin x) \hat{i} + (x \sin z + 2yz) \hat{j} + (xy \cos z + y^2) \hat{k}$ is conservative and find the corresponding scalar potential. <b>Answer:</b> $f = xy \sin z + \cos x + y^2z + c$ .	
T	7	What do you mean by an irrotational vector field? Show that $\vec{F} = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$ is conservative and find a potential function for it. <b>Answer:</b> $e^x \cos y + xyz + \frac{z^2}{2} + C$ .	
C	8	Check whether the vector field $\vec{F} = (e^{y+2z}) \hat{i} + (x e^{y+2z}) \hat{j} + (2x e^{y+2z}) \hat{k}$ is conservative or not. If yes, find the scalar potential function $\phi(x, y, z)$ such that $\vec{F} = \text{grad } \phi$ . <b>Answer:</b> Yes, $x e^{y+2z} + c$ .	S 2019 (7)



H	9	A vector field is given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ . Show that $\vec{F}$ is irrotational and find its scalar potential.  <b>Answer:</b> $\frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c$ .	
C	10	Find constants a, b, c so that $\vec{V} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational.  <b>Answer:</b> $a = 4, b = 2, c = -1$ .	W 2019 (3)
H	11	Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.	
H	12	Show that $\vec{A} = (3x^2y)\hat{i} + (x^3 - 2yz^2)\hat{j} + (z^2 - 2y^2z)\hat{k}$ is irrotational but not solenoidal.	
C	13	Show that $r^n \vec{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$ , where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r =  \vec{r} $ .	
T	14	Prove in usual notation $\nabla \times (\nabla \phi) = \vec{0}$ and $\nabla \cdot (\nabla \times \vec{A}) = 0$ .	

### ❖ EXACT DIFFERENTIAL FORM:

- ✓ Any expression  $M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$  is called a differential form.
- ✓ A differential form is exact on a domain D in a space & for some scalar function f if

$$Mdx + Ndy + PdZ = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df.$$

- ✓ If  $Mdx + Ndy + PdZ = df$  on D, then  $F = M\hat{i} + N\hat{j} + P\hat{k}$  is the gradient field of f on D.
- ✓ Conversely, if  $F = \nabla f$ , then the form  $Mdx + Ndy + PdZ$  is exact.

### ❖ COMPONENT TEST FOR EXACTNESS OF $Mdx + Ndy + PdZ$ :

- ✓ The differential form  $Mdx + Ndy + PdZ$  is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- ✓ This is equivalent to saying that the field  $F = M\hat{i} + N\hat{j} + P\hat{k}$  is conservative.

## METHOD – 4: EXAMPLES OF EXACT DIFFERENTIAL FORM

C	1	Show that $ydx + xdy + 4dz$ is exact.	
H	2	Show that $2xdx + 2ydy + 2zdz$ is exact.	
T	3	Show that $yzdx + xzdy + xydz$ is exact.	
T	4	Show that $2xydx + (x^2 - z^2)dy - 2yzdz$ is exact.	
C	5	Show that $yzdx - 2xzdy + 3xydz$ is not exact.	
H	6	Show that $y^2zdx - 2x^2dy + 3xdz$ is not exact.	

## ❖ PARAMETRIZATION OF CURVES:

- ✓ Parametrization of the curve C in the space can be represented by a vector function  $\vec{r}(t) = (x(t), y(t))$ , where x & y are Cartesian coordinates and  $t \in \mathbb{R}$  is called parameter.

Curve	Equation	Parametrization
Line passing through $(x_1, y_1)$ & $(x_2, y_2)$	$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$	$x(t) = tx_2 + (1 - t)x_1$ $y(t) = ty_2 + (1 - t)y_1, t \in \mathbb{R}$
Parabola	$y^2 = 4ax$	$x = at^2$ & $y = 2at, t \in \mathbb{R}$
	$x^2 = 4by$	$x = 2bt$ & $y = bt^2, t \in \mathbb{R}$
Circle with center $(h, k)$ and radius r	$(x - h)^2 + (y - k)^2 = r^2$	$x = h + r \cos \theta$ & $y = k + r \sin \theta$
An Ellipse having center $(h, k)$	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	$x = h + a \cos \theta$ & $y = k + b \sin \theta$
Hyperbola bhaving center $(h, k)$	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	$x = h + a \sec \theta$ & $y = k + b \tan \theta$

	$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$	$x = h + b \tan \theta \text{ \& } y = k + \sec \theta$
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### METHOD – 5: EXAMPLES OF PARAMETRIZATION OF CURVES

T	<b>1</b>	Find the parametric representations of the line $9x - 5y = 7$ . <b>Answer:</b> $\bar{r}(t) = (t, \frac{9t - 7}{5}), t \in \mathbb{R}$ .	
C	<b>2</b>	Find the parametric representations of the parabola $y = (x - 2)^2$ . <b>Answer:</b> $\bar{r}(t) = (t + 2, t^2), t \in \mathbb{R}$ .	
H	<b>3</b>	Find the parametric representations of the circle $x^2 + y^2 = 4$ . <b>Answer:</b> $\bar{r}(t) = (2 \cos t, 2 \sin t), t \in \mathbb{R}$ .	
C	<b>4</b>	Find the parametric representations of the line passing through $(-2, 3)$ & $(4, 7)$ . <b>Answer:</b> $\bar{r}(t) = (x(t), y(t)) = (-2 + 6t, 3 + 4t), t \in \mathbb{R}$ .	
H	<b>5</b>	Find the parametric representations of the parabola $x - 2 = y^2$ . <b>Answer:</b> $\bar{r}(t) = (t^2 + 2, t), t \in \mathbb{R}$ .	

### ❖ ARC LENGTH OF CURVE IN SPACE:

- ✓ The position vector of the curve C in space at any point t is denoted by  $\bar{r}(t) = (x(t), y(t), z(t))$ . The arc length of curve C from a to b is defined as:

$$L = \int_a^b |\bar{r}'(t)| dt = \int_a^b \sqrt{\bar{r}'(t) \cdot \bar{r}'(t)} dt.$$

- ✓ Consider the parameterization of the curve C as  $\bar{r}(t) = (x(t), y(t), z(t))$  then the arc length of C from a to b can be expressed in the component form as:

$$L = s(t) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2 + \{z'(t)\}^2} dt.$$

✓ Consider the curve  $y = f(x)$ , the arc length from  $x = a$  to  $x = b$  can be expressed as

$$L = s(t) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

✓ Consider the curve  $r = f(\theta)$  then the arc length of curve  $r$  from  $\theta = \theta_1$  to  $\theta = \theta_2$  can be expressed as:

$$L = s(t) = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

#### METHOD - 6: EXAMPLES OF ARC LENGTH

C	<b>1</b>	Find the length of the curve $y = \log (\sec x)$ from $x = 0$ to $x = \pi/3$ . <b>Answer:</b> $\log (2 + \sqrt{3})$ .	
H	<b>2</b>	Find the arc length of the curve $\vec{r}(t) = (2t, 3 \sin 2t, 3 \cos 2t)$ from $t = 0$ to $t = 2\pi$ . <b>Answer:</b> $4\sqrt{10}\pi$ .	
T	<b>3</b>	Find length of $y = \int_0^x \sqrt{\cos 2t} dt$ from $x = 0$ to $x = \frac{\pi}{4}$ . <b>Answer:</b> 1.	
T	<b>4</b>	Find the arc length of the curve $\vec{r}(t) = \frac{2\sqrt{2}}{3} t^{\frac{3}{2}} \hat{i} + \frac{t^2}{2} \hat{j} + (t + 3) \hat{k}$ from $t = 0$ to $t = 2$ . <b>Answer:</b> 4.	
C	<b>5</b>	Find the length of curve of the portion of circular helix $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$ from $t = 0$ to $t = \pi$ . <b>Answer:</b> $\sqrt{2}\pi$ .	W 2019 (3)

H	6	Find the circumference $L$ of the circle of radius " $a$ " centered at $O$ given by $x = a \cos \theta, y = a \sin \theta, a > 0$ . <b>Answer:</b> $2\pi a$ .	
C	7	Find the length of the asteroid $x = a(\cos \theta)^3, y = a(\sin \theta)^3$ in the first quadrant. <b>Answer:</b> $\frac{3a}{2}$ .	
T	8	Find the perimeter of the cardioid $r = a(1 + \cos \theta); a > 0$ . <b>Answer:</b> $8a$ .	
H	9	Find circumference of the circle $r = a \cos \theta$ ; where $a > 0$ . <b>Answer:</b> $\pi a$ .	
C	10	Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$ . <b>Answer:</b> $\frac{\sqrt{5}}{2} (e^{4\pi} - 1)$ .	

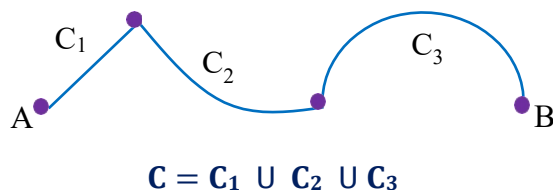
## ❖ SOME PRELIMINARY CONCEPTS:

- ✓ Suppose  $C$  is a curve parametrized by  $x = f(t), y = g(t), a \leq t \leq b$ ,  $A$  and  $B$  are the points  $(f(a), g(a))$  and  $(f(b), g(b))$  respectively.

- (1). **Smooth curve:**  $C$  is a smooth curve, if  $f'$  and  $g'$  are continuous on the closed interval  $[a, b]$  and not simultaneously zero on the open interval  $(a, b)$ .



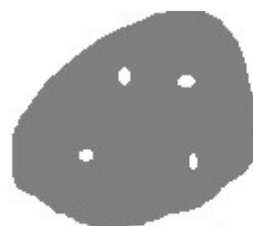
- (2). **Piecewise smooth curve:**  $C$  is piecewise smooth, if it consists of a finite number of smooth curves  $C_1, C_2, \dots, C_n$  joined end to end; i.e.  $C = C_1 \cup C_2 \cup \dots \cup C_n$ .



**(3). Connected domain:** A domain  $D$  is connected, if we can connect any two points in the region with a path that lies completely in  $D$ .



Connected domain

connected domain but  
not simply connected domain

**(4). Simply connected domain:** A domain  $D$  is simply connected, if it is connected and it contains no holes.

#### ❖ PARAMETRIC EQUATION IN 3D:

- ✓ Parametric equation of line passing through the points  $(a, b, c)$  and  $(x, y, z)$  is given by

$$\vec{r} = \langle a + t(x - a), \quad b + t(y - b), \quad c + t(z - c) \rangle$$

#### ❖ LINE INTEGRALS:

- ✓ Any integral which is to be evaluated along a curve is called a line integral.
- ✓ Let  $\vec{F}(\vec{r}) = (f_1, f_2, f_3)$  be a vector function defined at every point of a curve  $C$ . If  $\vec{r} = (x, y, z)$  is the position vector of a point  $P(x, y, z)$  on the curve  $C$  and then the line integral of  $\vec{F}(\vec{r})$  over a  $C$  is defined by

$$\int_C \{ \vec{F}(\vec{r}) \} \cdot d\vec{r} = \int_C \{ f_1 dx + f_2 dy + f_3 dz \}.$$

- ✓ If the curve C is represented by a parametric representation  $\vec{r}(t) = (x(t), y(t), z(t))$  then line integral along the curve C from  $t = a$  to  $t = b$  is

$$\int_C \{\vec{F}(\vec{r})\} d\vec{r} = \int_a^b \left\{ \vec{F}(\vec{r}(t)) \right\} \frac{d\vec{r}}{dt} dt = \int_a^b \left\{ f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right\} dt.$$

- ✓ If C is a closed curve then the line integral is denoted by  $\oint_C \{\vec{F}(\vec{r})\} d\vec{r}$ .

### ❖ VECTOR FIELDS AND APPLICATIONS AS WORK:

- ✓ If  $\vec{F}$  is force acting on a particle moving along the arc AB of curve C, then the line integral  $\int_A^B \{\vec{F} d\vec{r}\}$  represents the work done in displacing the particle from the point A to B.

### ❖ CIRCULATION AND FLUX:

- ✓ **Flow and Circulation:** Consider smooth curve  $\vec{r}(t) = (x(t), y(t), z(t))$  in the domain of continuous velocity field  $\vec{F}(\vec{r}) = (F_1, F_2, F_3)$ . The flow along the curve from  $t = a$  to  $t = b$  is defined as:

$$\int_C \{\vec{F}(\vec{r})\} d\vec{r} = \int_a^b \left\{ \vec{F}(\vec{r}(t)) \right\} \frac{d\vec{r}}{dt} dt = \int_a^b \left\{ f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right\} dt.$$

- ✓ The flow integral along the closed curve is called **circulation** around the curve.
- ✓ **Flux:** Let C is the smooth closed curve in the domain of a continuous vector field  $\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$  in the plane. Consider  $\hat{n}$  as the outward pointing unit normal vector on C. Then the line integral over C of  $\vec{F} \cdot \hat{n}$  gives the rate at which a fluid is entering or leaving a region enclosed by C. Thus the flux of  $\vec{F}$  across C is

$$\int_C \vec{F} \cdot \hat{n} ds, \text{ Where } \hat{n} = \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j}, \text{ then } \vec{F} \cdot \hat{n} = M(x, y) \frac{dy}{ds} - N(x, y) \frac{dx}{ds} \text{ i. e.}$$

$$\int_C \vec{F} \cdot \hat{n} ds = \oint_C (M dy - N dx).$$

## ❖ LINE INTEGRALS INDEPENDENT OF PATH:

- ✓ The line integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of path, if  $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$  for any two paths  $C_1$  and  $C_2$  in some domain  $D$  with the same initial and terminal points.
- ✓ The necessary and sufficient condition that  $\int_A^B \vec{F} \cdot d\vec{r}$  be independent of path is the  **$\text{curl } \vec{F} = \vec{0}$**
- ✓ In short, if  $\vec{F}$  is the gradient of some scalar potential function  $\phi$ , [ $\text{grad } \phi = \vec{F}$ ] then  $\int_A^B \vec{F} \cdot d\vec{r}$  be independent of path and  $\int_A^B \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$ .
- ✓ If  $\oint_C \vec{F} \cdot d\vec{r} = 0$  then  $\vec{F}$  is conservative (irrotational).
- ✓ Integrals in the differential form are evaluated as below if it is independent of path.

$$\int_A^B \{ M dx + N dy + P dz \} = \int_A^B \left\{ \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right\} = \phi(B) - \phi(A).$$

Where  $\phi$  is scalar potential function.

## METHOD - 7: EXAMPLES OF LINE INTEGRAL

H	<b>1</b>	Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment $C$ joining the origin to the point $(1, 1, 1)$ . <b>Answer: 0.</b>	
H	<b>2</b>	Integrate $f(x, y, z) = x - 3y^2 + z$ over the curve $C = C_1 + C_2$ , where $C_1$ is the line segment joining $(0, 0, 0)$ to $(1, 1, 0)$ and $C_2$ is the line segment joining $(1, 1, 0)$ to $(1, 1, 1)$ . <b>Answer:</b> $\frac{-\sqrt{2}-3}{2} - \frac{1}{2}$	
C	<b>3</b>	Integrate $f(x, y, z) = x - yz^2$ over the curve $C = C_1 + C_2$ , where $C_1$ is the line segment joining $(0, 0, 1)$ to $(1, 1, 0)$ and $C_2$ is the curve $y = x^2$ joining $(1, 1, 0)$ to $(2, 4, 0)$ . <b>Answer:</b> $\frac{5\sqrt{3}}{12} + \frac{17^{\frac{3}{2}} - 5^{\frac{3}{2}}}{12}$	S 2019 (7)



H	4	Evaluate $\int_C \{\bar{F}\bar{r}\} d\bar{r}$ along the parabola $y^2 = x$ between the points (0,0) and (1, 1) where $\bar{F} = x^2 \hat{i} + xy \hat{j}$ . <b>Answer:</b> $\frac{7}{12}$ .	
C	5	If $\bar{F} = 3xy \hat{i} - y^2 \hat{j}$ ; evaluate $\int_C \{\bar{F}\bar{r}\} d\bar{r}$ where C is the arc of the parabola $y = 2x^2$ From (0, 0) to (1, 2). <b>Answer:</b> $\frac{-7}{6}$ .	
H	6	Find the work done in moving a particle in the force field $\bar{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$ along the straight line from (0, 0, 0) to (2, 1, 3). <b>Answer:</b> 16.	
C	7	Prove that $\int_C \{\bar{F}\bar{r}\} d\bar{r} = 3\pi$ , where $\bar{F} = z \hat{i} + x \hat{j} + y \hat{k}$ and C is the arc of the curve $\bar{r} = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$ from $t = 0$ to $t = 2\pi$ .	
C	8	Evaluate $\oint_C \bar{F} d\bar{r}$ ; where $\bar{F} = (x^2 - y^2) \hat{i} + 2xy \hat{j}$ and C is the curve given by the parametric equation $C : \bar{r}(t) = t^2 \hat{i} + t \hat{j}; 0 \leq t \leq 2$ . <b>Answer:</b> $\frac{64}{3}$ .	S 2019 (3)
H	9	Let the vector function $\bar{F} = yz \hat{i} + xz \hat{j} + xy \hat{k}$ and $\bar{r}(t) = t \hat{i} + t^2 \hat{j} + t^3 \hat{k}; 0 \leq t \leq 1$ . Evaluate the line integral $\int_C \{\bar{F}\bar{r}\} d\bar{r}$ . <b>Answer:</b> 1.	
H	10	Find the work done when a force $\bar{F} = (x^2 - y^2 + x) \hat{i} - (2xy + y) \hat{j}$ moves a particle in the XY-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$ . <b>Answer:</b> $\frac{-2}{3}$ .	
C	11	Find the work done in moving a particle in the force field $\bar{F} = 3x^2 \hat{i} + (2xz - y) \hat{j} + z \hat{k}$ along the curve $x^2 = 4y$ and $3x^2 = 8z$ from $x = 0$ to $x = 2$ . <b>Answer:</b> $\frac{441}{40}$ .	

C	12	Find the work done when a force $\vec{F} = (x^2 - y^2 + 2x)\hat{i} - (2xy + y)\hat{j}$ moves a particle in the XY-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$ . Is the work done different when the path is the straight line $y = x$ .  <b>Answer:</b> $\frac{-1}{6}$ , no.	
H	13	Find the work done by the force $\vec{F} = (3x^2 - 3x)\hat{i} + 3z\hat{j} + \hat{k}$ along the straight line $(t, t, t)$ , $0 \leq t \leq 1$ .  <b>Answer:</b> 2.	
C	14	Find work done in moving a particle from (1, 0, 1) to (2, 1, 2) along the straight line AB in the force field $\vec{F} = x^2\hat{i} + (x - y)\hat{j} + (y + z)\hat{k}$  <b>Answer:</b> $\frac{16}{3}$ .	
T	15	Let $\vec{F} = 2xyz\hat{i} + (x^2z + 2y)\hat{j} + yx^2\hat{k}$ If $\vec{F}$ is conservative, find its scalar potential $\phi$ . Find the work done in moving a particle under force field from (0, 1, 1) to (1, 2, 0).  <b>Answer:</b> 1. $\phi = x^2yz + y^2 + c$ , 2. 3.	
H	16	Let $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ If $\vec{F}$ is conservative, find its scalar potential $\phi$ . Find the work done in moving a particle under force field from (1, 1, 0) to (2, 0, 1).  <b>Answer:</b> 1. $\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$ , 2. $\frac{7}{3}$ .	
H	17	Find the circulation of the field $\vec{F} = (x - y)\hat{i} + y\hat{j} + z\hat{k}$ around the closed curve $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$ , $0 \leq t \leq 2\pi$ .  <b>Answer:</b> $\pi$ .	
C	18	If $\vec{F} = (2x - y - 2z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$ calculate the circulation of $\vec{F}$ along the circle in the xy-plane of 3 unit radius and Centre at the origin.  <b>Answer:</b> $18\pi$ .	

T	19	Find the flux of the field $\vec{F} = x^2 \hat{i} + y \hat{j}$ around and across the closed curve $\vec{r}(t) = (\cos t) \hat{i} + (\sin t) \hat{j}$ , $0 \leq t \leq 2\pi$ . <b>Answer: <math>\pi</math>.</b>	
C	20	Find the flux of $\vec{F} = 3xy \hat{i} + (x - y) \hat{j}$ through the parabolic arc $y = x^2$ between $(-1, 1)$ and $(4, 16)$ . <b>Answer: <math>\frac{7465}{6}</math>.</b>	
T	21	$\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ is independent of path joining the points $(1, 2)$ and $(3, 4)$ . Hence, evaluate the integral. <b>Answer: 254.</b>	W 2019 (4)
C	22	If $\vec{F} = (2xy + z^3) \hat{i} + x^2 \hat{j} + 3xz^2 \hat{k}$ Show that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path integration. Hence find the integral when C is any path joining $(1, -2, 1)$ , to $(3, 1, 4)$ . <b>Answer: 202.</b>	

#### ❖ FUNDAMENTAL THEOREM OF LINE INTEGRALS:

- ✓ Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by  $\vec{r}(t)$ . Let  $f$  be a differentiable function with a continuous gradient vector  $\mathbf{F} = \nabla f$  on a domain containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f[\mathbf{B}] - f[\mathbf{A}].$$

#### METHOD – 8: EXAMPLES ON FUNDAMENTAL THEOREM OF LINE INTEGRALS

T	1	Find the work done by the conservative field $\mathbf{F} = yz \hat{i} + xz \hat{j} + xy \hat{k} = \nabla f$ , where $f(x, y, z) = xyz$ , along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$ . <b>Answer: 3.</b>	
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C	2	Suppose the force field $F = \nabla f$ is the gradient of the function $f(x, y, z) = -\frac{1}{x^2+y^2+z^2}$ . Find the work done by $F$ in moving an object along a smooth curve $C$ joining $(1, 0, 0)$ to $(0, 0, 2)$ that does not pass through the origin. <b>Answer:</b> $-\frac{3}{4}$ .	
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## ❖ GREEN'S THEOREM:

- ✓ If  $M(x, y)$  &  $N(x, y)$  and their partial derivatives  $\frac{\partial M}{\partial y}$  &  $\frac{\partial N}{\partial x}$  are continuous in region  $R$  of  $XY$ -plane bounded by a closed curve  $C$  then

$$\oint_C (M \, dx + N \, dy) = \iint_R \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dx \, dy.$$

- ✓ Green's theorem is useful for changing a line integral around a closed curve  $C$  into a double integral over the region  $R$  enclosed by  $C$ .
- ✓ Let 'A' be the area of the plane region  $R$  bounded by closed curve  $C$ . Let  $M = -y$  &  $N = x$ . Then  $\frac{\partial M}{\partial y} = -1$  and  $\frac{\partial N}{\partial x} = 1$ . So by Green's theorem, we have

$$\oint_C (M \, dx + N \, dy) = \iint_R \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dx \, dy = 2 \iint_R dx \, dy$$

$$\therefore \text{Area of region } R = \iint_R dx \, dy = \frac{1}{2} \oint_C (-y \, dx + x \, dy).$$

- ✓ In polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

$$\begin{aligned} \therefore \iint_R dx \, dy \text{ (Area of region } R) &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\ &= \frac{1}{2} \oint_C [(-r \sin \theta)(-r \sin \theta \, d\theta + \cos \theta \, dr) + (r \cos \theta)(r \cos \theta \, d\theta + \sin \theta \, dr)] \\ &= \frac{1}{2} \oint_C (r^2 \sin^2 \theta \, d\theta + r^2 \cos^2 \theta \, d\theta) = \frac{1}{2} \oint_C r^2 \, d\theta. \end{aligned}$$

$$\therefore \text{Area of region R} = \frac{1}{2} \oint_C r^2 d\theta.$$

### METHOD -9: EXAMPLES OF GREEN'S THEOREM

C	1	State Green's theorem and use it to evaluate the integral $\oint_C \{y^2 dx + x^2 dy\}$ , where C is the triangle bounded by $x = 0$ , $x + y = 1$ & $y = 0$ . <b>Answer: 0.</b>	
H	2	Using Green's theorem, evaluate the integral $\oint_C \{x y^3 dx + (x^2 - y^2) dy\}$ , where C is the triangle bounded by $x = 0$ , $x + y = 1$ & $y = 0$ . <b>Answer: <math>\frac{17}{60}</math>.</b>	
T	3	Using Green's theorem evaluate $\oint_C \{x^2 y dx + x^2 dy\}$ where C is the boundary of the triangle whose vertices are $(0, 0)$ , $(1, 0)$ , $(1, 1)$ . <b>Answer: <math>\frac{1}{4}</math>.</b>	
C	4	State the Green's theorem and also evaluate the following integral $\oint_C \{(6y + x)dx + (y + 2x)dy\}$ where C: $(x - 2)^2 + (y - 3)^2 = 4$ . <b>Answer: <math>-16\pi</math>.</b>	
T	5	Using Green's theorem evaluates the line integral $\oint_C \{\sin y dx + \cos x dy\}$ counter clock wise, where C is the boundary of the triangle with vertices $(0, 0)$ , $(\pi, 0)$ & $(\pi, 1)$ . <b>Answer: <math>\pi \cos 1 - \pi - 1</math>.</b>	
H	6	Evaluate $\oint_C \{(x^2 + 2y)dx + (4x + y^2)dy\}$ , by Green's theorem where C is the boundary of the region by $y = 0$ , $y = 2x$ & $x + y = 3$ . <b>Answer: 6.</b>	
T	7	Use Green's theorem to evaluate $\oint_C \{x^2 y dx + y^3 dy\}$ , where c is the closed path formed by $y = x$ and $y = x^3$ from $(0, 0)$ to $(1, 1)$ . <b>Answer: <math>-\frac{1}{12}</math>.</b>	

T	8	Apply Green's theorem to find the outward flux of a vector field $\vec{F} = \frac{1}{xy}(x\hat{i} + y\hat{j})$ across the curve bounded by $y = \sqrt{x}$ , $2y = 1$ and $x = 1$ . <b>Answer: 0.</b>	S 2019 (4)
C	9	Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by Green's theorem. <b>Answer: <math>\pi ab</math>.</b>	
H	10	Find the area of a circle of radius 'r' using Green's theorem. <b>Answer: <math>\pi r^2</math>.</b>	
T	11	Find area of asteroid $\frac{x^2}{a^3} + \frac{y^2}{b^3} = \frac{2}{a^3}$ by Green's theorem. <b>Answer: <math>\frac{3\pi a^2}{8}</math>.</b>	
T	12	Find area of region bounded by the parabola $y = x^2$ & line $y = x + 2$ using Green's theorem. <b>Answer: <math>\frac{9}{2}</math>.</b>	
C	13	Verify Green's theorem for the function $\vec{F} = (x + y)\hat{i} + 2xy\hat{j}$ and C is the rectangle in the xy-plane bounded by $x = 0$ , $y = 0$ , $x = a$ & $y = b$ . <b>Answer: <math>ab(b - 1)</math>.</b>	
H	14	Verify Green's theorem for $\vec{F} = x^2\hat{i} + xy\hat{j}$ under square bounded by $x = 0$ , $x = 1$ , $y = 0$ & $y = 1$ . <b>Answer: <math>\frac{1}{2}</math>.</b>	
H	15	Verify Green's theorem for $\oint_C \{ (3x - 8y^2) dx + (4y - 6xy) dy \}$ , where C is the boundary of the triangle with vertices $(0, 0)$ , $(1, 0)$ & $(0, 1)$ . <b>Answer: <math>\frac{5}{3}</math>.</b>	
H	16	Verify Green's Theorem for $\oint_C \{ (x^2 - 2xy) dx + (x^2y - 3) dy \}$ , where C is the boundary of the region bounded by the parabola $x^2 = y$ and line $y = x$ . <b>Answer: <math>\frac{1}{4}</math>.</b>	

T	17	Verify Green's Theorem in the plane for $\oint_C \{ (3x^2 - 8y^2) dx + (4y - 6xy) dy \}$ , where C is the boundary of the region defined by $y^2 = x$ & $x^2 = y$ . <b>Answer:</b> $\frac{3}{2}$ .	
T	18	Verify Green's theorem for the field $\vec{F} = (x - y) \hat{i} + x \hat{j}$ and the region R bounded by the unit circle C: $\vec{r}(t) = (\cos t) \hat{i} + (\sin t) \hat{j}$ ; $0 \leq t \leq 2\pi$ . OR Verify Green's theorem for the field $\vec{F} = (x - y) \hat{i} + x \hat{j}$ and C is $x^2 + y^2 = 1$ . <b>Answer:</b> $2\pi$ .	
C	19	Verify tangential form of Green's theorem for $\vec{F} = (x - \sin y) \hat{i} + (\cos y) \hat{j}$ , where C is the boundary of the region bounded by the lines $y = 0$ , $x = \pi/2$ and $y = x$ . <b>Answer:</b> 1.	W 2019 (7)







## UNIT-2 » LAPLACE TRANSFORM

### ❖ INTRODUCTION:

- ✓ Pierre Simon Marquis De Laplace (1749-1827) was French mathematician.
- ✓ Laplace transform converts a function of some domain into a function of another domain, without changing the value of a function.
- ✓ For example: From time domain function to frequency domain function
- ✓ Laplace transform is very flexible tool for solving differential equations.
- ✓ LT reduce the problem of differential equation into problem of an algebraic equation. Algebraic equations are easier to solve compare to differential equation.
- ✓ Another advantage of LT is that it solves problem directly i.e. by using Laplace transform we can find particular solution of a differential equation without determining the general solution.
- ✓ We can find solutions of a system of ODE, PDE and integral equations.

### ❖ LAPLACE TRANSFORM(LT):

- ✓ Let  $f(t)$  be a given function defined for all  $t \geq 0$ ,
- ✓ The Laplace transform of  $f(t)$  is denoted by  $f\{f(t)\}$  or  $\bar{f}(s)$  or  $F(s)$  where  $s$  is a parameter (real or complex),
- ✓ The Laplace transform of  $f(t)$  is defined as

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{provided the integral exist.}$$

### ❖ SUFFICIENT CONDITION FOR EXISTENCE OF LT:

- ✓ The LT of  $f(t)$  exists when the following two conditions are satisfied:
- (1).  $f(t)$  should be piecewise continuous function

(2).  $f(t)$  should be of exponential order of  $\alpha$ . i.e.  $\exists M \ \& \ \alpha \ni |f(t)| \leq M \cdot e^{\alpha t}, t \geq 0$ .

✓  $f\{\tan t\}$  does not exist because  $\tan t$  is not piecewise continuous.

✓  $f\{e^{t^2}\}$  does not exist because  $e^{t^2}$  is not of exponential order.

#### ❖ LINEARITY PROPERTY OF LT:

$$\checkmark \quad L\{a \cdot f(t) + b \cdot g(t)\} = a \cdot L\{f(t)\} + b \cdot L\{g(t)\}$$

#### ❖ LT OF SOME ELEMENTARY FUNCTIONS:

$$(1). f\{k\} = \frac{k}{s}, \quad s > 0$$

$$(2). f\{e^{at}\} = \frac{1}{s-a}, \quad s > a \quad \& \quad f\{e^{-at}\} = \frac{1}{s+a}, \quad s > -a$$

$$(3). f\{\sin at\} = \frac{a}{s^2 + a^2}, \quad s > 0 \text{ and } a \text{ is a constant}$$

$$(4). f\{\cos at\} = \frac{s}{s^2 + a^2}, \quad s > 0 \text{ and } a \text{ is a constant}$$

$$(5). f\{\sinh at\} = \frac{a}{s^2 - a^2}, \quad s^2 > a^2 \text{ or } (s > |a|)$$

$$(6). f\{\cosh at\} = \frac{s}{s^2 - a^2}, \quad s^2 > a^2 \text{ or } (s > |a|)$$

$$(7). f\{t^n\} = \frac{n!}{s^{(n+1)}}; n \text{ is integer}$$

$$= \frac{\Gamma(n+1)}{s^{(n+1)}}; n > -1$$

#### ❖ USEFUL FORMULAE:

$$(1). \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin bx - b \cos bx\} + c$$

$$(2). \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \{a \cos bx + b \sin bx\} + c$$

$$(3). \sin nx = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos nx = \frac{e^{ix} + e^{-ix}}{2}$$

$$(4). \Gamma(n+1) = \int_0^{\infty} e^{-x} x^n \, dx; n > -1.$$

(5).  $\Gamma(n+1) = n \cdot \Gamma(n), n > 0$

(6).  $\Gamma(n+1) = n!$ , if  $n$  is positive integer.

### METHOD – 1: EXAMPLES ON DEFINITION OF LT

H	1	Prove that $f\{k\} = \frac{k}{s}$ ; $k$ is constant & $s > 0$ .	
C	2	Prove that $f\{e^{at}\} = \frac{1}{s-a}$ , $s > a$ .	
C	3	Prove that $f\{\sin at\} = \frac{a}{s^2+a^2}$ & $f\{\cos at\} = \frac{s}{s^2+a^2}$ , $s > 0$ and $a$ is a constant.	
C	4	Prove that $f\{\sinh at\} = \frac{a}{s^2-a^2}$ & $f\{\cosh at\} = \frac{s}{s^2-a^2}$ , $s^2 > a^2$ .	
C	5	Prove that $f\{t^n\} = \frac{n!}{s^{(n+1)}}$ ; $n$ is integer $= \frac{\Gamma(n+1)}{s^{(n+1)}}$ ; $n > -1$	
H	6	Find the Laplace transform of $f(t) = \begin{cases} 0 & ; 0 \leq t < 3 \\ 4 & ; t \geq 3 \end{cases}$ . <b>Answer:</b> $\frac{4e^{-3s}}{s}$	
C	7	Given that $f(t) = \begin{cases} t+1 & ; 0 \leq t < 2 \\ 3 & ; t \geq 2 \end{cases}$ . Find $f\{f(t)\}$ . <b>Answer:</b> $\frac{-e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2}$	
H	8	Find the Laplace transform of $f(t)$ defined as $f(t) = \begin{cases} t & 0 < t < k \\ k & t > k \end{cases}$ . <b>Answer:</b> $\frac{1}{s^2} [1 - \frac{e^{-sk}}{k}]$	W 2019 (3)
H	9	Find the Laplace transformation of $f(x) = \begin{cases} e^t & ; 0 < t < 1 \\ 0 & ; t > 1 \end{cases}$ . <b>Answer:</b> $\frac{e^{1-s}}{1-s} - \frac{1}{1-s}$	

H	10	Find the Laplace transform of $f(t) = \begin{cases} 0 & ; 0 < t < \pi \\ \sin t & ; t > \pi \end{cases}$ .	
		<b>Answer:</b> $\frac{-e^{-\pi s}}{s^2 + 1}$	

## ❖ SOME IMPORTANT FORMULAE:

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos^3 A = \frac{\cos 3A + 3 \cos A}{4}$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\Gamma(n + 1) = n \Gamma(n), \quad n > 0$$

$$\Gamma(n + 1) = n!, \quad n \text{ is positive integer}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)! \sqrt{\pi}}{n! 4^n}, \quad n = 0, 1, 2, 3, \dots$$

## METHOD - 2: EXAMPLES ON LT OF SIMPLE FUNCTIONS

H	1	Find the Laplace transform of $t^3 + e^{-3t} + t^2$ .	
		<b>Answer:</b> $\frac{3!}{s^4} + \frac{1}{s+3} + \frac{\sqrt{\pi}}{2s^{3/2}}$	
H	2	Find the Laplace transform of $t^5 + e^{-100t} + \cos 5t$ .	
		<b>Answer:</b> $\frac{5!}{s^6} + \frac{1}{s+100} + \frac{s}{s^2+25}$	

H	<b>3</b>	Find the Laplace transform of $\sin \frac{t}{2} + 2t + t_3$ .	
		<b>Answer:</b> $\frac{2}{(4s^2 + 1)} + \frac{1}{s - \log 2} + \frac{4\Gamma(1/3)}{9s^{7/3}}$	
C	<b>4</b>	Find the Laplace transform of $100^t + 2t^{10} + \sin 10t$ .	
		<b>Answer:</b> $\frac{1}{s - \log_e 100} + \frac{2 \cdot 10!}{s^{11}} + \frac{10}{s^2 + 100}$	
H	<b>5</b>	Find $f[(2t - 1)^2]$ .	
		<b>Answer:</b> $\frac{8}{s^3} - \frac{4}{s^2} + \frac{1}{s}$	
H	<b>6</b>	Find the Laplace transformation (i) $\sin(\omega t + \alpha)$ (ii) $\cos(\omega t + b)$	
		<b>Answer:</b> (i) $\cos a \frac{\omega}{s^2 + \omega^2} + \sin a \frac{s}{s^2 + \omega^2}$ (ii) $\cos b \frac{s}{s^2 + \omega^2} - \sin b \frac{\omega}{s^2 + \omega^2}$	
H	<b>7</b>	Find $f\{\sin 2t \cos 2t\}$ .	
		<b>Answer:</b> $\frac{2}{s^2 + 16}$	
C	<b>8</b>	Find $f\{\sin 2t \sin 3t\}$ .	
		<b>Answer:</b> $\frac{1}{2} \left[ \frac{s}{s^2 + 1} - \frac{s}{s^2 + 25} \right]$	
C	<b>9</b>	Find the Laplace transform of $\sin^2 3t$ .	
		<b>Answer:</b> $\frac{18}{s(s^2 + 36)}$	
H	<b>10</b>	Find Laplace transform of $\cos^2(at)$	
		<b>Answer:</b> $\frac{s^2 + 2a^2}{s(s^2 + 4a^2)}$	
H	<b>11</b>	Find $f\{\cos^2 t\}$ .	
		<b>Answer:</b> $\frac{s^2 + 2}{s(s^2 + 4)}$	
C	<b>12</b>	Find Laplace transform of $\sin^3(at)$ .	
		<b>Answer:</b> $\frac{6a^3}{(s^2 + a^2)(s^2 + 9a^2)}$	

H	13	Find the Laplace transform of (i) $\sin^3 2t$ (ii) $\cos^3 2t$ . Answer: i $\frac{48}{(s^2 + 4)(s^2 + 36)}$ ii $\frac{s^3 + 28s}{(s^2 + 4)(s^2 + 36)}$	
T	14	Find $f\{\cos t \cos 2t \cos 3t\}$ . Answer: $\frac{1}{4s} + \frac{s}{4(s^2 + 36)} + \frac{s}{4(s^2 + 16)} + \frac{s}{4(s^2 + 4)}$	
T	15	Find the Laplace transformation of $f(t) = \cosh^2 3t$ . Answer: $\frac{1}{2s} + \frac{s}{2(s^2 - 36)}$	

## ❖ FIRST SHIFTING THEOREM FOR LT:

$$\text{If } L\{f(t)\} = F(s), \quad L\{e^{at} f(t)\} = F(s - a)$$

$$\text{Note: } L\{e^{-at} f(t)\} = F(s + a).$$

## METHOD – 3: EXAMPLES ON FIRST SHIFTING THEOREM

C	1	Find $f(e^{-3t} t^{3/2})$ . Answer: $\frac{3\sqrt{\pi}}{4(s + 3)^{5/2}}$	
H	2	By using first shifting theorem, obtain the value of $f\{(t + 1)^2 e^t\}$ . Answer: $\frac{2}{(s - 1)^3} + \frac{2}{(s - 1)^2} + \frac{1}{s - 1}$	
H	3	Find $f(e^{2t} \sin 3t)$ . Answer: $\frac{3}{s^2 - 4s + 13}$	
C	4	Obtain Laplace transform of $e^{2t} \sin^2 t$ Answer: $\frac{2}{(s - 2)(s^2 - 4s + 8)}$	
H	5	Find Laplace transform of $e^{-2t}(\sin 4t + t^2)$ . Answer: $\left(\frac{4}{s^2 + 4s + 20} + \frac{2}{(s + 2)^3}\right)$	

H	<b>6</b>	Find Laplace transform of $e^{-3t}(2 \cos 5t - 3 \sin 5t)$ . <b>Answer:</b> $\frac{2s - 9}{s^2 + 6s + 34}$	
H	<b>7</b>	Find Laplace transform of $e^{4t}(\sin 2t \cos t)$ . <b>Answer:</b> $\frac{1}{2} \left[ \frac{3}{(s^2 - 8s + 25)} + \frac{1}{(s^2 - 8s + 17)} \right]$	
C	<b>8</b>	Find the Laplace transformation of $f(t) = \frac{\cos zt \sin t}{e^{2t}}$ . <b>Answer:</b> $\frac{1}{2} \left( \frac{3}{(s^2 + 4s + 13)} - \frac{1}{(s^2 + 4s + 5)} \right)$	
C	<b>9</b>	Find $f\{\cosh 2t \cos 2t\}$ . <b>Answer:</b> $\frac{1}{2} \left[ \frac{s - 2}{s^2 - 4s + 8} + \frac{s + 2}{s^2 + 4s + 8} \right]$	
H	<b>10</b>	Find the Laplace transform of $f(t) = t^2 \sinh \pi t$ . <b>Answer:</b> $\frac{1}{(s - \pi)^3} - \frac{1}{(s + \pi)^3}$	
T	<b>11</b>	Find $f\{e^{-4t} \sinh t \sin t\}$ . <b>Answer:</b> $\frac{2(s + 4)}{(s^2 + 6s + 10)(s^2 + 10s + 26)}$	

## ❖ DIFFERENTIATION OF LT: (MULTIPLICATION BY T)

$$\text{If } L\{f(t)\} = F(s), \quad L\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n} \{F(s)\}; \quad n = 1, 2, 3, \dots$$

## METHOD – 4: EXAMPLES ON DIFFERENTIATION OF LT

H	<b>1</b>	Find the value of $f\{t e^{-t}\}$ . <b>Answer:</b> $\frac{1}{(s + 1)^2}$	
H	<b>2</b>	Find the value of $f\{t \sin at\}$ . <b>Answer:</b> $\frac{2as}{(s^2 + a^2)^2}$	

C	3	Find the value of $f\{t \cos^2 t\}$ . <b>Answer:</b> $\frac{1}{2s^2} + \frac{s^2 - 4}{2(s^2 + 4)^2}$	
H	4	Find the value of $f\{t \sin^3 t\}$ . <b>Answer:</b> $\frac{3s}{2} \left[ \frac{1}{(s^2 + 1)^2} + \frac{1}{(s^2 + 9)^2} \right]$	
H	5	Find $f\{t \sin 3t \cos 2t\}$ <b>Answer:</b> $\frac{5s}{(s^2 + 25)^2} + \frac{s}{(s^2 + 1)^2}$	
C	6	Find the Laplace transform of $t^2 \sin \pi t$ . <b>Answer:</b> $\frac{2\pi(3s^2 - \pi^2)}{(\pi^2 + s^2)^3}$	
H	7	Find $f\{t(\sin t - t \cos t)\}$ . <b>Answer:</b> $\frac{8s}{(s^2 + 1)^3}$	
C	8	Obtain $f\{e^{at} t \sin at\}$ . <b>Answer:</b> $\frac{2a(s - a)}{[(s - a)^2 + a^2]^2}$	
H	9	Find $f(t e^{-t} \cos ht)$ . <b>Answer:</b> $\frac{s^2 + 2s + 2}{s^2(s + 2)^2}$	
H	10	Find the Laplace transform of $t e^{4t} \cos 2t$ . <b>Answer:</b> $\frac{s^2 - 8s + 12}{(s^2 - 8s + 20)^2}$	
T	11	Find the Laplace transform of $t^2 e^t \sin 4t$ . <b>Answer:</b> $\frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$	

## ❖ INTEGRATION OF LT: (DIVISION BY T)

$$\text{If } L\{f(t)\} = F(s), \quad L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) \, ds$$



$$f \left\{ \frac{f(t)}{t^2} \right\} = \int_s^\infty \int_s^\infty \{ F(s) \} ds ds$$

## METHOD – 5: EXAMPLES ON INTEGRATION OF LT

H	<b>1</b>	Find $f \left\{ \frac{1 - e^t}{t} \right\}$ . <b>Answer:</b> $\log \left( \frac{s - 1}{s} \right)$	
C	<b>2</b>	Find $f \left( \frac{e^{-bt} - e^{-at}}{t} \right)$ . <b>Answer:</b> $\log \left( \frac{s + a}{s + b} \right)$	
H	<b>3</b>	Find $f \left\{ \frac{\sin \omega t}{t} \right\}$ . <b>Answer:</b> $\tan^{-1} \left( \frac{\omega}{s} \right)$	
C	<b>4</b>	Find the Laplace transform of $\frac{1 - \cos Zt}{t}$ . <b>Answer:</b> $\log \left( \sqrt{1 + \frac{4}{s^2}} \right)$	
H	<b>5</b>	Find the Laplace transform of $\frac{\cos at - \cos bt}{t}$ . <b>Answer:</b> $\log \sqrt{\frac{s^2 + b^2}{s^2 + a^2}}$	
C	<b>6</b>	Find $f \left\{ \frac{e^t \sin t}{t} \right\}$ . <b>Answer:</b> $\cot^{-1}(s - 1)$	
C	<b>7</b>	Find $f \left\{ \frac{1 - \cos t}{t^2} \right\}$ . <b>Answer:</b> $\left[ -\frac{s}{2} \log \left( 1 + \frac{1}{s^2} \right) + \cot^{-1}(s) \right]$	

T	<b>8</b>	Find $f\left\{\frac{\sin^2 t}{t^2}\right\}$ .  <b>Answer:</b> $-\frac{1}{4}\left[-s \log\left(1 + \frac{4}{s^2}\right) + 4 \cot^{-1}\left(\frac{s}{2}\right)\right]$	
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## ❖ LT OF DERIVATIVES:

✓ If  $f\{f(t)\} = F(s)$ , then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$f\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

⋮

$$f\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{n-1}(0)$$

## METHOD - 6: EXAMPLES ON LT OF DERIVATIVES

H	<b>1</b>	Find $f\{f'(t)\}$ if $f(t) = \sin^2 t$ .  <b>Answer:</b> $\frac{2}{(s^2 + 4)}$	
H	<b>2</b>	Find $f\{f(t)\}$ and $f\{f'(t)\}$ when $f(t) = t \cos t$  <b>Answer:</b> $\frac{s(s^2 - 1)}{(s^2 + 1)^2}$	
C	<b>3</b>	Find $f\{f(t)\}$ and $f\{f'(t)\}$ when $f(t) = \frac{\sin t}{t}$ .  <b>Answer:</b> $\cot^{-1} s, \quad s \cot^{-1} s - 1$	
C	<b>4</b>	Find $f\{f(t)\}$ and $f\{f'(t)\}$ when $f(t) = \frac{\sin t}{e^{5t}}$ .  <b>Answer:</b> $\frac{1}{(s+5)^2 + 1}, \quad \frac{s}{s^2 + 10s + 26}$	
H	<b>5</b>	Find $f\{f'(t)\}$ if $f(t) = e^{2t} \sin 3t$ .  <b>Answer:</b> $\frac{3s}{s^2 - 4s + 13}$	

T	6	Find $f\{f'(t)\}$ if $f(t) = \frac{1 - \cos 2t}{t}$ .  <b>Answer:</b> $s \log \left( \sqrt{1 + \frac{4}{s^2}} \right)$	
C	7	Find $f\{f(t)\}$ and $f\{f'(t)\}$ if $f(t) = \begin{cases} t; & 0 \leq t < 3 \\ 6; & t \geq 3 \end{cases}$ .  <b>Answer:</b> $\frac{1}{s^2} + e^{-3s} \left( \frac{3}{s} - \frac{1}{s^2} \right), \quad \frac{1}{s} + e^{-3s} \left( 3 - \frac{1}{s} \right)$	
H	8	Find $f\{f'(t)\}$ if $f(t) = \begin{cases} t+1; & 0 \leq t \leq 2 \\ 3; & t > 2 \end{cases}$ .  <b>Answer:</b> $\frac{1}{s} (1 - e^{-2s})$	

## ❖ LT OF INTEGRALS:

$$\text{If } L\{f(t)\} = F(s), \quad L\left\{\int_0^t f(t) dt\right\} = \frac{F(s)}{s}$$

$$f\left\{\int_0^t \int_0^t f(t) dt dt\right\} = \frac{F(s)}{s^2}$$

## METHOD – 7: EXAMPLES ON LT OF INTEGRALS

C	1	Find the Laplace transform of $\int_0^t e^{-2t} t^3 dt$ .  <b>Answer:</b> $\frac{6}{s(s+2)^4}$	
H	2	Find $f\left\{\int_0^t e^{-t} \cos t dt\right\}$ .  <b>Answer:</b> $\frac{s+1}{s[s^2+2s+2]}$	

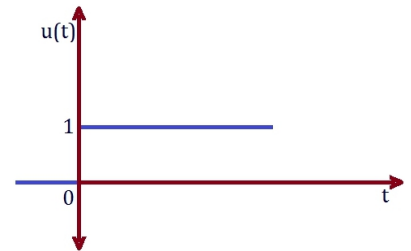
T	3	$\text{Find } f \left\{ \int_0^t e^t(t + \sin t) dt \right\}.$ $\text{Answer: } \frac{1}{s} \left\{ \frac{1}{(s-1)^2} + \frac{1}{s^2 - 2s + 2} \right\}$	
H	4	$\text{Find } f \left\{ \int_0^t t \cosh t dt \right\}.$ $\text{Answer: } \frac{s^2 + 1}{s(s^2 - 1)^2}$	
C	5	$\text{Find the Laplace transformation of } f(t) = \int_0^t e^{-4t} t \sin 3t dt.$ $\text{Answer: } \frac{6(s+4)}{s(s^2 + 8s + 25)^2}$	
C	6	$\text{Find the Laplace transformation of } f(t) = e^{-3t} \int_0^t t \sin 3t dt.$ $\text{Answer: } \frac{6}{(s^2 + 6s + 18)^2}$	
H	7	$\text{Find the Laplace transformation of } \int_0^t e^{-t} t \cos t dt.$ $\text{Answer: } \frac{s+2}{[s^2 + 2s + 2]^2}$	
H	8	$\text{Find the Laplace transformation of } f(t) = \int_0^t \frac{\sin t}{t} dt.$ $\text{Answer: } \frac{\cot^{-1} s}{s}$	W 2019 (2)
C	9	$\text{Find the Laplace transformation of } f(t) = \int_0^t \frac{e^t \sin t}{t} dt.$ $\text{Answer: } \frac{\cot^{-1}(s-1)}{s}$	

C	10	$\text{Find } f \left\{ \int_0^t \int_0^t \sin at \, dt \, dt \right\}.$ <p><b>Answer:</b> <math>\frac{a}{s^2 (s^2 + a^2)}</math></p>	
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## ❖ LT OF UNIT STEP FUNCTION:

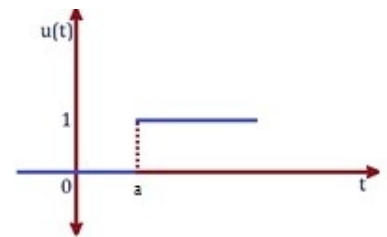
- ✓ Unit step function is defined as

$$\begin{aligned} u(t) &= 0; & t < 0 \\ &= 1; & t > 0 \end{aligned}$$



- ✓ Delayed or Displaced unit step function  $u(t - a)$  represent the function  $f(t)$  which is displaced by a distance  $h$  to the right

$$\begin{aligned} u(t - a) &= 0; & t < a \\ &= 1; & t > a \end{aligned}$$



- ✓ Laplace transform of the unit step function  $u(t)$  is

$$f\{u(t)\} = \frac{1}{s}$$

- ✓ Laplace transform of the displaced unit step function  $u(t - a)$  is

$$f\{u(t - a)\} = \frac{e^{-as}}{s}$$

- ✓ Instead of  $u(t - a)$ , we can also write  $H(t - a)$ , which is referred as Heaviside's unit step function.

## ❖ SECOND SHIFTING THEOREM:

$$\text{If } f\{f(t)\} = F(s), \quad f\{f(t - a) \cdot u(t - a)\} = e^{-as} f\{f(t)\}$$

$$L\{f(t) \cdot u(t - a)\} = e^{-as} L\{f(t + a)\}$$

## METHOD – 8: EXAMPLES ON SECOND SHIFTING THEOREM

H	<b>1</b>	Find the Laplace transform of $e^t u(t - 2)$ . <b>Answer:</b> $\frac{e^{-2(s-1)}}{s-1}$	
C	<b>2</b>	Find the Laplace transform of $e^{-3t} u(t - 2)$ . <b>Answer:</b> $\frac{e^{-2(s+3)}}{s+3}$	
H	<b>3</b>	Find the Laplace transform of $t^2 u(t - 3)$ . <b>Answer:</b> $e^{-3s} \left( \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right)$	W 2019 (1)
H	<b>4</b>	Find the Laplace transform of $\cos t u(t - \pi)$ . <b>Answer:</b> $\frac{-se^{-\pi s}}{s^2 + 1}$	
C	<b>5</b>	Find $f(e^{-t} \sin t u(t - \pi))$ . <b>Answer:</b> $-\frac{e^{-\pi(s+1)}}{s^2 + 2s + 2}$	
C	<b>6</b>	Find the Laplace transform of $(t - 1)^2 u(t - 1)$ . <b>Answer:</b> $\frac{2e^{-s}}{s^3}$	
H	<b>7</b>	Find the $f\{(t - 4)^2 u(t - 4)\}$ . <b>Answer:</b> $\frac{2e^{-4s}}{s^3}$	
H	<b>8</b>	Find the Laplace transform of $e^{3(t-2)} u(t - 2)$ . <b>Answer:</b> $\frac{e^{-2s}}{s-3}$	
C	<b>9</b>	Define: Unit step function. Use it to find the Laplace transform of $f(t) = \begin{cases} (t-1)^2; & t \in (0,1] \\ 1 & ; t \in (1,\infty) \end{cases}$ <b>Answer:</b> $\frac{1}{s^3} [2(1 - e^{-s}) - 2s + s^2(1 + e^{-s})]$	S 2019 (4)
H	<b>10</b>	Express the following in terms of unit step function and hence find $F(s)$ . $f(t) = \begin{cases} \sin 2t & ; 2\pi < t < 4\pi \\ 0 & ; \text{otherwise} \end{cases}$ <b>Answer:</b> $\frac{2(e^{-2\pi s} - e^{-4\pi s})}{s^2 + 4}$	

## ❖ LT OF DIRAC'S DELTA FUNCTION:

- ✓ A Dirac's delta Function or Unit Impulse Function  $\delta(t - a)$  is defined as

$$\delta(t - a) = \lim_{\varepsilon \rightarrow 0} f(t)$$

- ✓ Where,  $f(t)$  is an impulse function, which is defined as

$$0 \quad ; \quad t < a$$

$$f(t) = \frac{1}{\varepsilon} \quad ; \quad a \leq t \leq a + \varepsilon$$

$$0 \quad ; \quad t > a + \varepsilon$$

- ✓ Laplace transform of unit impulse function  $\delta(t - a)$  is  $f\{\delta(t - a)\} = e^{-as}$ .
- ✓ Laplace transform of function  $f(t) \delta(t - a)$  is  $L\{f(t) \delta(t - a)\} = e^{-as}f(a)$ .
- ✓  $f\{\delta(t)\} = 1$

## METHOD – 9: EXAMPLES ON LT OF DIRAC'S DELTA FUNCTIONS

H	1	Find the $f\{\delta(t - 3)\}$ . <b>Answer: <math>e^{-3s}</math></b>	
C	2	Find the $f\{t \delta(t - 2)\}$ . <b>Answer: <math>2e^{-2s}</math></b>	
H	3	Find the $f\{e^{3t} \delta(t - 3)\}$ . <b>Answer: <math>e^{-3s+9}</math></b>	
C	4	Find the $f\{e^{3t} \delta(t - 3) + t^2 u(t - 3)\}$ . <b>Answer: <math>e^{-3s+9} + e^{-3s} \left( \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right)</math></b>	
T	5	Find the $f\{t^2 \delta(t - 4) + t u(t - 4)\}$ . <b>Answer: <math>e^{-4s} \left( \frac{1}{s^2} + \frac{4}{s} + 16 \right)</math></b>	

## ❖ LT OF PERIODIC FUNCTION:

- ✓ A function  $f(t)$  is known as periodic function if for a constant  $T > 0$ ,  $f(T + t) = f(t)$  for all  $t$ .
- ✓ In general,  $f(nT + t) = f(t)$ , where  $n$  is integer &  $T$  is the period of the function.

- ✓ Let  $f(t)$  is a piecewise continuous periodic function with period  $T$ . The Laplace transform of  $f(t)$  is

$$F(s) = L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt, \quad \text{Where } s > 0.$$

#### METHOD – 10: EXAMPLES ON LT OF PERIODIC FUNCTIONS

H	1	Find the Laplace transform of the periodic function defined by $f(t) = \frac{t}{2}, 0 < t < 3,$ if $f(t + 3) = f(t)$ . <b>Answer:</b> $\frac{1}{2s^2} \left[ 1 - \frac{3s}{e^{3s} - 1} \right]$	
H	2	Find the Laplace transform of the periodic function defined by $f(t) = e^t, 0 < t < 2\pi,$ if $f(t + 2\pi) = f(t)$ . <b>Answer:</b> $\frac{e^{(1-s)2\pi} - 1}{(1-s)(1 - e^{-2s\pi})}$	
C	3	Find the Laplace transformation of $f(t) = \frac{t}{T}, 0 < t < T;$ if $f(t) = f(t + T)$ . <b>Answer:</b> $\frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1 - e^{-Ts})}$	
C	4	Find the Laplace transform of the half wave rectifier $f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}, \quad f(t) = f\left(t + \frac{2\pi}{\omega}\right).$ <b>Answer:</b> $\frac{1}{((s^2 + \omega^2)(1 - e^{-\pi s/\omega}))}$	
H	5	Find the Laplace transform of $f(t) =  \sin \omega t , t \geq 0$ . <b>Answer:</b> $\frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}$	



## ❖ INVERSE LAPLACE TRANSFORM(ILT):

✓ If  $F(s) = \mathcal{L}\{f(t)\}$  then  $f(t)$  is known as the Inverse Laplace transform of  $F(s)$ . The Inverse Laplace transform of  $F(s)$  is denoted as  $\mathcal{L}^{-1}\{F(s)\} = f(t)$ .

## ❖ LINEARITY PROPERTY OF ILT:

$$\mathcal{L}^{-1}\{a \cdot F(s) + b \cdot G(s)\} = a \cdot \mathcal{L}^{-1}\{F(s)\} + b \cdot \mathcal{L}^{-1}\{G(s)\}$$

## ❖ ILT OF SOME ELEMENTARY FUNCTIONS:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \quad \text{OR} \quad \frac{t^{n-1}}{\Gamma(n)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \sinh at$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a} \sinh at$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$$

## METHOD - 11: EXAMPLES ON ILT

H	1	Find $\mathcal{L}^{-1}\left\{\frac{6s}{s^2-16}\right\}$ . <b>Answer: 6 cosh (4t)</b>	
H	2	Find $\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$ . <b>Answer: 2 cosh(2t) - <math>\frac{5}{2}</math> sinh(2t)</b>	
C	3	Find $\mathcal{L}^{-1}\left(\frac{4s+15}{16s^2-25}\right)$ . <b>Answer: <math>\frac{1}{4} \cosh\left(\frac{5t}{4}\right) + \frac{3}{4} \sinh\left(\frac{5t}{4}\right)</math></b>	
H	4	Find $\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+9}\right\}$ . <b>Answer: 3 cos(3t) + <math>\frac{4}{3}</math> sin(3t)</b>	

C	5	Find $f^{-1}\left\{\frac{3s-8}{3s^2+25}\right\}$ . Answer: $\frac{4}{3}\cos\left(\frac{5t}{2}\right) - \frac{4}{5}\sin\left(\frac{5t}{2}\right)$	
C	6	Find $f^{-1}\left\{\frac{3(s-1)}{2s^5}\right\}$ . Answer: $\frac{3}{2}\left[1 - t^2 + \frac{t^4}{24}\right]$	
T	7	Find $f^{-1}\left\{\frac{\sqrt{s}-1}{s}\right\}$ . Answer: $1 + t - 4\sqrt{\frac{t}{\pi}}$	
H	8	Find $f^{-1}\left\{\frac{3}{s-3}\right\}$ . Answer: $3e^{3t}$	
T	9	Find $f^{-1}\left\{\frac{1}{s - \log 3}\right\}$ . Answer: $3^t$	
C	10	Find $f^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$ . Answer: $e^{-t} + 1$	

## ❖ FIRST SHIFTING THEOREM FOR ILT:

$$\text{If } L^{-1}\{F(s)\} = f(t), \quad L^{-1}\{F(s-a)\} = e^{at} f(t) = e^{at} L^{-1}\{F(s)\}.$$

## METHOD - 12: EXAMPLES ON FIRST SHIFTING THEOREM FOR ILT

H	1	Find $f^{-1}\left\{\frac{10}{(s-2)^4}\right\}$ . Answer: $\frac{5}{3}e^{2t}t^3$	
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C	2	Find $f^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$ .  <b>Answer:</b> $\frac{1}{\sqrt{2\pi t}} e^{(3t/2)}$	
C	3	Find $f^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\}$ .  <b>Answer:</b> $\frac{t^2(9-2t)}{6e^t}$	
T	4	Find $f^{-1}\left\{\frac{s}{(2s+1)^2}\right\}$ .  <b>Answer:</b> $e^{-\frac{t}{2}}\left[\frac{1}{4} - \frac{t}{8}\right]$	
H	5	Find $f^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$ .  <b>Answer:</b> $\frac{(\cos t - 2 \sin t)}{e^{2t}}$	
H	6	Find $f^{-1}\left\{\frac{2s+3}{s^2-4s+13}\right\}$ .  <b>Answer:</b> $\frac{e^{2t}}{12} (6 \cos 3t + 7 \sin 3t)$	
C	7	Find $f^{-1}\left\{\frac{s+7}{s^2+8s+25}\right\}$ .  <b>Answer:</b> $\frac{(\sin 3t + \cos 3t)}{e^{4t}}$	
H	8	Find $f^{-1}\left\{\frac{1}{s^2+s+1}\right\}$ .  <b>Answer:</b> $\frac{2}{\sqrt{3} e^{t/2}} \sin\left(\frac{\sqrt{3}t}{2}\right)$	
C	9	Find $f^{-1}\left\{\frac{s}{s^2+s+1}\right\}$ .  <b>Answer:</b> $e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2}\right)\right]$	
T	10	Find $f^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$ .  <b>Answer:</b> $e^t(3 \cosh 2t + 5 \sinh 2t)$	

## ❖ SECOND SHIFTING THEOREM FOR ILT:

✓ If  $L^{-1}\{F(s)\} = f(t)$ ,  $L^{-1}\{e^{-as} F(s)\} = f(t-a) \cdot u(t-a) = L^{-1}\{F(s-a)\} \cdot u(t-a)$ .

## METHOD – 13: EXAMPLES ON SECOND SHIFTING THEOREM FOR ILT

H	1	Find $f^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}$ .  <b>Answer:</b> $e^{3(t-2)} u(t-2)$	
C	2	Find the inverse Laplace transform of $\frac{se^{-2s}}{s^2 + \pi^2}$ .  <b>Answer:</b> $\cos \pi(t-2) \cdot u(t-2)$	
H	3	Find the inverse Laplace transform of $\frac{se^{-\pi s}}{s^2 - \pi^2}$ .  <b>Answer:</b> $\cosh \pi(t-\pi) \cdot u(t-\pi)$	
T	4	Find $f^{-1}\left\{\frac{e^{-2\pi s} - e^{-8\pi s}}{s^2 + 1}\right\}$ .  <b>Answer:</b> $\sin(t-2\pi) \cdot u(t-2\pi) - \sin(t-8\pi) \cdot u(t-8\pi)$	
C	5	Find $f^{-1}\left\{e^{-s}\left\{\frac{\sqrt{s-1}}{s}\right\}\right\}$ .  <b>Answer:</b> $\{1 + (t-1) - 4\sqrt{\frac{(t-1)}{\pi}}\} u(t-1)$	
H	6	Find $f^{-1}\left\{\frac{e^{-2s}}{(s+2)(s+3)}\right\}$ .  <b>Answer:</b> $\{e^{-2(t-2)} - e^{-3(t-2)}\} \cdot u(t-2)$	
C	7	Find $f^{-1}\left\{\frac{e^{-2s}}{(s^2+2)(s^2-3)}\right\}$ .  <b>Answer:</b> $\frac{1}{5} \left[ \frac{1}{\sqrt{3}} \sinh \sqrt{3}(t-2) - \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-2) \right] u(t-2)$	
H	8	Find $f^{-1}\left\{\frac{e^{-2s}}{s^2 + 8s + 25}\right\}$ .  <b>Answer:</b> $\frac{e^{-4(t-2)}}{3} \sin 3(t-2) \cdot u(t-2)$	

C	9	Find $f^{-1} \left\{ \frac{s e^{-2s}}{s^2 + 2s + 2} \right\}$ .  <b>Answer:</b> $e^{-(t-2)}[\cos(t-2) - \sin(t-2)]u(t-2)$	
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## ❖ DIFFERENTIATION OF ILT: (MULTIPLICATION BY S)

✓ If  $f^{-1} \{ F(s) \} = f(t)$  &  $f(0) = 0$ ,

$$L^{-1} \{ s \cdot F(s) \} = f'(t) = \frac{d}{dt} [L^{-1} \{ F(s) \}]$$

✓ If  $f(0) = f^k(0) = 0$ , where  $k = 1, 2, 3, \dots, n-1$ ,

$$L^{-1} \{ s^n \cdot F(s) \} = f^{(n)}(t) = \frac{d^n}{dt^n} [L^{-1} \{ F(s) \}]$$

## METHOD - 14: EXAMPLES ON DIFFERENTIATION OF ILT

H	1	Find $f^{-1} \left\{ \frac{s}{s^2 - a^2} \right\}$ .  <b>Answer:</b> $\cosh at$	
H	2	Find $f^{-1} \left\{ \frac{s}{4s^2 - 1} \right\}$ .  <b>Answer:</b> $\frac{1}{4} \cosh \frac{t}{2}$	
C	3	Find $f^{-1} \left\{ \frac{s}{(s+3)^4} \right\}$ .  <b>Answer:</b> $e^{-3t} t^2 \frac{(1-t)}{2}$	
H	4	Find $f^{-1} \left\{ \frac{s^2}{(s-2)^2} \right\}$ .  <b>Answer:</b> $4e^{2t}(1+t)$	
C	5	Find $f^{-1} \left\{ \frac{s^2}{(s+4)^3} \right\}$ .  <b>Answer:</b> $e^{-4t}(8t^2 - 8t + 1)$	

## ❖ INTEGRATION OF ILT: (DIVISION BY S)

$$\text{If } f^{-1}\{F(s)\} = f(t), \quad L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\} dt$$

$$f^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^t L^{-1}\{F(s)\} dt dt$$

## METHOD – 15: EXAMPLES ON INTEGRATION OF ILT

H	1	Find $f^{-1}\left\{\frac{1}{s(s+2)}\right\}$ .  Answer: $\frac{1 - e^{-2t}}{2}$	
C	2	Find $f^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\}$ .  Answer: $\frac{1 - \cos at}{a^2}$	
C	3	Find $f^{-1}\left\{\frac{1}{s(s^2 + 2s + 2)}\right\}$ .  Answer: $\frac{1 - e^{-t}(\sin t + \cos t)}{2}$	
T	4	Find $f^{-1}\left\{\frac{1}{s(s^2 - 3s + 3)}\right\}$ .  Answer: $e^{(3t/2)} \left[ \frac{1}{\sqrt{3}} \sin\left(\frac{t\sqrt{3}}{2}\right) - \frac{1}{3} \cos\left(\frac{t\sqrt{3}}{2}\right) \right] + \frac{1}{3}$	
C	5	Find $f^{-1}\left\{\frac{1}{s^2(1 + s^2)}\right\}$ .  Answer: $t - \sin t$	

## ❖ ILT OF DERIVATIVE:

$$\checkmark \text{ If } f^{-1}\{F(s)\} = f(t), \quad f^{-1}\{F'(s)\} = -t \cdot f^{-1}\{F(s)\}$$

$$L^{-1}\{F(s)\} = \frac{-1}{t} L^{-1}\{F'(s)\}$$

## METHOD – 16: EXAMPLES ON ILT OF DERIVATIVE

C	1	Find $f^{-1} \{ F'(s) \}$ if $F(s) = \frac{1}{s+1}$ .  <b>Answer:</b> $-t e^{-t}$	
H	2	Find $f^{-1} \{ \log \left( \frac{s+a}{s+b} \right) \}$ .  <b>Answer:</b> $\frac{e^{-bt} - e^{-at}}{t}$	
T	3	Find the inverse transform of the function $\ln \left( 1 + \frac{w^2}{s^2} \right)$ .  <b>Answer:</b> $\frac{2}{t} (1 - \cos wt)$	
C	4	Find $f^{-1} \{ \log \left( \frac{s^2+9}{(s+2)^2} \right) \}$ .  <b>Answer:</b> $\frac{2(e^{-2t} - \cos 3t)}{t}$	
H	5	Find $f^{-1} \{ \tan^{-1} \left( \frac{t}{s} \right) \}$ .  <b>Answer:</b> $\frac{\sin 2t}{t}$	
C	6	Find $f^{-1} \{ \tan^{-1} \left( \frac{s+a}{b} \right) \}$ .  <b>Answer:</b> $\frac{e^{-at} \sin bt}{-t}$	

## ❖ ILT OF INTEGRAL:

$$\text{If } f^{-1} \{ F(s) \} = f(t), \quad f^{-1} \left\{ \int_s^\infty F(s) \, ds \right\} = \frac{f^{-1} \{ F(s) \}}{t}$$

$$L^{-1} \{ F(s) \} = t L^{-1} \left\{ \int_s^\infty F(s) \, ds \right\}$$

## METHOD – 17: EXAMPLES ON ILT OF INTEGRAL

C	1	Find $f^{-1} \left\{ \int_s^{\infty} \frac{1}{s+1} ds \right\}$ .  Answer: $\frac{e^{-t}}{t}$	
H	2	Find $f^{-1} \left\{ \frac{1}{(s+1)^2} \right\}$ using ILT of integral.  Answer: $t e^{-t}$	
C	3	Find $f^{-1} \left\{ \frac{zs}{(s^2+1)^2} \right\}$ .  Answer: $t \sin t$	
H	4	Find $f^{-1} \left\{ \frac{s}{(s^2-a^2)^2} \right\}$ .  Answer: $\frac{t}{2a} \sinh at$	

## ❖ CONVOLUTION PRODUCT:

✓ The convolution of two function  $f$  and  $g$  is denoted by  $f * g$  and is defined as

$$f * g = \int_0^t f(u) \cdot g(t-u) du = \int_0^t f(t-u) \cdot g(u) du$$

## ❖ CONVOLUTION THEOREM:

✓ If  $f^{-1}\{F(s)\} = f(t)$  and  $f^{-1}\{G(s)\} = g(t)$ , then

$$L^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) \cdot g(t-u) du = f(t) * g(t)$$

## METHOD – 18: EXAMPLES ON CONVOLUTION THEROREM

C	1	State convolution theorem and using it find $f^{-1} \left\{ \frac{1}{(s+1)(s+3)} \right\}$ .  Answer: $\frac{e^{-t} - e^{-3t}}{2}$	
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H	2	Using the convolution theorem, obtain the value of $f^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\}$ .  <b>Answer:</b> $\frac{1 - \cos at}{a^2}$	W 2019 (4)
C	3	State convolution theorem and use it to evaluate $f^{-1}\left\{\frac{a}{s^2(s^2 + a^2)}\right\}$ .  <b>Answer:</b> $\frac{at - \sin at}{a^2}$	
C	4	State Convolution Theorem and Use to it Evaluate $f^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}$ .  <b>Answer:</b> $\frac{1}{2a^3}(\sin at - at \cos at)$	
H	5	Using convolution theorem, obtain $f^{-1}\left\{\frac{1}{(s^2 + 4)^2}\right\}$ .  <b>Answer:</b> $\frac{1}{16}(\sin 2t - 2t \cos 2t)$	
C	6	Apply convolution theorem to Evaluate $f^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\}$ .  <b>Answer:</b> $\frac{t \sin at}{2a}$	
H	7	Find the inverse Laplace transform of $\frac{s}{(s^2 + 1)^2}$ .  <b>Answer:</b> $\frac{t \sin t}{2}$	
C	8	State convolution theorem and using it find $f^{-1}\left\{\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right\}$ .  <b>Answer:</b> $\frac{a \sin at - b \sin bt}{a^2 - b^2}$	W 2019 (4)
H	9	State convolution theorem and using it to find $f^{-1}\left\{\frac{s^2}{(s^2 + 9)(s^2 + 4)}\right\}$ .  <b>Answer:</b> $\frac{3 \sin 3t - 2 \sin 2t}{5}$	
H	10	Using the convolution theorem, Find $f^{-1}\left(\frac{1}{(s-2)(s+2)^2}\right)$ .  <b>Answer:</b> $\frac{e^{2t} - e^{-2t} - 4te^{-2t}}{16}$	

C	<b>11</b>	Using the convolution theorem, Find the inverse Laplace transform of $\frac{s}{(s+1)(s-1)^2}$ . <b>Answer:</b> $\frac{te^t}{2} + \frac{e^t}{4} - \frac{e^{-t}}{4}$	
C	<b>12</b>	Find $f^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$ . <b>Answer:</b> $\frac{e^{-2t}t\sin t}{2}$	
T	<b>13</b>	Find $f^{-1}\left\{\frac{s(s+1)}{(s^2+1)(s^2+2s+2)}\right\}$ . <b>Answer:</b> $\frac{e^{-t}[2\sin t - 6\cos t] + 2\sin t + 6\cos t}{10}$	
T	<b>14</b>	Using the convolution theorem, Find $f^{-1}\left\{\frac{1}{s(s+a)^3}\right\}$ . <b>Answer:</b> $\frac{e^{-st}}{-a}\left[\frac{t^2}{2} + \frac{t}{a} + \frac{1}{a^2}\right] + \frac{1}{a^3}$	

## ❖ PARTIAL FRACTION METHOD:

- ✓ **Case 1:** If the denominator has non-repeated linear factors  $(s-a)$ ,  $(s-b)$ ,  $(s-c)$ , then

$$\frac{F(s)}{(s-a)(s-b)(s-c)} = \frac{A}{(s-a)} + \frac{B}{(s-b)} + \frac{C}{(s-c)}$$

- ✓ **Case 2:** If the denominator has repeated linear factors  $(s-a)$ , ( $n$  times), then

$$\frac{F(s)}{(s-a)^n} = \frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \dots + \frac{A_n}{(s-a)^n}$$

- ✓ **Case 3:** If the denominator has non-repeated quadratic factors  $(s^2+as+b)$  &  $(s^2+cs+d)$ , then

$$\frac{F(s)}{(s^2+as+b)(s^2+cs+d)} = \frac{As+B}{(s^2+as+b)} + \frac{Cs+D}{(s^2+cs+d)}$$

- ✓ **Case 4:** If the denominator has repeated quadratic factors  $(s^2+as+b)$ -( $n$  times), then

$$\frac{F(s)}{(s^2+as+b)^n} = \frac{As+B}{(s^2+as+b)} + \frac{Cs+D}{(s^2+as+b)^2} + \dots + \frac{\#s+\#^*}{(s^2+as+b)^n}$$

✓ **Case 5:**

$$\frac{F(s)}{(s+a)(s+b)^n} = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \frac{B_3}{(s+b)^3} + \dots + \frac{B_n}{(s+b)^n}$$

$$\frac{F(s)}{(s+a)(s+b)^2} = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2}$$

✓ **Case 6:**

$$\frac{F(s)}{(s+a)(s^2+as+b)^n} = \frac{A}{s+a} + \frac{Bs+C}{s^2+as+b} + \frac{Ds+E}{(s^2+as+b)^2} + \dots + \frac{\#s+\#^*}{(s^2+as+b)^n}$$

$$\frac{F(s)}{(s+a)(s^2+as+b)^1} = \frac{A}{s+a} + \frac{Bs+C}{s^2+as+b}$$

**METHOD – 19: EXAMPLES ON PARTIAL FRACTION METHOD**

C	<b>1</b>	Find $f^{-1} \left\{ \frac{5s^2 + 3s - 16}{(s-1)(s+3)(s-2)} \right\}$ . <b>Answer:</b> $2e^t + e^{-3t} + 2e^{2t}$	
H	<b>2</b>	Find $f^{-1} \left\{ \frac{3s^2 + 2}{(s+1)(s+2)(s+3)} \right\}$ . <b>Answer:</b> $\frac{5}{2}e^{-t} - 14e^{-2t} + \frac{29}{2}e^{-3t}$	
H	<b>3</b>	Find $f^{-1} \left\{ \frac{2s+3}{(s+2)(s+1)^2} \right\}$ . <b>Answer:</b> $-e^{-2t} + e^{-t} + te^{-t}$	
C	<b>4</b>	Find $f^{-1} \left\{ \frac{3s+1}{(s+1)(s^2+2)} \right\}$ . <b>Answer:</b> $-\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$	
C	<b>5</b>	Find $f^{-1} \left\{ \frac{1}{s(s^2-3s+3)} \right\}$ . <b>Answer:</b> $\frac{1}{3} + e^{\frac{3t}{2}} \left[ \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{3} \cos\left(\frac{\sqrt{3}t}{2}\right) \right]$	

H	<b>6</b>	Find the inverse Laplace transform of $\frac{5s + 3}{(s^2 + 2s + 5)(s - 1)}$ . <b>Answer:</b> $-e^{-t} \cos 2t + \frac{3}{2}e^{-t} \sin 2t + e^t$	
H	<b>7</b>	Find $f^{-1} \left\{ \frac{s + 4}{s(s - 1)(s^2 + 4)} \right\}$ . <b>Answer:</b> $-1 + e^t - \frac{1}{2} \sin 2t$	
H	<b>8</b>	Find $f^{-1} \left\{ \frac{s}{(s^2 + 1)(s^2 + 4)} \right\}$ . <b>Answer:</b> $\frac{1}{3} (\cos t - \cos 2t)$	
C	<b>9</b>	Find $f^{-1} \left\{ \frac{2s^2 - 1}{(s^2 + 1)(s^2 + 4)} \right\}$ . <b>Answer:</b> $\frac{3}{2} \sin 2t - \sin t$	
C	<b>10</b>	Find $f^{-1} \left\{ \frac{s^3}{s^4 - 81} \right\}$ . <b>Answer:</b> $\frac{\cos 3t + \cosh 3t}{2}$	
T	<b>11</b>	Find $f^{-1} \left\{ \frac{1}{s^4 - 81} \right\}$ . <b>Answer:</b> $\frac{\sinh 3t - \sin 3t}{54}$	
T	<b>12</b>	Find $f^{-1} \left\{ \frac{s^3 + 2s^2 + 2}{s^3(s^2 + 1)} \right\}$ . <b>Answer:</b> $\sin t + t^2$	

## ❖ NOTATION:

- ✓ Let  $y$  is a function of independent variable  $t$  i.e.  $y(t)$
- ✓ Let  $Y$  is a function of independent variable  $s$  i.e.  $Y(s)$
- ✓ Now  $f\{y(t)\} = Y(s)$ ,  $f^{-1}\{Y(s)\} = y(t)$

## ❖ PROCEDURE TO SOLVE ODES WITH CONSTANT COEFFICIENTS USING LT:

(1). Apply LT on both the sides of ODE

(2). Find the value of  $Y(s)$  from above equation

(3). Apply ILT on both the sides of  $Y(s)$

(4). Find  $y(t)$  which is a solution of given ODE

❖ FORMULAE IN NEW NOTATION:

$$f\left\{\frac{d^n y(t)}{dt^n}\right\} = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0)$$

$$f^{-1}\left\{s^n Y(s)\right\} = \frac{d^n}{dt^n} f^{-1}\{Y(s)\}$$

$$f\left\{\frac{y(t)}{t^2}\right\} = \int_s^\infty \int_s^\infty \{Y(s)\} ds ds,$$

$$f^{-1}\left\{\frac{Y(s)}{s^2}\right\} = \int_0^t \int_0^t [f^{-1}\{Y(s)\}] dt dt$$

$$f\left\{\frac{dy}{dt}\right\} = f\{y'(t)\} = s Y(s) - y(0)$$

$$f\left\{\frac{d^2 y}{dt^2}\right\} = f\{y''(t)\} = s^2 Y(s) - s y(0) - y'(0)$$

⋮

$$f\left\{\frac{d^n y}{dt^n}\right\} = f\{y^{(n)}(t)\}$$

$$= s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - s^{n-3} y''(0) - \dots - s^{n-n} y^{(n-1)}(0)$$

METHOD – 20: EXAMPLES ON SOLVE ODES WITH CONSTANT COEFFICIENTS

C	1	Solve using Laplace transform: $\frac{dy}{dt} - 2y = 4$ , at $t = 0, y = 1$ .  <b>Answer:</b> $y(t) = 3e^{2t} - 2$	
T	2	Solve using Laplace transform $y'' + 6y = 1$ , $y(0) = 2$ , $y'(0) = 0$ .  <b>Answer:</b> $y(t) = \frac{11}{6} \cos \sqrt{6}t + \frac{1}{6}$	
T	3	Using the method of Laplace transform solve the IVP : $y'' + 3y' + 2y = e^t$ , $y(0) = 1$ and $y'(0) = 0$ .  <b>Answer:</b> $y(t) = \frac{1}{6}e^t - \frac{3}{2}e^{-t} + \frac{2}{3}e^{-2t}$	

C	4	Using Laplace transform solve a differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}$ , where $y(0) = 0, y'(0) = -1$ . <b>Answer:</b> $y(t) = \frac{1}{2}e^{-t} - 2e^{-2t} + \frac{3}{2}e^{-3t}$	S 2019 (7)
H	5	Use the Laplace transform to solve the following initial value problem: $y'' - 3y' + 2y = 12e^{-2t}$ , $y(0) = 2$ and $y'(0) = 6$ . <b>Answer:</b> $y(t) = e^{-2t} + 7e^{2t} - 6e^t$	
T	6	Using the method of Laplace transform solve the IVP : $y'' - 4y' + 3y = 6t - 8$ , $y(0) = 0$ and $y'(0) = 0$ . <b>Answer:</b> $y(t) = 2t + e^t - e^{3t}$	
H	7	Using Laplace transform solve the IVP: $y'' + y = \sin 2t$ , $y(0) = 2$ , $y'(0) = 1$ . <b>Answer:</b> $y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t$	
C	8	Using the Laplace transforms, find the solution of the initial value problem $y'' + 25y = 10 \cos 5t$ , $y(0) = 2, y'(0) = 0$ . <b>Answer:</b> $y(t) = t \sin 5t + 2 \cos 5t$	W 2019 (4)
C	9	Solve the initial value problem: $y'' - 2y' = e^t \sin t$ , $y(0) = y'(0) = 0$ , using Laplace transform. <b>Answer:</b> $y(t) = -\frac{1}{4} + \frac{1}{4}e^{2t} - \frac{1}{2}e^t \sin t$	
T	10	Solve the differential equation using Laplace Transformation method $y'' - 3y' + 2y = 4t + e^{3t}$ , $y(0) = 1$ and $y'(0) = -1$ . <b>Answer:</b> $y(t) = 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$	

## ❖ PROCEDURE TO SOLVE SYSTEM OF ODES USING LT:

- (1). Apply LT on both the sides of both ODEs
- (2). Find the value of  $Y(s)$  &  $X(s)$  by solving above two equations
- (3). Apply ILT on both the sides of  $Y(s)$  &  $X(s)$
- (4). Find  $y(t)$  &  $x(t)$  which is a solution of given ODEs

## METHOD – 21: EXAMPLES ON SOLVE SYSTEM OF ODES

H	1	By using Laplace transform solve a system of differential equations $\frac{dx}{dt} = 1 - y$ , $\frac{dy}{dt} = -x$ , where $x(0) = 1, y(0) = 0$ . <b>Answer:</b> $y(t) = 1 - e^t$ , $x(t) = e^t$	S 2019 (7)
H	2	Solve the ODEs using LT: $\frac{dy}{dt} + x = 2$ , $\frac{dx}{dt} + y = 1$ , where $y(0) = 1$ , $x(0) = 1$ . <b>Answer:</b> $y(t) = 1 + \sinh t$ , $x(t) = 2 - \cosh t$	
C	3	Solve the ODEs using LT: $\frac{dy}{dt} + x = \cos t$ , $\frac{dx}{dt} + y = \sin t$ , where $y(0) = 2$ , $x(0) = 0$ . <b>Answer:</b> $y(t) = \sin t + 2 \cosh t$ , $x(t) = -2 \sinh t$	
T	4	Solve the ODEs using LT: $\frac{dy}{dt} + x = e^{-t}$ , $\frac{dx}{dt} + 2y = e^t$ , where $y(0) = 1$ , $x(0) = 1$ . <b>Answer:</b> $y(t) = e^{-t} + e^t - \frac{e^{t\sqrt{2}}}{2} - \frac{e^{-t\sqrt{2}}}{2}$ , $x(t) = 2e^{-t} - e^t + \frac{e^{t\sqrt{2}}}{\sqrt{2}} - \frac{e^{-t\sqrt{2}}}{\sqrt{2}}$	
T	5	Solve the ODEs using LT: $\frac{dy}{dt} + 2x + y = 0$ , $\frac{dx}{dt} - 2y + 5x = t$ , where $y(0) = 0$ , $x(0) = 0$ . <b>Answer:</b> $y(t) = \frac{4}{27} - \frac{2t}{9} - \frac{2e^{-3t}}{9} (\frac{1}{3} + t)$ , $x(t) = \frac{1}{27} + \frac{t}{9} - \frac{e^{-3t}}{9} (\frac{1}{3} + 2t)$	
H	6	Solve the ODEs using LT: $\frac{dx}{dt} + \frac{dy}{dt} + x - y = 1$ , $\frac{dx}{dt} + \frac{dy}{dt} + x + y = e^t$ where $y(0) = 0$ , $x(0) = 1$ . <b>Answer:</b> $y(t) = \frac{(e^{-t} - 1)}{2}$ , $x(t) = \frac{(e^{-t} + 1)}{2}$	

C	7	<p>Solve the ODEs using LT:</p> $\frac{dx}{dt} + \frac{dy}{dt} + x - y = e^{-t}, \quad \frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^t$ <p>where <math>y(0) = 0, \quad x(0) = 1.</math></p> <p><b>Answer:</b> <math>y(t) = \sinh t - e^{-3t}, \quad x(t) = 2e^{-3t}</math></p>	
T	8	<p>Solve the ODEs using LT:</p> $\frac{d^2x}{dt^2} + 2x = y, \quad \frac{d^2y}{dt^2} + 2y = x$ <p>where <math>y(0) = 2, \quad x(0) = 4, \quad y'(0) = 0, \quad x'(0) = 0.</math></p> <p><b>Answer:</b> <math>y(t) = -\cos(\sqrt{3}t) + 3\cos t, \quad x(t) = \cos(\sqrt{3}t) + 3\cos t</math></p>	

## ❖ PROCEDURE TO SOLVE ODES WITH VARIABLE COEFFICIENTS USING LT:

- (1). Apply LT on both the sides of ODE
- (2). Find the value of  $Y(s)$  by solving step-1 ODE using IF
- (3). Find the value of  $c$  & Apply ILT on both the sides of  $Y(s)$
- (4). Find  $y(t)$  which is a solution of given ODE

## ❖ FORMULAE:

$$f\left\{t \cdot \frac{dy}{dt}\right\} = L\{t \cdot y'\} = -\frac{d}{ds}[sY(s) - y(0)] = -Y(s) - s \frac{dY}{ds}$$

$$f\left\{t^2 \cdot \frac{d^2y}{dt^2}\right\} = L\{t^2 \cdot y''\} = -\frac{d}{ds}\left[s^2 Y(s) - sy(0) - y'(0)\right] = -2sY(s) - s \frac{dY}{ds} + y'(0)$$

Let  $\frac{dY}{ds} + PY = Q$ , IF =  $e^{\int P ds}$  & Solution is  $Y \cdot IF = \int (IF \cdot Q) ds + c$

## METHOD - 22: EXAMPLES ON SOLVE ODES WITH VARIABLE COEFFICIENTS

H	1	<p>Solve using Laplace transform:</p> $ty'' - y' = -1, \quad y(0) = 0.$ <p><b>Answer:</b> <math>1 + c \cdot t^2</math></p>	
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C	<b>2</b>	Solve using Laplace transform: $y'' + 3ty' - 6y = 2, \quad y(0) = 1 \quad \& \quad y'(0) = 0.$ <b>Answer: <math>1 + 4t^2</math></b>	
H	<b>3</b>	Solve using Laplace transform: $ty'' - ty' + y = 4, \quad y(0) = 4 \quad \& \quad y'(0) = -2.$ <b>Answer: <math>4 - 2t</math></b>	
T	<b>4</b>	Solve using Laplace transform: $ty'' + (2t - 1)y' - 2y = 0, \quad y(0) = 1.$ <b>Answer: <math>e^{-2t} + c \left( \frac{t}{2} + \frac{e^{-2t}}{4} - \frac{1}{4} \right)</math></b>	

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## UNIT-3 » FOURIER INTEGRALS

## ❖ INTRODUCTION:

- ✓ We know that periodic functions (signals) defined on the finite interval  $(-l, l)$  can be expressed by Fourier series.
- ✓ Non-periodic functions (signals) cannot be expressed by this series.
- ✓ Many engineering problems deal with non-periodic functions.
- ✓ Fourier integral can be considered as a limiting case of Fourier series.
- ✓ Fourier integral is a formula for the decomposition of a non-periodic function into harmonic components.

## ❖ FOURIER INTEGRALS:

- ✓ Fourier integral of  $f(x)$  is given by

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega, \quad \text{where}$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \{f(x) \cos \omega x\} dx, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \{f(x) \sin \omega x\} dx$$

## METHOD – 1: EXAMPLES ON FOURIER INTEGRALS

C	1	Using Fourier integral, Prove that	
		$\int_0^{\infty} \left\{ \frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} \right\} d\omega = \begin{cases} 0 & ; x < 0 \\ \frac{\pi}{2} & ; x = 0 \\ \pi e^{-x} & ; x > 0. \end{cases}$	

C	2	<p>Find the Fourier integral representation of <math>f(x) = \begin{cases} 1 &amp; ;  x  &lt; 1 \\ 0 &amp; ;  x  &gt; 1. \end{cases}</math></p> <p>Hence calculate the followings:</p> <p>(a) <math>\int_0^{\infty} \left\{ \frac{\sin \lambda \cos \lambda x}{\lambda} \right\} d\lambda</math>      (b) <math>\int_0^{\infty} \left\{ \frac{\sin \omega}{\omega} \right\} d\omega .</math></p> <p><b>Answer:</b> <math>f(x) = \int_0^{\infty} \frac{2 \sin \omega}{\pi \omega} \cos \omega x d\omega</math>      a) <math>\begin{cases} \frac{\pi}{2} &amp; ;  x  &lt; 1 \\ 0 &amp; ;  x  &gt; 1 \end{cases}</math>      b) <math>\frac{\pi}{2}</math></p>	
H	3	<p>Find the Fourier integral representation of <math>f(x) = \begin{cases} 2 &amp; ;  x  &lt; 2 \\ 0 &amp; ;  x  &gt; 2. \end{cases}</math></p> <p><b>Answer:</b> <math>f(x) = \int_0^{\infty} \frac{4 \sin 2\omega}{\pi \omega} \cos \omega x d\omega</math></p>	
C	4	<p>Find the Fourier integral representation of <math>f(x) = \begin{cases} e^{kx} &amp; ; x &lt; 0 \\ e^{-kx} &amp; ; x &gt; 0 \end{cases} .</math></p> <p><b>Answer:</b> <math>f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{k^2 + \omega^2} \sin \omega d\omega</math></p>	
T	5	<p>Find the Fourier integral representation of <math>f(x) = \begin{cases} 2 &amp; ; x &lt; 0 \\ e^{-x} \sin x &amp; ; x &gt; 0. \end{cases}</math></p> <p><b>Answer:</b> <math>f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{(2 - \omega^2) \cos \omega x + 2\omega \sin 2\omega}{\omega^4 + 4} d\omega</math></p>	

### ❖ FOURIER COSINE INTEGRAL

✓ Fourier cosine integral of  $f(x)$  is given by

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega, \quad \text{where} \quad A(\omega) = \frac{2}{\pi} \int_0^{\infty} \{ f(x) \cos \omega x \} dx, \quad B(\omega) = 0.$$

## METHOD – 2: EXAMPLES ON FOURIER COSINE INTEGRALS

C	1	Find the Fourier cosine integral of $f(x) = e^{-kx}$ ( $x > 0, k > 0$ ).	
		<b>Answer:</b> $f(x) = \int_0^{\infty} \frac{2k}{\pi(k^2 + \omega^2)} \cos \omega x \, d\omega$	
H	2	Find the Fourier cosine integral of $f(x) = \frac{\pi}{2} e^{-x}, x \geq 0$ .	
		<b>Answer:</b> $f(x) = \int_0^{\infty} \frac{1}{(1 + \omega^2)} \cos \omega x \, d\omega$	
H	3	Find the Fourier integral representation of $f(x) = \begin{cases} \cos x & ;  x  < \pi \\ 0 & ;  x  > \pi. \end{cases}$	
		<b>Answer:</b> $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega \sin \pi \omega \cos \omega x}{1 - \omega^2} d\omega$	
C	4	Find the Fourier integral representation of $f(x) = \begin{cases} 1 - x^2 & ;  x  \leq 1 \\ 0 & ;  x  > 1. \end{cases}$	
		<b>Answer:</b> $f(x) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin \omega - \omega \cos \omega}{\omega^3} \cos \omega x \, d\omega$	

## ❖ FOURIER SINE INTEGRAL

✓ Fourier sine integral of  $f(x)$  is given by

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega, \quad \text{where} \quad B(\omega) = \frac{2}{\pi} \int_0^{\infty} \{ f(x) \sin \omega x \} dx, \quad A(\omega) = 0.$$

## METHOD – 3: EXAMPLES ON FOURIER SINE INTEGRALS

C	1	Using Fourier integral prove that	
		$\int_0^{\infty} \frac{1 - \cos \omega \pi}{\omega} \sin \omega x \, d\omega = \begin{cases} \pi/2 & ; 0 < x < \pi \\ 0 & ; x > \pi. \end{cases}$	

C	2	Using Fourier Cosine integral representation show that $\int_0^{\infty} \frac{\cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi e^{-kx}}{2k}.$	W 2019 (4)
H	3	Express $f(x) = \begin{cases} 1 & ; 0 \leq x \leq \pi \\ 0 & ; x > \pi \end{cases}$ as a Fourier Sine integral and hence evaluate $\int_0^{\infty} \frac{1 - \cos \lambda \pi}{\lambda} \sin \lambda x d\lambda.$ <b>Answer:</b> $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \omega \pi}{\omega} \sin \omega x d\omega, \quad \begin{cases} \pi/2 & ; 0 \leq x \leq \pi \\ 0 & ; x > \pi \end{cases}$	
H	4	Find Fourier sine integral representation of $f(x) = e^{-x} ; x > 0.$ <b>Answer:</b> $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{1 + \omega^2} \sin \omega x d\omega$	
T	5	Find Fourier cosine and sine integral of $f(x) = \begin{cases} \sin x & ; 0 \leq x \leq \pi \\ 0 & ; x > \pi. \end{cases}$ <b>Answer:</b> $\int_0^{\infty} \frac{2(1 + \cos \omega \pi)}{\pi (1 - \omega^2)} \cos \omega x d\omega ; A(1) = 0,$ $\int_0^{\infty} \frac{2 \sin \omega \pi}{\pi (1 - \omega^2)} \sin \omega x d\omega ; B(1) = 1$	

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## UNIT-4 » DIFFERENTIAL EQUATION OF FIRST ORDER

### ❖ INTRODUCTION:

- ✓ A differential equation is a mathematical equation which involves differentials or differential coefficients. Differential equations are very important in engineering problem. Most common differential equations are Newton's Second law of motion, Series RL, RC, and RLC circuits, etc.
- ✓ Mathematical modeling reduces many Natural phenomenon (real world problem) to differential equation(s).
- ✓ In this chapter, we will study, the method of obtaining the solution of ordinary differential equation of first order.

### ❖ DEFINITION: DIFFERENTIAL EQUATION:

- ✓ An equation which involves differential co-efficient is called a Differential Equation.

$$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + y = 0$$

### ❖ DEFINITION: ORDINARY DIFFERENTIAL EQUATION:

- ✓ An equation which involves function of single variable and ordinary derivatives of that function then it is called an Ordinary Differential Equation.

$$\frac{dy}{dx} + y = 0$$

### ❖ DEFINITION: PARTIAL DIFFERENTIAL EQUATION:

- ✓ An equation which involves function of two or more variables and partial derivatives of that function then it is called a Partial Differential Equation.

$$\frac{y}{x} + \frac{y}{t} = 0$$

### ❖ DEFINITION: ORDER OF DIFFERENTIAL EQUATION:

- ✓ The order of highest derivative which appeared in a differential equation is "Order of D.E".

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0 \text{ has order 1.}$$

❖ **DEFINITION: DEGREE OF DIFFERENTIAL EQUATION:**

- ✓ When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a “Degree of D.E.”.

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0 \text{ has degree 2.}$$

❖ **NOTE:**

- ✓ To determine the degree, the D.E has to be expressed in a polynomial form in the derivatives. If the D.E. cannot be expressed in a polynomial form in the derivatives, the degree of D.E. is not defined.

**METHOD – 1: EXAMPLES ON ORDER AND DEGREE OF DIFFERENTIAL EQUATION**

C	<b>1</b>	Find order and degree of $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$ . <b>Answer: 2, 4</b>	
H	<b>2</b>	Find order and degree of $y = x \frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$ . <b>Answer: 1, 2</b>	
C	<b>3</b>	Find order and degree of $\left(\frac{d^2y}{dx^2}\right)^3 = \left[x + \sin\left(\frac{dy}{dx}\right)^2\right]$ . <b>Answer: 2, Undefined</b>	
H	<b>4</b>	Define order and degree of the differential equation. Find order and degree of differential equation $\sqrt{x^2 \frac{d^2y}{dx^2} + 2y} = \frac{d^3y}{dx^3}$ . <b>Answer: 3, 2</b>	
H	<b>5</b>	Find order and degree of differential equation $dy = (y + \sin x)dx$ . <b>Answer: 1, 1</b>	



❖ **SOLUTION OF A DIFFERENTIAL EQUATION:**

- ✓ A solution or integral or primitive of a differential equation is a relation between the variables which does not involve any derivative(s) and satisfies the given differential equation.

❖ **GENERAL SOLUTION (G.S.):**

- ✓ A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the General solution or complete integral or complete primitive.

❖ **PARTICULAR SOLUTION:**

- ✓ The solution obtained from the general solution by giving a particular value to the arbitrary constants is called a particular solution.

❖ **SINGULAR SOLUTION:**

- ✓ A solution which cannot be obtained from a general solution is called a singular solution.

❖ **LINEAR DIFFERENTIAL EQUATION:**

- ✓ A differential equation is called “LINEAR DIFFERENTIAL EQUATION” if the dependent variable and every derivatives in the equation occurs in the first degree only and they should not be multiplied together.

- ✓ Examples:

$$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + y = 0 \text{ is linear.}$$

$$\frac{d^2y}{dx^2} + y \frac{dy}{dx} + y = 0 \text{ is non – linear.}$$

$$\frac{d^2y}{dx^2} + x^2 \left( \frac{dy}{dx} \right)^2 + y = 0 \text{ is non – linear.}$$

- ✓ A Linear Differential Equation of first order is known as Leibnitz’s linear Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x) + c \quad \text{or} \quad \frac{dx}{dy} + P(y)x = Q(y) + c$$

❖ **TYPE OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATION:**

- ✓ Variable Separable Equation
- ✓ Homogeneous Differential Equation
- ✓ Linear(Leibnitz's) Differential Equation
- ✓ Bernoulli's Equation
- ✓ Exact Differential Equation

❖ **PROCEDURE FOR SOLVING D.E. BY VARIABLE SEPARABLE METHOD:**

(1). If a differential equation of type  $\frac{dy}{dx} = f(x, y)$ .

(2). Convert it into  $M(x)dx = N(y)dy$ .

(3). Integrate both side to get general solution of a Variable Separable Equation,

$$\int M(x)dx = \int N(y)dy + c \quad \text{Where, } c \text{ is an arbitrary constant.}$$

❖ **NOTE:**

- ✓ For convenience, the arbitrary constant can be chosen in any suitable form for the answers.  
e. g. in the form  $\log c, \tan^{-1} c, e^c, \sin c$ , etc.

❖ **PROCEDURE TO REDUCE A HOMOGENOUS D.E. IN VARIABLE SEPARABLE EQUATION:**

- ✓ If a differential equation of type  $\frac{dy}{dx} = f(x, y)$  can be converted into  $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$ , then to convert it into variable separable equation follow this procedure:

(1). Take  $\frac{y}{x} = t \Rightarrow y = xt$

(2). Differentiate both the sides, w.r.to  $x \Rightarrow \frac{dy}{dx} = x \frac{dt}{dx} + t$

(3). Use step-1 and Step-2 in given D.E.

- ✓ Then the given Homogenous D.E. will be converted into variable separable equation.

## METHOD – 2: EXAMPLES ON VARIABLE SEPARABLE METHOD

C	<b>1</b>	Solve: $9y y' + 4x = 0$ . <b>Answer:</b> $\frac{9}{2}y^2 = -2x^2 + c$	
C	<b>2</b>	Solve: $xy' + y = 0$ ; $y(2) = -2$ . <b>Answer:</b> $x \cdot y = -4$	
H	<b>3</b>	Solve $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ by variable separable method. <b>Answer:</b> $e^y = e^x + \frac{x^3}{3} + c$	
H	<b>4</b>	Solve: $L \frac{dI}{dt} + RI = 0, I(0) = I_0$ . <b>Answer:</b> $I = I_0 \cdot e^{-\frac{R}{L}t}$	
T	<b>5</b>	Solve: $(1+x)ydx + (1-y)xdy = 0$ . <b>Answer:</b> $\log(xy) + x - y = c$	
H	<b>6</b>	Solve the following differential equation using variable separable method. $3e^x \tan y dx + (1+e^x) \sec^2 y dy = 0$ <b>Answer:</b> $(1+e^x)^3 = \frac{c}{\tan y}$	
H	<b>7</b>	Solve: $e^x \tan y dx + (1-e^x) \sec^2 y dy = 0$ . <b>Answer:</b> $(1-e^x) = c \cdot \tan y$	
C	<b>8</b>	Solve: $\sin hx \cos y dx = \cos hx \sin y dy$ . <b>Answer:</b> $\cos hx \cos y = c$	
T	<b>9</b>	Solve: $xy' = y^2 + y$ . <b>Answer:</b> $\frac{y}{y+1} = x \cdot c$	
C	<b>10</b>	Solve: $xy \frac{dy}{dx} = 1 + x + y + xy$ . <b>Answer:</b> $y - \log(1+y) = \log x + x + c$	
H	<b>11</b>	Solve: $\tan y \frac{dy}{dx} = \sin(x+y) + \sin(x-y)$ . <b>Answer:</b> $\sec y = -2 \cos x + c$	
H	<b>12</b>	Solve: $1 + \frac{dy}{dx} = e^{x+y}$ . <b>Answer:</b> $-(e^{-x-y}) = x + c$	

T	13	Solve: $\frac{dy}{dx} = \cos x \cos y - \sin x \sin y$ . <b>Answer:</b> $\cot(x + y) + \operatorname{cosec}(x + y) = x + c$	
C	14	Solve: $\cos(x + y) dy = dx$ . <b>Answer:</b> $-\cot(x + y) + \operatorname{cosec}(x + y) = y + c$	W 2019 (3)
H	15	Solve: $x \frac{dy}{dx} = y + x e^{\frac{y}{x}}$ . <b>Answer:</b> $\log x + e^{-\frac{y}{x}} = c$	
C	16	Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ . <b>Answer:</b> $\sin(y/x) = x \cdot c$	

## ❖ LEIBNITZ'S DIFFERENTIAL EQUATION:

	Form - 1	Form -2
Form of differential equation	$\frac{dy}{dx} + P(x)y = Q(x)$	$\frac{dx}{dy} + P(y)x = Q(y)$
Integrating factor	<b>I. F.</b> = $e^{\int P(x) dx}$	<b>I. F.</b> = $e^{\int P(y) dy}$
Solution	$y (\text{I. F.}) = \int Q(x) (\text{I. F.}) dx + c$	$x (\text{I. F.}) = \int Q(y) (\text{I. F.}) dy + c$

## ❖ PROCEDURE FOR SOLVING D.E. USING LEIBNITZ'S METHOD

**Type – I:**  $\frac{dy}{dx} + P(x)y = Q(x)$

(1). Find  $P(x)$  and  $Q(x)$  by comparing given D.E. with the above equation.

(2). Find I. F. =  $e^{\int P(x) dx}$ .

(3). Find the general solution by solving  $y (\text{I. F.}) = \int Q(x) (\text{I. F.}) dx + c$ .

**Type – II :**  $\frac{dx}{dy} + P(y)x = Q(y)$

(1). Find  $P(y)$  and  $Q(y)$  by comparing given D.E. with the above equation.

(2). Find I. F. =  $e^{\int P(y) dy}$ .

(3). Find the general solution by solving  $x \text{ (I. F. )} = \int Q(y) \text{ (I. F. )} dy + c$ .

### METHOD – 3: EXAMPLES ON LEIBNITZ'S DIFFERENTIAL EQUATION

C	1	Solve: $y' + y \sin x = e^{\cos x}$ . <b>Answer:</b> $y e^{-\cos x} = x + c$	
H	2	Solve: $\frac{dy}{dx} + \frac{1}{x^2} y = 6 e^{\frac{1}{x}}$ . <b>Answer:</b> $y e^{\frac{-1}{x}} = 6x + c$	
H	3	Solve: $y' + 6x^2 y = \frac{e^{-2x^3}}{x^2}$ , $y(1) = 0$ . <b>Answer:</b> $y e^{2x^3} = -\frac{1}{x} + 1$	
C	4	Solve: $(x+1) \frac{dy}{dx} - y = (x+1)^2 e^{3x}$ . <b>Answer:</b> $\frac{y}{x+1} = \frac{e^{3x}}{3} + c$	S 2019 (4)
H	5	Find the general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} - \sqrt{y} = 0$ . <b>Answer:</b> $\frac{1}{\sqrt{xy}} = \frac{1}{3} x^{\frac{3}{2}} + c$	S 2019 (3)
H	6	Solve: $\frac{dy}{dx} + y = x$ . <b>Answer:</b> $y e^x = e^x x - e^x + c$	
C	7	Solve: $\frac{dy}{dx} + 2y \tan x = \sin x$ . <b>Answer:</b> $y \sec^2 x = \sec x + c$	W 2019 (3)
H	8	Solve: $x \frac{dy}{dx} + (1+x)y = x^3$ . <b>Answer:</b> $x y e^x = x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c$	
H	9	Solve: $\frac{dy}{dx} + (\cot x)y = 2 \cos x$ . <b>Answer:</b> $y \cdot \sin x = -\frac{\cos 2x}{2} + c$	

H	10	Solve: $\frac{dy}{dx} + y \tan x = \sin 2x, y(0) = 0$ . <b>Answer:</b> $y \cdot \sec x = -2 \cos x + 2$	
T	11	Solve: $\frac{dy}{dx} + \frac{4x}{x^2 + 1} y = \frac{1}{(x^2 + 1)^3}$ . <b>Answer:</b> $y \cdot (x^2 + 1)^2 = \tan^{-1} x + c$	
C	12	Solve the differential equation $(1 + y^2)dx = (e^{-\tan^{-1} y} - x)dy$ . <b>Answer:</b> $x \cdot e^{-\tan^{-1} y} = -\frac{e^{-2 \tan^{-1} y}}{2} + c$	S 2019 (4)
C	13	Solve the differential equation $(2x^3 + 4y)dx - xdy = 0$ . <b>Answer:</b> $\frac{y}{x^4} = -\frac{2}{x} + c$	S 2019 (3)

#### ❖ BERNOULLI'S DIFFERENTIAL EQUATION:

- ✓ A differential equation of the form  $\frac{dy}{dx} + P(x)y = Q(x)y^n$  OR  $\frac{dx}{dy} + P(y)x = Q(y)x^n$  is known as Bernoulli's Differential Equation. Where,  $n \in \mathbb{R} - \{0, 1\}$  such differential equation can be converted into linear differential equation and accordingly can be solved.

#### ❖ PROCEDURE TO REDUCE BERNOULLI'S D.E INTO LINEAR DIFFERENTIAL EQUATION:

- ✓ **Case 1:** A differential equation of the form  $\frac{dy}{dx} + P(x)y = Q(x)y^n$

(1). Divide above equation both sides by  $y^n$ ,

$$\Rightarrow y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x) \quad \text{---(1)}$$

(2). Take  $y^{1-n} = v$

(3). Differentiate both the sides, w.r.to x,

$$\Rightarrow (1 - n)y^{-n} \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow y^{-n} \frac{dy}{dx} = \frac{1}{(1 - n)} \frac{dv}{dx}$$

(4). Substitute in Eq.(1), then

$$\Rightarrow \frac{1}{(1 - n)} \frac{dv}{dx} + P(x)v = Q(x) \Rightarrow \frac{dv}{dx} + P(x)(1 - n)v = Q(x)(1 - n)$$

Which is Linear Differential equation and accordingly can be solved.

✓ **Case 2:** A differential of form  $\frac{dy}{dx} + P(x)f(y) = Q(x) g(y)$

**(1).** Step-1: Divide above equation both sides by "g(y)",

$$\Rightarrow \frac{1}{g(y)} \frac{dy}{dx} + P(x) \frac{f(y)}{g(y)} = Q(x) \quad (2)$$

**(2).** Take  $\frac{f(y)}{g(y)} = v$

**(3).** Differentiate both the sides w.r.to x, then Eq<sup>n</sup> (2) becomes Linear Differential equation and accordingly can be solved.

#### METHOD – 4: EXAMPLES ON BERNOULLI'S DIFFERENTIAL EQUATION

C	<b>1</b>	Solve: $\frac{dy}{dx} + \frac{1}{x} y = x^3 y^3$ . Answer: $\frac{1}{x^2 y^2} = -x^2 + c$	
T	<b>2</b>	Solve the following Bernoulli's equation $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ . Answer: $\frac{1}{xy} = \frac{1}{2x^2} + c$	
H	<b>3</b>	Solve: $\frac{dy}{dx} + y = -\frac{x}{y}$ . Answer: $y^2 e^{2x} = -x e^{2x} + \frac{e^{2x}}{2} + c$	
H	<b>4</b>	Solve: $e^y \frac{dy}{dx} + e^y = e^x$ . Answer: $e^{x+y} = \frac{e^{2x}}{2} + c$	
H	<b>5</b>	Solve: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ . Answer: $\frac{e^{-y}}{x} = \frac{1}{2x^2} + c$	S 2019 (3)
C	<b>6</b>	Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ . Answer: $\frac{\sin y}{1+x} = e^x + c$	

C	7	Solve: $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$ .  <b>Answer:</b> $\tan y \cdot e^{\frac{x^2}{2}} = 2x^2 e^{\frac{x^2}{2}} - 4e^{\frac{x^2}{2}} + c$	W 2019 (4)
H	8	Solve: $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ .  <b>Answer:</b> $e^{x^2} \tan y = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$	
H	9	Solve: $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$ .  <b>Answer:</b> $\frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + c$	
C	10	Solve: $(x^3 y^2 + x y)dx = dy$ .  <b>Answer:</b> $\frac{e^{\frac{x^2}{2}}}{y} = 2e^{\frac{x^2}{2}} - x^2 e^{\frac{x^2}{2}} + c$	
T	11	Solve: $x \frac{dy}{dx} + y = y^2 \log x$ .  <b>Answer:</b> $\frac{1}{xy} = \frac{\log x}{x} + \frac{1}{x} + c$	
C	12	Solve: $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$ .  <b>Answer:</b> $-e^{-2x} y^2 = 2 \log y + c$	W 2019 (3)

#### ❖ EXACT DIFFERENTIAL EQUATION:

- ✓ A differential equation of the form  $M(x,y)dx + N(x,y)dy = 0$  is said to be Exact Differential Equation.
- ✓ The necessary and sufficient condition for differential equation to be exact is  $\frac{M}{y} = \frac{N}{x}$
- ✓ Where first order continuous partial derivative of M and N must be exist at all points of  $f(x,y)$ .

#### ❖ PROCEDURE TO SOLVE EXACT DIFFERENTIAL EQUATION:

- ✓ If A differential equation of the form  $M(x,y)dx + N(x,y)dy = 0$ .

(1). Find  $M(x,y)$  and  $N(x,y)$  by comparing the given D.E. with above equation.



(2). Find  $\frac{M}{y}$  and  $\frac{N}{x}$

(3). If  $\frac{M}{y} = \frac{N}{x}$  then the given D.E. is Exact.

(4). Find the general solution by solving

$$\int_{y=\text{constant}} M dx + \int (\text{terms of } N \text{ free from } x) dy = c$$

Where,  $c$  is an arbitrary constant.

#### METHOD – 5: EXAMPLES ON EXACT DIFFERENTIAL EQUATION

H	1	Solve: $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$ .  Answer: $\frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$	
C	2	Check whether the given differential equation is exact or not $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$ .  Answer: Exact, $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$	
H	3	Solve: $(x^2 + y^2)dx + 2xydy = 0$ .  Answer: $\frac{x^3}{3} + y^2x = c$	
H	4	Solve: $2xy dx + x^2 dy = 0$ .  Answer: $x^2y = c$	
C	5	Solve: $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}$ .  Answer: $\frac{x^3}{3} - \frac{x^2}{2} - xy^2 = c$	
H	6	Solve: $ye^x dx + (2y + e^x)dy = 0$ .  Answer: $ye^x + y^2 = c$	W 2019 (4)
H	7	Solve: $(e^y + 1) \cos x dx + e^y \sin x dy = 0$ .  Answer: $(e^y + 1) \sin x = c$	

H	8	Write a necessary and sufficient condition for the differential equation $M(x, y)dx + N(x, y)dy = 0$ to be exact differential equation. Hence check whether the differential equation $[(x + 1)e^x - e^y]dx - xe^y dy = 0$ is exact or not. <b>Answer:</b> $x(e^x - e^y) = c$	S 2019 (3)
C	9	Solve: $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$ . <b>Answer:</b> $y \sin x + x \sin y + xy = c$	
H	10	Solve: $\frac{y}{x} dx + (1 + 2y \log x) dy = 0, x > 0$ . <b>Answer:</b> $y^2 \log x + y = c$	
H	11	Solve: $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$ . <b>Answer:</b> $e^{xy^2} + x^4 - y^3 = c$	

#### ❖ A FIRST ORDER NON-LINEAR DIFFERENTIAL EQUATION BUT NOT OF FIRST DEGREE

- ✓ A differential equation of first order but of higher degree is, of the form  $f(x, y, y') = 0$  or  $f(x, y, p) = 0$  where  $\frac{dy}{dx} = y' = p$  and degree of  $p$  is more than one.

- ✓ For example,  $\left(\frac{dy}{dx}\right)^2 (\sin x + e^x) + 2y \left(\frac{dy}{dx}\right) + xy^2 = 0$  is of order one and degree 2.

- ✓ The general form of a first order differential equation of degree  $n$  is

$$p^n + a_1(x, y)p^{n-1} + a_2(x, y)p^{n-2} + \dots + a_{n-1}(x, y)p + a_n(x, y) = 0 \quad \dots (1)$$

- ✓ Here, coefficients  $a_1(x, y), a_2(x, y), \dots, a_n(x, y)$  are all functions of  $x$  and  $y$ .
- ✓ Equation (1) can be solved by reducing it into first order and first degree equation, by

(1). Equation solving for  $p$ .

(2). Equation solving for  $y$ .

(3). Equation solving for  $x$ .

#### ❖ EQUATIONS SOLVABLE FOR $p$

- ✓ If L.H.S. of eq. (1), which is an  $n^{\text{th}}$  degree polynomial in  $p$ , can be factorised into the form  $(p - b_1)(p - b_2) \dots (p - b_n) = 0$

Where,  $b_1, b_2, \dots, b_n$  are all functions of  $x$  and  $y$ .

✓ Therefore,  $p = b_1, p = b_2, \dots, p = b_n$

$$\Rightarrow \frac{dy}{dx} = b_1(x, y), \frac{dy}{dx} = b_2(x, y), \dots, \frac{dy}{dx} = b_n(x, y)$$

Solving these  $n$  equations, we obtain  $f_1(x, y, c) = 0, f_2(x, y, c) = 0, \dots, f_n(x, y, c) = 0$ .

#### ❖ PROCEDURE FOR EQUATIONS SOLVABLE FOR $p$

(1). Take  $\frac{dy}{dx} = p$ .

(2). Factorize the equation.

(3). Equate all the factors with zero.

(4). Find the solution.

#### METHOD – 6: EXAMPLES ON EQUATION SOLVABLE FOR $p$

C	<b>1</b>	Solve: $p^2 + px + py + xy = 0$ . <b>Answer:</b> $(2y + x^2 - c)(x + \ln  y  - c)$	
H	<b>2</b>	Solve: $p^2y + p(x - y) - x = 0$ . <b>Answer:</b> $(y - x - c) \left( \frac{y^2}{2} + \frac{x^2}{2} - c \right) = 0$	
H	<b>3</b>	Solve: $\left( \frac{dy}{dx} \right) - 5y + 6 = 0$ . <b>Answer:</b> $(y - 3x - c)(y - 2x - c) = 0$	
C	<b>4</b>	Solve: $x^2 \left( \frac{dy}{dx} \right) + xy \frac{dy}{dx} - 6y^2 = 0$ . <b>Answer:</b> $(yx^3 - c)(y - cx^2) = 0$	
H	<b>5</b>	Solve: $x + yp^2 = p(1 + xy)$ . <b>Answer:</b> $\left( y - \frac{x^2}{2} - c \right) \left( \frac{y^2}{2} - x - x \right) = 0$	
T	<b>6</b>	Solve: $4y^2p^2 + 2pxy(3x + 1) + 3x^2 = 0$ . <b>Answer:</b> $(2y^2 + x^2 - 2c)(y^2 + x^3 - c) = 0$	

## ❖ EQUATIONS SOLVABLE FOR y

✓ The given equation is  $f(x, y, p) = 0 \dots \dots (1)$  Solving it for y,

(1). Let  $y = F(x, p)$

(2). Differentiate Eq. (1) w.r.t. x, gives

$$\frac{dy}{dx} = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \frac{dp}{dx} \Rightarrow p = \phi(x, p, \frac{dp}{dx})$$

(3). Solve the above equation and get a relation of type  $\psi(x, p, c) = 0$

(4). Eliminate p between  $f(x, y, p) = 0$  and  $\psi(x, p, c) = 0$ , which will give you the required solution.

✓ When the elimination of p is not possible then we obtain the values of x and y in terms of p as a parameter and these together give us the required solution.

## METHOD – 7: EXAMPLES ON EQUATION SOLVABLE FOR y

C	1	Solve: $3x^4 \left(\frac{dy}{dx}\right)^2 - x \left(\frac{dy}{dx}\right) - y = 0$ . <b>Answer:</b> $xy = c(3cx - 1)$	
H	2	Solve: $y = x + a \tan^{-1} p$ . <b>Answer:</b> $x = \frac{a}{2} \ln \frac{p-1}{\sqrt{p^2+1}} - \frac{a}{2} \tan^{-1} p + c$ $y = \frac{a}{2} \ln \frac{p-1}{\sqrt{p^2+1}} - \frac{a}{2} \tan^{-1} p + c$	
H	3	Solve: $p^3 + p = e^y$ . <b>Answer:</b> $x = -\frac{1}{p} + 2 \tan^{-1} p + c$ $y = \ln p + \ln(p^2 + 1)$	
C	4	Solve: $x = y + a \ln p$ . <b>Answer:</b> $x = -a \ln \left(\frac{p-1}{p}\right) + c$ $y = c - a \ln(p-1)$	

## ❖ SOLVABLE FOR x

✓ The given equation is  $f(x, y, p) = 0 \dots \dots (1)$  Solving it for y,

(1). Let  $x = F(y, p)$

(2). Differentiate Eq. (1) w.r.t. y, gives

$$\frac{dx}{dy} = \frac{1}{p} = \frac{F}{y} \cdot 1 + \frac{F \phi}{p} \frac{dp}{dy} \Rightarrow p = \phi(x, p, \frac{dp}{dx})$$

(3). Suppose  $y(x, p, c) = 0$

(4). Eliminate p between  $x = F(y, p)$  and  $y(x, p, c) = 0$ , which will give you the required solution.

✓ When the elimination of p is not possible then we can express  $x = x(p, c)$  and  $y = y(p, c)$  which will be the parametric representation of the solution.

## METHOD – 8: EXAMPLES ON EQUATION SOLVABLE FOR x

C	1	<p>Solve: <math>p = \tan(x - \frac{p}{1+p^2})</math>.</p> <p><b>Answer:</b> <math>x = \tan^{-1} p + \frac{p}{1+p^2}</math></p> <p><math>y = c - \frac{1}{1+p^2}</math></p>	
H	2	<p>Solve: <math>p^2 - py + x = 0</math>.</p> <p><b>Answer:</b> <math>x = py - p^2</math></p> <p><math>y = -\frac{\sin^{-1} p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}}</math></p>	
T	3	<p>Solve: <math>x = y + a \ln p</math>.</p> <p><b>Answer:</b> <math>x = c - a \ln(1-p) + a \ln p</math></p> <p><math>y = c - [2p + 2 \ln(p-1)]</math></p>	

## ❖ CLAIRAUT'S EQUATION

✓ An equation of the form

$$y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \quad \text{or} \quad y = xp + f(p)$$

is called Clairaut's equation, where  $p = \frac{dy}{dx}$  and  $f$  is a known function of  $p$ .

### ❖ PROCEDURE FOR SOLVING CLAIRAUT'S EQUATION

✓ The given equation is  $f(x, y, p) = 0 \dots \dots \dots$  (1) Solving it for  $y$ ,

(1). Convert into the form  $y = xp + f(p)$ .

(2). For **Complete Solution**, replace “p” by “c”.

(3). For **Singular Solution**, differentiate complete solution with respect to  $c$ .

(4). Find value of  $c$  and substitute the value of  $c$  into complete solution.

### METHOD – 9: EXAMPLES ON CLAIRAUT'S EQUATION

H	1	Solve: $px + \log(p^2 + 1)$ , where $p = \frac{dy}{dx}$ .  <b>Answer:</b> $cx + \log(c^2 + 1)$ .	
C	2	Solve $y = x\frac{dy}{dx} + (\frac{dy}{dx} + 1)^3$ .  <b>Answer:</b> $y = cx + (c + 1)^3$ .	
H	3	Solve $y = 2xp + y^2p^3$ .  <b>Answer:</b> $y^2 = xc + \frac{1}{8}c^3$ .	
C	4	Solve: $(px - y)(py + x) = 2p$ .  <b>Answer:</b> $y^2 = cx^2 - \frac{2c}{c+1}$ .	
T	5	Solve: $e^{4x}(p - 1) + e^{2y}p^2 = 0$ .  <b>Answer:</b> $e^{2y} = ce^{2x} + c^2$ .	



## UNIT-5 » ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

### ❖ INTRODUCTION:

- ✓ Many engineering problems such as Oscillatory phenomena, Bending of beams, etc. leads to the formulation and solution of Linear Ordinary Differential equations of second and higher order.
- ✓ In this chapter we will study, the method of obtaining the solution of Linear Ordinary Differential equations (homogeneous and nonhomogeneous) of second and higher order.

### ❖ HIGHER ORDER LINEAR DIFFERENTIAL EQUATION:

- ✓ A linear differential equation with more than one order is known as Higher Order Linear Differential Equation.
- ✓ A general linear differential equation of the  $n^{\text{th}}$  order is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R(x) \quad (A)$$

Where  $P_0, P_1, P_2, \dots$  are functions of  $x$  and  $P_0 \neq 0$ .

### ❖ HIGHER ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT CO-EFFICIENT:

- ✓ The  $n^{\text{th}}$  order linear differential equation with constant co-efficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R(x) \quad (B)$$

Where  $a_0, a_1, a_2, \dots$  are constants and  $a_0 \neq 0$ .

### ❖ NOTATIONS:

- ✓ Eq. (B) can be written in operator form by taking  $D \equiv \frac{d}{dx}$  as below,

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R(x) \quad \dots \dots \dots (C) \quad \text{OR}$$

$$[f(D)] y = R(x) \quad \dots \dots \dots (D)$$

### ❖ NOTE:

- ✓ An  $n^{\text{th}}$  order linear differential equation has  $n$  linear independent solution.

### ❖ AUXILIARY EQUATION:

- ✓ The auxiliary equation for  $n^{\text{th}}$  order linear differential equation  $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R(x)$  is derived by replacing  $D$  by  $m$  and equating with 0 i.e.  

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

### ❖ COMPLIMENTARY FUNCTION (C.F. - $y_c$ ):

- ✓ A general solution of  $[f(D)] y = 0$  is called complimentary function of  $[f(D)] y = R(x)$ .

### ❖ PARTICULAR INTEGRAL (P.I. - $y_p$ ):

- ✓ A particular integral of  $[f(D)] y = R(x)$  is  $y = \frac{1}{f(D)} R(x)$ .

### ❖ GENERAL SOLUTION $[y(x)]$ OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATION:

- ✓ G.S. = C.F + P.I. i.e.  $y(x) = y_c + y_p$

### ❖ NOTE:

- ✓ In case of higher order homogeneous differential equation, complimentary function is same as general solution.

### ❖ FORMULAE:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad (a + b)^3 = a^3 + b^3 + 3ab(a + b)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad (a - b)^3 = a^3 - b^3 - 3ab(a - b)$$

### ❖ PROCEDURE FOR FINDING G.S.(C.F.) OF HOMOGENEOUS LINEAR ODE WITH CONSTANT CO-EFFICIENTS:

- (1). Consider,  $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R(x)$
- (2). The Auxiliary equation is  $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} y + \dots + a_n y = 0$
- (3). Find roots of auxiliary equation i.e.  $m_1, m_2, m_3, \dots \dots \dots$
- (4). Write down the C.F. as per following table:



Case	Nature of the “n” roots	L.I. solutions	General Solutions
1)	$m_1 \neq m_2 \neq m_3 \neq \dots$	$e^{m_1x}, e^{m_2x}, e^{m_3x}, \dots$	$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \dots$
2)	$m_1 = m_2 = m_3 = m$	$e^{mx}, xe^{mx}, x^2e^{mx}$	$y = (c_1 + c_2x + c_3x^2)e^{mx}$
3)	$m_1 = m_2 = m_3 = m$ $m_4 \neq m_5, \dots$	$e^{mx}, xe^{mx}, x^2e^{mx},$ $e^{m_4x}, e^{m_5x}, \dots$	$y = (c_1 + c_2x + c_3x^2)e^{mx}$ $+ c_4e^{m_4x} + c_5e^{m_5x} + \dots$
4)	$m = p \pm iq$	$e^{px} \cos qx, e^{px} \sin qx,$	$y = e^{px}(c_1 \cos qx + c_2 \sin qx)$
5)	$m_1 = m_2 = p \pm iq$	$e^{px} \cos qx, xe^{px} \cos qx,$ $e^{px} \sin qx, xe^{px} \sin qx,$	$y = e^{px}[(c_1 + c_2x) \cos qx$ $+ (c_3 + c_4x) \sin qx]$

#### METHOD – 1: EXAMPLES ON HL ODE WITH CONSTANT CO-EFFICIENTS

C	<b>1</b>	Solve: $y'' + y' - 2y = 0$ . <b>Answer:</b> $y = c_1e^{-2x} + c_2e^x$	
H	<b>2</b>	Solve: $y'' - 9y = 0$ . <b>Answer:</b> $y = c_1e^{3x} + c_2e^{-3x}$	
H	<b>3</b>	Solve $y''' - 6y'' + 11y' - 6y = 0$ <b>Answer:</b> $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$	
H	<b>4</b>	Solve: $y'' - 6y' + 9y = 0$ . <b>Answer:</b> $y = (c_1 + c_2x) e^{3x}$	
C	<b>5</b>	Solve $y''' - 3y'' + 3y' - y = 0$ <b>Answer:</b> $y = (c_1 + c_2x + c_3x^2) e^x$	
C	<b>6</b>	Solve: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$ . <b>Answer:</b> $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$	
H	<b>7</b>	Solve: $\frac{d^4y}{dx^4} - 18\frac{d^2y}{dx^2} + 81y = 0$ <b>Answer:</b> $y = (c_1 + c_2x)e^{3x} + (c_3 + c_4x)e^{-3x}$	

C	8	Solve: $16y'' - 8y' + 5y = 0$ . Answer: $y = e^{\frac{x}{4}} \left( c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} \right)$	
H	9	Solve: $(D^3 - 2D^2 + 4D - 8)y = 0$ . Answer: $y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$	
T	10	Solve: $(D^4 - 1)y = 0$ . Answer: $y = c_1 e^{-x} + c_2 e^x + (c_3 \cos x + c_4 \sin x)$	
C	11	Solve: $y''' - y = 0$ . Answer: $y = c_1 e^x + e^{-\frac{1}{2}x} \left( c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right)$	
C	12	Solve: $y'' - 5y' + 6y = 0$ ; $y(1) = e^2, y'(1) = 3e^2$ . Answer: $y = e^{3x-1}$	
H	13	Solve: $y'' + 4y' + 4y = 0$ ; $y(0) = 1, y'(0) = 1$ . Answer: $y = (1 + 3x)e^{-2x}$	
T	14	Solve: $y'' - 4y' + 4y = 0$ ; $y(0) = 3, y'(0) = 1$ . Answer: $y = (3 - 5x)e^{2x}$	
H	15	Solve: $y''' - y'' + 100y' - 100y = 0$ ; $y(0) = 4, y'(0) = 11, y''(0) = -299$ . Answer: $y = e^x + \sin 10x + 3 \cos 10x$	
T	16	Solve: $(D^4 + K^4)y = 0$ . Answer: $y = e^{\frac{k}{\sqrt{2}}x} \left\{ c_1 \cos \frac{k}{\sqrt{2}}x + c_2 \sin \frac{k}{\sqrt{2}}x \right\} + e^{-\frac{k}{\sqrt{2}}x} \left\{ c_3 \cos \frac{k}{\sqrt{2}}x + c_4 \sin \frac{k}{\sqrt{2}}x \right\}$	

### ❖ PARTICULAR INTEGRAL

✓ Consider the differential equation  $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R(x)$

$$\Rightarrow f(D)y = R(x)$$

$$\therefore \text{Particular Integral} = y_p = \frac{1}{f(D)} R(x)$$

❖ **METHOD OF FINDING THE PARTICULAR INTEGRAL:**

- ✓ There are many methods of finding the particular integral  $\frac{1}{f(D)} R(x)$ , we shall discuss following four main methods:

- (1). General Methods
- (2). Direct or Short-cut Methods involving operators
- (3). Method of Undetermined Co-efficient
- (4). Method of Variation of parameters

❖ **GENERAL METHODS:**

Particular Integral may be obtained by following two ways:

**(1) Method of Factors:**

- ✓ The operator  $\frac{1}{f(D)}$  may be factorized into n linear factors; then the P.I. will be

$$\text{P.I.} = \frac{1}{f(D)} R(x) = \frac{1}{(D - m_1)(D - m_2)\dots\dots\dots(D - m_n)} R(x)$$

- ✓ Now, we know that,

$$\frac{1}{D - m_n} R(x) = e^{m_n x} \int R(x) e^{-m_n x} dx$$

- ✓ On operating with the first symbolic factor, beginning at the right, the particular integral will have form

$$\text{P.I.} = \frac{1}{(D - m_1)(D - m_2)\dots\dots\dots(D - m_{n-1})} e^{m_n x} \int R(x) e^{-m_n x} dx$$

- ✓ Then, on operating with the second and remaining factors in succession, taking them from right to left, one can find the desired particular integral.

**(2) Method of Partial Fractions:**

- ✓ The operator  $\frac{1}{f(D)}$  may be factorized into n linear factors; then the P.I. will be

$$\text{P.I.} = \frac{1}{f(D)} R(x) = \left( \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \dots + \frac{A_n}{D - m_n} \right) R(x)$$

$$= A_1 \frac{1}{D - m_1} R(x) + A_2 \frac{1}{D - m_2} R(x) + \dots + A_n \frac{1}{D - m_n} R(x)$$

Using  $\frac{1}{D - m_n} R(x) = e^{m_n x} \int R(x) e^{-m_n x} dx$ , we get

$$P.I. = A_1 e^{m_1 x} \int R(x) e^{-m_1 x} dx + A_2 e^{m_2 x} \int R(x) e^{-m_2 x} dx + \dots + A_n e^{m_n x} \int R(x) e^{-m_n x} dx$$

✓ Out of these two methods, this (2<sup>nd</sup>) method is generally preferred.

#### ❖ DIRECT OR SHORTCUT METHOD:

✓ **Case-1:**  $R(x) = e^{ax}$

$$\text{If } f(a) \neq 0, \quad P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

$$\text{If } f(a) = 0, \quad P.I. = \frac{1}{f(D)} e^{ax} = \frac{x}{f'(a)} e^{ax} \quad \text{where } f'(a) \neq 0$$

$$\text{If } f(a) = 0, \quad P.I. = \frac{1}{f(D)} e^{ax} = \frac{x^2}{2! f''(a)} e^{ax} \quad \text{where } f''(a) \neq 0$$

✓ **Note:** If  $R(x) = \sinh x$  or  $\cosh x$  or constant, then **Case-1** is preferable.

✓ **Case-2:**  $R(x) = \sin(ax + b)$

$$P.I. = \frac{1}{f(D^2)} \sin(ax + b)$$

$$\text{If } f(-a^2) \neq 0, \quad P.I. = \frac{1}{f(-a^2)} \sin(ax + b)$$

$$\text{If } f(-a^2) = 0, \quad P.I. = \frac{x}{f'(-a^2)} \sin(ax + b), \quad \text{where } f'(-a^2) \neq 0$$

$$\text{If } f(-a^2) = 0, \quad P.I. = \frac{x^2}{2! f''(-a^2)} \sin(ax + b), \quad \text{where } f''(-a^2) \neq 0 \text{ and so on ...}$$

✓ **Case-3:**  $R(x) = \cos(ax + b)$

$$P.I. = \frac{1}{f(D^2)} \cos(ax + b)$$

$$\text{If } f(-a^2) \neq 0, \quad P.I. = \frac{1}{f(-a^2)} \cos(ax + b)$$

$$\text{If } f(-a^2) = 0, \quad \text{P.I.} = \frac{x}{f'(-a^2)} \cos(ax + b), \quad \text{where } f'(-a^2) \neq 0$$

$$\text{If } f'(-a^2) = 0, \quad \text{P.I.} = \frac{x^2}{f''(-a^2)} \cos(ax + b), \quad \text{where } f''(-a^2) \neq 0 \text{ and so on ...}$$

✓ **Case-4:**  $R(x) = x^m$  = polynomial of degree  $m$ ;  $m > 0$

$$\text{P.I.} = \frac{1}{f(D)} x^m$$

In this case, convert  $f(D)$  in the form of  $1 + \phi(D)$  or  $1 - \phi(D)$  form.

$$\text{P.I.} = \frac{1}{1 + \phi(D)} x^m = [1 + \phi(D)]^{-1} x^m = [ \{1 - \phi(D) + [\phi(D)]^2 - \dots \} ] x^m$$

✓ **Case-5:**  $R(x) = e^{ax} V(x)$ , where  $V(x)$  is a function of  $x$ .

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D + a)} V(x)$$

❖ **NOTE:**

$$\frac{1}{1 + x} = (1 + x)^{-1} = 1 - x + x^2 - \dots \quad \& \quad \frac{1}{1 - x} = (1 - x)^{-1} = 1 + x + x^2 + \dots$$

❖ **PROCEDURE FOR FINDING G.S. OF NON-HOMOGENEOUS LINEAR ODE WITH CONSTANT CO-EFFICIENTS:**

- (1). Start with the Auxiliary equation  $a_0 m^n y + a_1 m^{n-1} y + a_2 m^{n-2} y + \dots + a_n y = 0$
- (2). Find roots of auxiliary equation i.e.  $m_1, m_2, m_3, \dots \dots \dots$
- (3). Write down the C.F. as per table of method-1.
- (4). Find out P.I. as per above case (1 to 5).
- (5). Write the G.S. = C.F. + P.I.

#### METHOD - 2: EXAMPLES ON NHL ODE WITH CONSTANT CO-EFFICIENTS

C	1	Solve: $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 12y = e^{6x}$ . Answer: $y = (c_1 e^{-4x} + c_2 e^{3x}) + \frac{1}{30} e^{6x}$	
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H	2	Solve: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 35y = 12e^{5x}$ . <b>Answer:</b> $y = c_1e^{-7x} + c_2e^{5x} + xe^{5x}$	
H	3	Solve: $(D^2 - 2D + 1)y = 10e^x$ . <b>Answer:</b> $y = (c_1 + c_2x)e^x + 5x^2e^x$	
H	4	Solve: $(D^2 - 4)y = 1 + e^x$ ; Where $D = \frac{d}{dx}$ <b>Answer:</b> $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{e^x}{3}$	
C	5	Solve: $y'' - 6y' + 9y = 6e^{3x} - 5 \log 2$ . <b>Answer:</b> $y = (c_1 + c_2x)e^{3x} + 3x^2e^{3x} - \frac{5}{9} \log 2$	
C	6	Solve: $y''' - 3y'' + 3y' - y = 4e^t$ . <b>Answer:</b> $y = (c_1 + c_2t + c_3t^2)e^t + \frac{2t^3}{3}$	W 2019 (4)
H	7	Solve: $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = e^x$ . <b>Answer:</b> $y = c_1e^x + c_2e^{2x} + c_3e^{-3x} - \frac{xe^x}{4}$	S 2019 (4)
H	8	Solve: $(D^2 - 49)y = \sinh 3x$ . <b>Answer:</b> $y = c_1e^{-7x} + c_2e^{7x} - \frac{1}{40} \sinh 3x$	
C	9	Find the complete solution of $\frac{d^3y}{dx^3} + 8y = \cosh(2x)$ . <b>Answer:</b> $y = c_1e^{-2x} + e^x[c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)] + \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$	
C	10	Solve $(D^2 + 4)y = \cos 2x$ . <b>Answer:</b> $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x$	W 2019 (4)
C	11	Solve: $(D^3 - 3D^2 + 9D - 27)y = \cos 3x$ . <b>Answer:</b> $y = c_1e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36}(\cos 3x + \sin 3x)$	
T	12	Solve: $(D^2 + 4)y = \sin 2x$ , given that $y = 0$ and $\frac{dy}{dx} = 2$ when $x = 0$ . <b>Answer:</b> $y = \frac{9}{8} \sin 2x - \frac{x}{4} \cos 2x$	

C	13	Solve: $(D^2 - 4D + 3)y = \sin 3x \cos 2x$ .  Answer: $y = c_1 e^x + c_2 e^{3x} + \frac{\sin x + 2 \cos x}{20} + \frac{10 \cos 5x - 11 \sin 5x}{884}$	
H	14	Solve complementary differential equation $\frac{d^4 y}{dx^4} - \frac{6dy}{dx} + 9y = \sin x \cos 2x$ .  Answer: $y = (c_1 + c_2 x)e^{3x} + \frac{\cos 3x}{36} - \frac{3 \cos x + 4 \sin x}{100}$	
H	15	Solve: $(D^2 + 9)y = \cos 3x + 2 \sin 3x$ .  Answer: $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3} \cos 3x + \frac{x}{6} \sin 3x$	
C	16	Solve: $y'' + 2y' + 3y = 2x^2$ .  Answer: $y = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + (\frac{2}{3}x^2 - \frac{8}{9}x + \frac{4}{27})$	
T	17	Solve: $(D^3 - D)y = x^3$ .  Answer: $y = c_1 e^x + c_2 e^{-x} + c_3 - \frac{x^4}{4} - 3x^2$	
T	18	Solve: $(D^3 - D^2 - 6D)y = x^2 + 1$ .  Answer: $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{x^3}{18} + \frac{x^2}{36} - \frac{25x}{108}$	
C	19	Solve: $(D^2 - 5D + 6)y = e^{2x} \sin 2x$ .  Answer: $y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^{2x}}{10} (\cos 2x - 2 \sin 2x)$	
T	20	Find the general solution of the following differential equation $\frac{d^3 y}{dx^3} - 2 \frac{dy}{dx} + 4y = e^x \cos x$  Answer: $y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x) + \frac{x e^x}{20} (3 \sin x - \cos x)$	
H	21	Solve: $(D^3 - D^2 + 3D + 5)y = e^x \cos 3x$ .  Answer: $y = c_1 e^{-x} + e^x (c_2 \cos 2x + c_3 \sin 2x) - \frac{e^x}{65} (3 \sin 3x + 2 \cos 3x)$	
C	22	Solve: $(D^2 - 2D + 1)y = x^2 e^{3x}$ .  Answer: $y = (c_1 + c_2 x) e^x + \frac{e^{3x}}{4} (x^2 - 2x + \frac{3}{2})$	

H	23	Solve: $(D^4 - 16)y = e^{2x} + x^4$ .  <b>Answer:</b> $y = c_1 e^{2x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x + \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{3}{32}$	
T	24	Solve: $\frac{d^4 y}{dt^4} - 2\frac{d^2 y}{dt^2} + y = \cos t + e^{2t} + e^t$ .  <b>Answer:</b> $y = (c_1 + c_2 t)e^{-t} + (c_3 + c_4 t)e^t + \left(-\frac{\cos t}{4} + \frac{e^{2t}}{9} + \frac{t^2 e^t}{8}\right)$	
C	25	Solve: $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x$ .  <b>Answer:</b> $y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16} \left(x^4 - \frac{3x^2}{4} + \frac{3}{2}\right) + \frac{e^{3x}}{25} + \frac{\cos 3x}{7}$	
H	26	Solve: $y'' + 4y = 8e^{-2x} + 4x^2 + 2$ ; $y(0) = 2, y'(0) = 2$ .  <b>Answer:</b> $y = \cos 2x + 2\sin 2x + e^{-2x} + x^2$	
H	27	Solve $(D^2 - 1)y = x e^x$ .  <b>Answer:</b> $y = c_1 e^x + c_2 e^{-x} + \frac{1}{4} e^x (x^2 - x)$	

#### ❖ METHOD OF UNDETERMINED CO-EFFICIENTS:

- ✓ This method can be used to find the particular integral only if linearly independent derivative of  $R(x)$  are finite in number i.e.

$R(x)$  can be of the form  $k, x^n, e^{ax}, \sin ax, \cos ax$  and combination of this terms; where  $k$  and  $a$  are constant.

- ✓ For example: If  $R(x) = \frac{1}{x}$  or  $\tan x$  or  $\sec x$  etc., then this method is not applicable.

#### ❖ TRIAL SOLUTION FROM $R(X)$ FOR P.I.:

Sr. No.	RHS of $f(D)y = R(x)$	Form of Trial Solution
1)	$R(x) = e^{ax}$	$Y_p = Ae^{ax}$
Example	$R(x) = e^{2x}$ $R(x) = e^{2x} - 3e^{-x}$	$Y_p = Ae^{2x}$ $Y_p = Ae^{2x} + Be^{-x}$



Sr. No.	RHS of $f(D)y = R(x)$	Form of Trial Solution
2)	$R(x) = \sin ax$ <b>or</b> $R(x) = \cos ax$	$Y_p = A \sin ax + B \cos ax$
Example	$R(x) = \cos 3x$ $R(x) = 2 \sin(4x - 5)$	$Y_p = A \sin 3x + B \cos 3x$ $Y_p = A \sin(4x - 5) + B \cos(4x - 5)$
3)	$R(x) = a + bx + cx^2 + dx^3$ $R(x) = ax^2 + bx$ $R(x) = ax + b$ $R(x) = c$	$Y_p = A + Bx + Cx^2 + Dx^3$ $Y_p = A + Bx + Cx^2$ $Y_p = A + Bx$ $Y_p = A$
4)	$R(x) = e^{ax} \sin bx$ <b>or</b> $R(x) = e^{ax} \cos bx$	$Y_p = e^{ax}(A \sin bx + B \cos bx)$
5)	$R(x) = x e^{ax}$ $R(x) = x^2 e^{ax}$	$Y_p = e^{ax}(A + Bx)$ $Y_p = e^{ax}(A + Bx + Cx^2)$
6)	$R(x) = x \sin ax$ $R(x) = x^2 \cos ax$	$Y_p = \sin ax (A + Bx) + \cos ax (C + Dx)$ $Y_p = \sin ax (A + Bx + Cx^2) + \cos ax (D + Ex + Fx^2)$

❖ **NOTE:**

- ✓ Before assuming trial solution for particular integral (P.I.) it is necessary to compare the terms of  $R(x)$  with Complementary function (C.F.).
- ✓ While comparing the terms following cases arise.
  - **Case - 1:** If no terms of  $R(x)$  present in C.F. then P.I. is assumed **from the table** depending on the nature of  $R(x)$ .
  - **Case - 2:** If a term **u** of  $R(x)$  is also present in C.F. then we multiply the corresponding term **u by x** while assuming trial solution for P.I.
  - **Case - 3:** If a term say  **$x^n u$**  of  $R(x)$  is also present in C.F. then we multiply the corresponding term  **$x^n u$  by  $x^n$**  while assuming trial solution for P.I.

## ❖ PROCEDURE FOR FINDING G.S. BY METHOD OF UNDETERMINED COEFFICIENTS:

- (1). Find C.F. as per table of method-1.
- (2). Assume the trial solution for P.I. from the above three cases according to the nature of  $R(x)$ .
- (3). Find P.I. by using the trial solution in given differential equation.
- (4). Write the G.S. = C.F. + P.I.

## METHOD – 3: EXAMPLES ON METHOD OF UNDETERMINED CO-EFFICIENTS

C	1	Solve: $y'' + 4y = 4e^{2x}$ Answer: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}e^{2x}$	
H	2	Solve: $y'' + 10y' + 25y = e^{-5x}$ . Answer: $y = (c_1 + c_2 x)e^{-5x} + \frac{x^2}{2}e^{-5x}$	
C	3	Solve: $y'' + 4y = 2\sin 3x$ . Answer: $y = c_1 \cos 2x + c_2 \sin 2x - \frac{2}{5}\sin 3x$	
C	4	Solve: $y'' + 9y = 2x^2$ . Answer: $y = c_1 \cos 3x + c_2 \sin 3x + \frac{2}{9}x^2 - \frac{4}{81}$	
H	5	Solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$ by method of undetermined coefficients. Answer: $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + x^3$	
T	6	Solve: $y'' + 4y' = 8x^2$ . Answer: $y = c_1 + c_2 e^{-4x} + \frac{x}{4} - \frac{x^2}{2} + \frac{2}{3}x^3$	
C	7	Solve: $y'' - 2y' + y = e^x + x$ . Answer: $y = (c_1 + x c_2)e^x + \frac{x^2 e^x}{2} + x + 2$	

H	8	Solve the following differential equation using the method of undetermined coefficient : $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}.$ <b>Answer:</b> $y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) - \frac{1}{2}x + \frac{1}{2}x^2 + e^{-x}$	W 2019 (7)
C	9	Use the method of undetermined coefficients to solve the differential equation $y'' - 2y' + y = x^2e^x$ . <b>Answer:</b> $y = (c_1 + x c_2)e^x + \frac{x^4e^x}{12}$	S 2019 (7)

❖ **EXISTENCE AND UNIQUENESS OF SOLUTIONS:**

$$y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1y' + p_0y = R(x)$$

- ✓ If  $p_1, p_2, \dots, p_n$  and  $R(x)$  are continuous functions on an open interval  $I$  such that  $x_0 \in I$  & initial conditions are  $y(x_0) = y_0, y'(x_0) = y'$ , then above ODE has a unique solution  $y = \phi(x)$  throughout the interval  $I$ .

❖ **LINEAR DEPENDENCE AND INDEPENDENCE OF SOLUTIONS:**

- ✓ Two solutions  $y_1(x)$  and  $y_2(x)$  of second order linear differential equations with constant coefficient are said to be

**(1).** Linearly independent if  $k_1y_1(x) + k_2y_2(x) = 0 \Rightarrow k_1 = k_2 = 0$

**(2).** Linearly dependent if  $k_1y_1(x) + k_2y_2(x) = 0 \Rightarrow k_1 \neq 0$  **or/and**  $k_2 \neq 0$

- ✓ If  $y_1$  and  $y_2$  are linearly independent, then  $y_1$  and  $y_2$  cannot be expressed in terms of each other.

❖ **WRONSKIAN:**

- ✓ Wronskian of the  $n$  functions  $y_1, y_2, \dots, y_n$  is defined and denoted by the determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

❖ **THEOREM:**

- ✓ Let  $y_1, y_2, \dots, y_n$  be differentiable functions defined on some interval  $I$  then  $y_1, y_2, \dots, y_n$  are linearly independent on  $I$  if and only if  $W(y_1, y_2, \dots, y_n) \neq 0$  at least one value of  $x \in I$ .
- ✓ If  $W(y_1, y_2, \dots, y_n) = 0$ , then no conclusion can be made about linearly dependent or independent of these functions.
- ✓ If  $y_1, y_2, \dots, y_n$  are linearly dependent on  $I$  then  $W(y_1, y_2, \dots, y_n) = 0$  for all  $x \in I$ .

**METHOD – 4: EXAMPLES ON LD & LI**

H	<b>1</b>	Check whether given functions are LD or LI: $x, \log x, x(\log x)^2$ ; $x > 0$ . <b>Answer: Linear Independent</b>	
C	<b>2</b>	Check whether given functions are LD or LI: $e^x, e^{-x}$ . <b>Answer: Linear Independent</b>	
H	<b>3</b>	Find the wronskian for $\cos 2x, \sin^2 x, \cos^2 x$ . <b>Answer: 0</b>	

❖ **METHOD OF VARIATION OF PARAMETERS:**

- ✓ The process of replacing the parameters by functions is called method of variation of parameters.

❖ **PROCEDURE FOR FINDING G.S. BY METHOD OF VARIATION OF PARAMETERS:**

- ✓ For **second** order differential equation

(1). Find the C. F. =  $c_1 y_1 + c_2 y_2$ .

(2). Find the Wronskian of  $y_1, y_2$  as  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ .

(3). Find  $u(x)$  and  $v(x)$  by evaluating the integrals

$$u(x) = \int \frac{-y_2 R(x)}{W} dx, \quad v(x) = \int \frac{y_1 R(x)}{W} dx.$$

(4). Find the P. I. =  $u(x)y_1 + v(x)y_2$ .

(5). Write the general solution  $y = C. F. + P. I.$

✓

For **third** order differential equation

(1). Find the C. F. =  $c_1 y_1 + c_2 y_2 + c_3 y_3$ .

(2). Find the Wronskian of  $y_1, y_2, y_3$  as  $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ .

(3). Find  $A(x), B(x)$  and  $C(x)$  by evaluating the integrals

$$A(x) = \int (y_2 y_3' - y_3 y_2') \frac{R(x)}{W} dx$$

$$B(x) = \int (y_3 y_1' - y_1 y_3') \frac{R(x)}{W} dx$$

$$C(x) = \int (y_1 y_2' - y_2 y_1') \frac{R(x)}{W} dx$$

(4). Find the P. I. =  $A(x)y_1 + B(x)y_2 + C(x)y_3$ .

(5). Write the general solution  $y = C. F. + P. I.$

#### METHOD – 5: EXAMPLES ON METHOD OF VARIATION OF PARAMETERS

C	1	<p>Solve: <math>(D^2 - 4D + 4)y = \frac{e^{2x}}{x^5}</math>.</p> <p><b>Answer:</b> <math>y = (c_1 + c_2 x)e^{2x} + \frac{1}{12} \frac{e^{2x}}{x^3}</math></p>	
H	2	<p>Solve: <math>y'' - 3y' + 2y = e^x</math>.</p> <p><b>Answer:</b> <math>y = (c_1 e^{2x} + c_2 e^x) - e^x - x e^x</math></p>	

H	3	Solve: $(D^2 - 1)y = x e^x$ .  <b>Answer:</b> $y = (c_1 e^x + c_2 e^{-x}) + \frac{e^x}{4} x^2 - \frac{e^x}{8} (1 - 2x)$	
T	4	Use variation of parameter to find general sol <sup>n</sup> of $y'' - 4y' + 4y = \frac{e^{2x}}{x}$ .  <b>Answer:</b> $y = e^{2x}(c_1 + c_2 x - x + x \log x)$	
H	5	Using variation of parameter method solve $(D^2 + 1)y = x \sin x$ .  <b>Answer:</b> $y = c_1 \cos x + c_2 \sin x - \frac{x^2}{4} \cos x + \frac{1}{8} (2x \sin x + \cos x)$	W 2019 (7)
C	6	Solve: $y'' + 2y' + y = e^{-x} \cos x$ .  <b>Answer:</b> $y = (c_1 + c_2 x - \cos x) e^{-x}$	
T	7	Solve: $(D^2 - 4D + 4)y = \frac{e^{2x}}{1 + x^2}$ .  <b>Answer:</b> $y = (c_1 + c_2 x) e^{2x} - e^{2x} \frac{1}{2} \log(1 + x^2) + x e^{2x} (\tan^{-1} x)$	
C	8	Find the solution of $y'' + a^2 y = \tan ax$ by variation of parameter.  <b>Answer:</b> $y = c_1 \cos ax + c_2 \sin ax - \frac{\cos ax \log( \sec ax + \tan ax )}{a^2}$	
H	9	Find solution of $\frac{d^2 y}{dx^2} + 9y = \tan 3x$ using variation of parameter.  <b>Answer:</b> $y = c_1 \cos 3x + c_2 \sin 3x - \frac{\cos 3x}{9} \log( \sec 3x + \tan 3x )$	
T	10	Find the solution of $y'' + 4y = 4 \tan 2x$ by variation of parameter.  <b>Answer:</b> $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log( \sec 2x + \tan 2x )$	
C	11	Solve the differential equation $y'' + 25y = \sec 5x$ by using the method of variation of parameters.  <b>Answer:</b> $y = c_1 \cos 5x + c_2 \sin 5x + \frac{x}{5} \sin 5x + \frac{1}{25} \cos 5x \log( \cos 5x )$	S 2019 (7)
H	12	Solve: $y'' + y = \sec x$ .  <b>Answer:</b> $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log( \cos x )$	
T	13	Solve differential equation using variation of parameter $y'' + 9y = \sec 3x$ .  <b>Answer:</b> $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x \log( \cos 3x )$	

H	14	Solve: $(D^2 + a^2)y = \operatorname{cosec} ax$ <b>Answer:</b> $y = c_1 \cos ax + c_2 \sin ax - \frac{x \cos ax}{a} + \frac{1}{a^2} \sin ax \log( \sin ax )$	
H	15	Solve the following differential equation $\frac{d^2y}{dx^2} + y = \sin x$ using the method of variation of parameters. <b>Answer:</b> $y = c_1 \cos x + c_2 \sin x - \frac{x \cos x}{2} + \frac{\cos x \sin 2x}{4} - \frac{\sin x \cos 2x}{4}$	
C	16	Solve: $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec} x$ . <b>Answer:</b> $y = c_1 + c_2 \cos x + c_3 \sin x + \log( \operatorname{cosec} x - \cot x ) - \cos x \{ \log( \sin x ) \} - x \sin x$	
C	17	Solve the differential equation $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3 \cos x$ by using the method of variation of parameters. <b>Answer:</b> $y = c_1 x^2 + c_2 x - x \cos x$	S 2019 (7)

### ❖ CAUCHY - EULER EQUATIONS:

✓ An equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = R(x)$$

is called Cauchy's homogeneous linear equation. Where  $a_1, a_2, \dots, a_n$  are constants and  $R(x)$  is a function of  $x$ .

### ❖ PROCEDURE TO SOLVE HOMOGENOUS CAUCHY EULER EQUATIONS:

- (1). Replace following terms in given differential equation to convert Cauchy Euler Equation into Linear Differential Equation.

$$x \frac{dy}{dx} = \nu y, \quad x^2 \frac{d^2y}{dx^2} = \nu(\nu-1)y, \quad x^3 \frac{d^3y}{dx^3} = \nu(\nu-1)(\nu-2)y, \dots \dots \dots$$

$$\text{where } D = \frac{d}{dz}$$

- (2). Obtained Linear Differential Equation with constant co-efficient.
- (3). Find C.F.

(4). Put  $z = \log x$  and get the general solution.

❖ **PROCEDURE TO SOLVE NON-HOMOGENOUS CAUCHY EULER EQUATIONS:**

(1). Replace following terms in given differential equation to convert Cauchy Euler Equation into Linear Differential Equation.

Let,  $x = e^z$

$$x \frac{dy}{dx} = D y, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y, \quad x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y, \dots \dots \dots$$

$$\text{where } D = \frac{d}{dz}$$

(2). Obtained Linear Differential Equation with constant co-efficient.

(3). Find C.F.

(4). Find P.I.

(5). Write the general solution  $y = C. F. + P. I.$

(6). Put  $z = \log x$  and get the general solution.

**METHOD – 6: EXAMPLES ON CAUCHY EULER EQUATIONS**

C	1	Solve: $(x^2 D^2 - 3xD + 4)y = 0$ ; $y(1) = 0$ , $y'(1) = 3$ . <b>Answer:</b> $y = 3x^2 \log x$	
H	2	Solve: $x^2 y'' + xy' + y = 0$ . <b>Answer:</b> $y = c_1 \cos(\log x) + c_2 \sin(\log x)$	
H	3	Solve: $x^2 y'' - 2.5xy' - 2y = 0$ . <b>Answer:</b> $y = c_1 x^4 + c_2 \frac{1}{\sqrt{x}}$	
T	4	Solve: $x^2 y'' - 4xy' + 6y = 21x^{-4}$ . <b>Answer:</b> $y = c_1 x^2 + c_2 x^3 + \frac{1}{2} x^{-4}$	
H	5	Solve: $(x^2 D^2 - 3xD + 4)y = x^2$ ; $y(1) = 1$ , $y'(1) = 0$ . <b>Answer:</b> $y = (1 - 2 \log x)x^2 + \frac{1}{2} x^2 (\log x)^2$	



C	6	Solve: $x^3 y''' + 2x^2 y'' + 2y = 10 \left(x + \frac{1}{x}\right)$ . <b>Answer:</b> $y = c_1 x^{-1} + x\{c_2 \cos(\log x) + c_3 \sin(\log x)\} + 5x + 2x^{-1} \log x$	
H	7	Solve: $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = x$ . <b>Answer:</b> $y = \frac{c_1}{x} + x(c_2 \cos \log x + c_3 \sin \log x) + \frac{x}{2}$	S 2019 (4)
H	8	Solve: $(x^2 D^2 - 3xD + 3)y = 3 \ln x - 4$ . <b>Answer:</b> $y = c_1 x + c_2 x^3 + \ln x$	
C	9	Solve: $x^2 D^2 y - xDy + y = \sin(\log x)$ . <b>Answer:</b> $y = (c_1 + c_2 \log x) e^{\log x} + \frac{1}{2} \cos(\log x)$	
T	10	Solve the following Cauchy-Euler equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \cdot \sin(\log x)$ . <b>Answer:</b> $y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4} \cos(\log x) + \frac{\log x}{4} \sin \log x$	
H	11	Solve: $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 \sin(\log x)$ . <b>Answer:</b> $y = (c_1 e^{-2 \log x} + c_2 e^{-\log x}) - \frac{x^2}{170} (7 \cos(\log x) - 11 \sin(\log x))$	
T	12	Solve completely the differential equation $x^2 \frac{d^2 y}{dx^2} - 6x \frac{dy}{dx} + 6y = x^{-3} \log x$ . <b>Answer:</b> $y = c_1 x^6 + c_2 x + \frac{1}{36x^3} \left(\log x + \frac{13}{36}\right)$	
C	13	Solve: $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$ . <b>Answer:</b> $y = x[c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x$	

#### ❖ HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION:

Reduction of order method for Linear second order O.D.E.

(1). Convert given D.E. into  $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$  and find  $P(x)$  &  $Q(x)$ .

(2). Find U.

$$U = \frac{1}{y_1^2} e^{-\int P dx}$$

(3). Find V.

$$V = \int U dx$$

(4). Second solution  $y_2 = V \cdot y_1$

(5). General solution is  $y = c_1 y_1 + c_2 y_2$ .

#### METHOD – 7: EXAMPLES ON FINDING SECOND SOLUTION

C	1	If $y_1 = x$ is one of solution of $x^2 y'' + xy' - y = 0$ find the second solution. <b>Answer:</b> $y_2 = x \log x$	W 2019 (4)
H	2	Find second solution of $x^2 y'' - 4x y' + 6y = 0$ , $y_1 = x^2$ ; $x > 0$ . <b>Answer:</b> $y_2 = x^3$	
T	3	Find second solution of $xy'' + 2y' + xy = 0$ , $y_1 = \frac{\sin x}{x}$ . <b>Answer:</b> $y_2 = -\frac{\cos x}{x}$	

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## UNIT - 6A – INTRODUCTION TO SOME SPECIAL FUNCTIONS

### ❖ INTRODUCTION:

- ✓ Special functions are particular mathematical functions which have some fixed notations due to their importance in mathematics. In this Unit we will study various type of special functions such as Gamma function, Beta function, Error function, Dirac Delta function etc. These functions are useful to solve many mathematical problems in advanced engineering mathematics.

### ❖ BETA FUNCTION:

- ✓ If  $m > 0, n > 0$ , then Beta function is defined by the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

And is denoted by  $\beta(m, n)$  or  $B(m, n)$ .

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

### ❖ PROPERTIES:

- (1). Beta function is a symmetric function i.e.  $B(m, n) = B(n, m)$ , where  $m > 0, n > 0$ .

$$(2). B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$(3). \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$(4). B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

❖ **GAMMA FUNCTION:**

✓ If  $n > 0$ , then Gamma function is denoted by  $\Gamma(n)$  and defined as

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

❖ **PROPERTIES:**

(1). Reduction formula for Gamma Function  $\Gamma(n+1) = n\Gamma(n)$  ; where  $n > 0$ .

(2). If  $n$  is a positive integer, then  $\Gamma(n+1) = n!$

(3). Second Form of Gamma Function

$$\int_0^{\infty} e^{-x^2} x^{2m-1} dx = \frac{1}{2} \Gamma(m)$$

(4). Relation Between Beta and Gamma Function,

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

(5). Legendre's duplication formula

$$\Gamma(n)\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \Gamma(2n) \quad \text{OR} \quad \Gamma(n+1)\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}} \Gamma(2n+1)$$

(6). Euler's formula:

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} ; 0 < n < 1$$

(7). Other formulae:

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)! \sqrt{\pi}}{n! 4^n}, \quad \text{for } n = 0, 1, 2, 3, \dots$$

$$\text{Examples: } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

**METHOD – 1: EXAMPLES ON SPECIAL FUNCTION**

C	<b>1</b>	Find $B(4,3)$ . <b>Answer:</b> $\frac{1}{60}$	
T	<b>2</b>	Find $B\left(\frac{1}{2}, \frac{1}{2}\right)$ . <b>Answer:</b> $\frac{5\pi}{2048}$	
H	<b>3</b>	State the relation between Beta and Gamma function.	
H	<b>4</b>	State Duplication (Legendre) formula.	
C	<b>5</b>	Find $\Gamma\left(\frac{1}{2}\right)$ . <b>Answer:</b> $\frac{15\sqrt{\pi}}{8}$	
H	<b>6</b>	Find $\Gamma\left(\frac{13}{2}\right)$ . <b>Answer:</b> $\frac{10395\sqrt{\pi}}{64}$	
T	<b>7</b>	Find $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ . <b>Answer:</b> $\frac{\pi}{2\sqrt{2}}$	

**❖ LEGENDRE'S EQUATION:**

- ✓ An equation of the form  $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$  is called Legendre's Equation, where  $n$  is non-negative real constant.

**❖ LEGENDRE'S POLYNOMIAL:**

- ✓ A solution of Legendre's equation is known as the Legendre's polynomial which is denoted by  $P_n(x)$  and defined as

$$P_n(x) = \sum_{r=0}^N \frac{1}{2^n} \cdot \frac{(-1)^r (2n-2r)! x^{n-2r}}{r! (n-r)! (n-2r)!}; \quad \text{Where } N = \begin{cases} \frac{n}{2} & ; n = \text{even} \\ \frac{n-1}{2} & ; n = \text{odd} \end{cases}$$

❖ **GENERATING FUNCTION OF THE LEGENDRE'S POLYNOMIAL:**

$$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}; |x| < 1, |t| < 1$$

❖ **RODRIGUES' FORMULA:**

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

**METHOD - 2: EXAMPLES ON LEGENDRE'S POLYNOMIAL**

C	<b>1</b>	Write Legendre's polynomial $P_n(x)$ of degree- $n$ and hence obtain $P_1(x)$ and $P_2(x)$ in powers of $x$ .	S 2019 (3)
H	<b>2</b>	Obtain Legendre's polynomials $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x)$ . <b>Answer:</b> $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x),$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$	
C	<b>3</b>	Express following polynomials in the form of Legendre's polynomials $P_n(x)$ . 1) $2x^3 - 3x + 5$ 2) $x^4 + x^3 - 2x + 1$ 3) $2 - 3x + 4x^2$ <b>Answer:</b> 1) $5P_0(x) - \frac{13}{5}P_1(x) + \frac{4}{5}P_3(x),$ 2) $\frac{6}{5}P_0(x) - \frac{9}{5}P_1(x) + \frac{4}{7}P_2(x) + \frac{2}{5}P_3(x) + \frac{8}{35}P_4(x),$ 3) $\frac{4}{3} + 2P_0(x) - 3P_1(x) + \frac{8}{3}P_2(x)$	
C	<b>4</b>	Prove that 1) $P_n(1) = 1$ and 2) $P_n(-1) = (-1)^n$	
H	<b>5</b>	Using Rodrigues' formula, find $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$ . <b>Answer:</b> $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1),$ $P_3(x) = \frac{1}{2}(5x^3 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$	

## ❖ BESSEL'S EQUATION:

- ✓ The linear second order differential equation  $x^2 y'' + xy' + (x^2 - n^2)y = 0$  is called Bessel's equation, where  $n$  is a non-negative real constant.
- ✓ The solutions of Bessel's equation are called Bessel functions.

## ❖ BESSEL'S FUNCTIONS OF THE FIRST KIND

- ✓ The Bessel's function of the first kind of order  $n$  is denoted by  $J_n(x)$  and defined as

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[ 1 - \frac{x^2}{2(n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \dots \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left( \frac{x}{2} \right)^{2k+n}$$

## ❖ PROPERTIES

- ✓  $J_{-n}(x) = (-1)^n J_n(x)$ , if  $n$  is a positive integer.
- ✓  $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$
- ✓  $\frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}(x)$ .

## METHOD - 3: EXAMPLES ON BESSEL'S FUNCTION

C	1	Write Bessel's function $J_p(x)$ of the first kind of order- $p$ and hence show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ .	S 2019 (3)
H	2	Prove that $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$	
C	3	Prove that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right)$ .	
H	4	Determine the value $J_{(-\frac{3}{2})}(x)$ .  <b>Answer:</b> $[-\sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} + \sin x \right)]$	

C	5	Prove that $J'_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x))$ .	
H	6	Prove that $J'_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ .	
H	7	Prove that $xJ'_n(x) = xJ_{n-1}(x) - nJ_n(x)$ .	
C	8	Prove that $J'_0(x) = -J_1(x)$ .	
H	9	Using Bessel's function of the first kind prove that $J_0(0) = 1$ .	

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## UNIT - 6B- SERIES SOLUTION OF DIFFERENTIAL EQUATION

### ❖ INTRODUCTION:

- ✓ If homogeneous linear differential equation has constant coefficients, it can be solved by algebraic methods, and its solutions are elementary functions known from calculus ( $e^x$ ,  $\cos x$ , etc ...), as we know from unit-4 and 5. However, if such an equation has variable coefficients it must usually be solved by other methods (for example Euler-Cauchy equation).
- ✓ There are some linear differential equations which do not come in this category. In such cases we have to find a convergent power series arranged according to powers of the independent variable, which will approximately express the value of the dependent variables.
- ✓ Before actually proceeding to solve linear ordinary differential equations with polynomial coefficient, we will look at some of the basic concepts which require for their study.

### ❖ POWER SERIES:

- ✓ An infinite series of the below form is called a power series in  $(x - x_0)$ .

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

### ❖ ANALYTIC FUNCTION:

- ✓ A function is said to be analytic at a point  $x_0$  if it can be expressed in a power series near  $x_0$ .

### ❖ ORDINARY POINT & SINGULAR POINT:

- ✓ If  $P_0(x) \frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x)y = 0$  be the given differential equation with variable coefficient,

Divide by  $P_0(x)$ ,  $\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)} \frac{dy}{dx} + \frac{P_2(x)}{P_0(x)} y = 0$

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0 \dots \dots \dots (1), \quad \text{where } p(x) = \frac{P_1(x)}{P_0(x)} \text{ and } q(x) = \frac{P_2(x)}{P_0(x)}$$

✓ In above equation (1)

- A point  $x_0$  is called an ordinary point of the differential equation if the functions  $P(x)$  and  $Q(x)$  both are analytic at  $x_0$ .
- If at least one of the functions  $P(x)$  or  $Q(x)$  is not analytic at  $x_0$  then  $x_0$  is called a singular point.

#### ❖ REGULAR SINGULAR POINT AND IRREGULAR SINGULAR POINT:

- ✓ A singular point  $x_0$  is called regular singular point if both  $(x - x_0)P(x)$  and  $(x - x_0)^2Q(x)$  are analytic at  $x_0$  otherwise it is called an irregular singular point.

#### METHOD – 1: EXAMPLES ON SINGULARITY OF DIFFERENTIAL EQUATION

C	1	Find singularity of $y'' + (x^2 + 1)y' + (x^3 + 2x^2 + 3x)y = 0$ . <b>Answer: No singular point</b>	
H	2	Find singularity of $y'' + e^x y' + \sin(x^2)y = 0$ . <b>Answer: No singular point</b>	
H	3	Classify ordinary points, singular points, regular-singular points and irregular-singular points (if exist) of the differential equation $y'' + xy' = 0$ . <b>Answer: Set of ordinary point = <math>\mathbb{R}</math>. There is no singular point.</b>	S 2019 (4)
H	4	Classify ordinary points, singular points, regular-singular points and irregular-singular points (if exist) of the differential equation $xy'' + y' = 0$ . <b>Answer: Set of ordinary point = <math>\mathbb{R} - \{0\}</math>. <math>x = 0</math> is a singular and also Regular singular point.</b>	S 2019 (4)
C	5	Find singularity of $x^3y'' + 5xy' + 3y = 0$ . <b>Answer: <math>x = 0</math> is an Irregular Singular Point.</b>	
H	6	Find singularity of $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ <b>Answer: <math>x = 1</math> &amp; <math>-1</math> are Regular Singular Points.</b>	
H	7	Find singularity of $x^3(x - 1)y'' + 3(x - 1)y' + 7xy = 0$ <b>Answer: <math>x = 1</math> is a Regular Singular Point &amp; <math>x = 0</math> is an Irregular Singular Point.</b>	

T	8	Find singularity of $(x^2 + 1)y'' + xy' - xy = 0$ . <b>Answer: <math>x = i, -i</math> are Regular Singular Points.</b>	
C	9	Find singularity of $2x(x - 2)^2y'' + 3xy' + (x - 2)y = 0$ . <b>Answer: <math>x = 0</math> is a Regular Singular Point &amp; <math>x = 2</math> is an Irregular Singular Point.</b>	
H	10	Find singularity of $x(x + 1)^2y'' + (2x - 1)y' + x^2y = 0$ . <b>Answer: <math>x = 0</math> is a Regular Singular Point &amp; <math>x = -1</math> is an Irregular Singular Point</b>	
T	11	$x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$ say true or false. <b>Answer: True</b>	
H	12	Classify the singular points of the equation $x^3(x - 2)y'' + x^3y' + 6y = 0$ <b>Answer: <math>x = 0</math> is an Irregular Singular Point &amp; <math>x = 2</math> is a Regular Singular Point.</b>	W 2019 (3)

#### ❖ POWER SERIES SOLUTION :

- ✓ A series solution of the below differential equation at an ordinary point is called power series solution.

$$r_0(x) \frac{d^2y}{dx^2} + r_1(x) \frac{dy}{dx} + r_2(x) y = 0$$

#### ❖ PROCEDURE FOR FINDING SERIES SOLUTION BY POWER SERIES METHOD:

- ✓ A power series solution of a differential equation

$$r_0(x) \frac{d^2y}{dx^2} + r_1(x) \frac{dy}{dx} + r_2(x) y = 0$$

at an ordinary point  $x_0 = 0$  can be obtained using the following steps.

- ✓ **Step-1:** Assume that below  $y$  is the solution of the given differential equation.

$$y = \sum_{k=0}^{\infty} a_k(x - x_0)^k = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

- ✓ **Step-2:** Differentiating  $y$  with respect to  $x$  we get,

$$\Rightarrow y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$\Rightarrow y'' = \frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$$

- ✓ **Step-3:** Substitute the expressions of  $y$ ,  $y'$ , and  $y''$  in the given differential equation.
- ✓ **Step-4:** Equate to zero the co-efficient of various powers of  $x$  and find  $a_2, a_3, a_4 \dots$  etc in terms of  $a_0$  and  $a_1$ .
- ✓ **Step-5:** Substitute the expressions of  $a_2, a_3, a_4, \dots$  in  $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$  which is required solution.

### METHOD – 2: EXAMPLES ON POWER SERIES METHOD

H	<b>1</b>	$y' + 2xy = 0$ . <b>Answer:</b> $a_0 - a_0x^2 + \frac{1}{2}a_0x^4 - \frac{1}{6}a_0x^6 + \dots$	
H	<b>2</b>	$y'' + y = 0$ . <b>Answer:</b> $a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 + \frac{1}{24}a_0x^4 + \frac{1}{120}a_1x^5 + \dots$	
C	<b>3</b>	$y'' + xy = 0$ in powers of $x$ . <b>Answer:</b> $a_0 + a_1x - \frac{1}{6}a_0x^3 - \frac{1}{12}a_1x^4 + \frac{1}{180}a_0x^6 + \dots$	W 2019 (7)
H	<b>4</b>	Find a power series solution of the differential equation $y'' - xy = 0$ near an ordinary point $x = 0$ . <b>Answer:</b> $a_0 + a_1x + \frac{1}{6}a_0x^3 + \frac{1}{12}a_1x^4 + \frac{1}{180}a_0x^6 + \dots$	S 2019 (7)
C	<b>5</b>	$y'' + x^2y = 0$ . <b>Answer:</b> $a_0 + a_1x - \frac{1}{12}a_0x^4 - \frac{1}{20}a_1x^5 + \frac{1}{672}a_0x^8 + \frac{1}{1440}a_1x^9 + \dots$	
H	<b>6</b>	$y'' = y'$ . <b>Answer:</b> $a_0 + a_1x + \frac{1}{2}a_1x^2 + \frac{1}{6}a_1x^3 + \frac{1}{24}a_1x^4 + \frac{1}{120}a_1x^5 + \dots$	

C	7	Find the power series solution about $x = 0$ of $y'' + xy' + x^2y = 0$ . <b>Answer:</b> $a_0 \left\{ 1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 + \dots \right\} + a_1 \left\{ x - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \dots \right\}$	
H	8	$y'' - 2xy' + 2py = 0$ . <b>Answer:</b> $a_0 + a_1x - p a_0x^2 + \frac{(1-p)}{3} a_1x^3 - \frac{p(2-p)}{6} a_0x^4 + \frac{(1-p)(3-p)}{30} a_1x^5 + \dots$	
T	9	$(1-x^2)y'' - 2xy' + 2y = 0$ . <b>Answer:</b> $a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 - \frac{1}{5}a_0x^6 + \dots$	
C	10	$\frac{d^2y}{dx^2}(1-x^2) - x\frac{dy}{dx} + py = 0$ . <b>Answer:</b> $a_0 + a_1x - \frac{p}{2} a_0x^2 + \frac{(1-p)}{6} a_1x^3 - \frac{p(4-p)}{24} a_0x^4 + \frac{(9-p)(1-p)}{120} a_1x^5 + \dots$	
C	11	$(1+x^2)y'' + xy' - 9y = 0$ . <b>Answer:</b> $a_0 + a_1x + \frac{9}{2}a_0x^2 + \frac{4}{3}a_1x^3 + \frac{15}{8}a_0x^4 - \frac{7}{16}a_0x^6 + \dots$	
H	12	$(x^2+1)y'' + xy' - xy = 0$ near $x = 0$ . <b>Answer:</b> $a_0 + a_1x + a_0\frac{x^3}{6} - a_1\frac{x^3}{6} + \left(\frac{a_1}{12}\right)x^4 - \left(\frac{3}{40}\right)a_0x^5 + \dots$	
H	13	$(x-2)y'' - x^2y' + 9y = 0$ . <b>Answer:</b> $a_0 \left( 1 + \frac{9x^2}{4} + \frac{9x^3}{24} + \frac{90x^4}{4} + \dots \right) + a_1 \left( x + \frac{18x^3}{24} + \frac{14x^4}{4} + \dots \right)$	
H	14	$(1-x^2)y'' - 2xy' + 2y = 0$ . <b>Answer:</b> $a_0 + a_1x - a_0x^2 - \frac{a_0}{3}x^4 + \dots$	

## ❖ FROBENIUS METHOD

Procedure for finding series solution by Frobenius method :

- ✓ A power series solution of a differential equation

$$r_0(x) \frac{d^2y}{dx^2} + r_1(x) \frac{dy}{dx} + r_2(x) y = 0$$

at a regular-singular point  $x_0$  can be obtained by using the following steps.

- ✓ **Step-1:** Assume that

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+k}$$

$$= (x - x_0)^k [ a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots ] \dots \dots \dots (1)$$

is the solution of the given differential equation.

- ✓ **Step-2:** Differentiating  $y$  with respect to  $x$  twice we get,

$$\Rightarrow y' = \frac{dy}{dx} = \sum_{n=0}^{\infty} (n+k) a_n (x - x_0)^{n+k-1}$$

$$\Rightarrow y'' = \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+k)(n+k-1) a_n (x - x_0)^{n+k-2}$$

- ✓ **Step-3:** Substitute the expressions of  $y$ ,  $y'$ , and  $y''$  in the given differential equation.

- ✓ **Step-4:** Equating to zero the coefficients of the lowest degree term in  $(x - x_0)$ , we obtained a quadratic equation in  $k$ , called the Indicial Equation of the given differential equation. The roots of the indicial equation are known as Indicial Roots.

- ✓ **Step-5:** Using the recurrence relation for each indicial root separately, two linearly independent solutions  $y_1(x)$  and  $y_2(x)$  of the given D.E. are obtained.

Therefore, the general solution of the given D.E. is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Where,  $C_1$  and  $C_2$  are arbitrary constants.

- ✓ Now, one of the solutions  $y_1(x)$  or  $y_2(x)$  is in the form of equation (1).  
 ✓ The form of the other solution depends upon the nature of the indicial roots. There are three cases:

#### Case-1: $k_1 - k_2 \neq \text{Integer}$

$$\text{Then } y_1 = (y)_{k=k_1} \text{ \& } y_2 = (y)_{k=k_2}$$

$$\text{The general solution is: } y = C_1 y_1 + C_2 y_2$$

**Case-2:** Repeated Roots i.e. When  $k_1 = k_2 = t$  (say)

$$\text{Then } y_1 = (y)_{k=t} \text{ \& } y_2 = (y)_{k=t}$$

$$\text{The general solution is: } y = C_1(y)_{k=t} + C_2 \left( \frac{\partial y}{\partial k} \right)_{k=t}$$

**Case-3:**  $k_1 - k_2$  differs by an integer i.e.  $k_1 - k_2 = \text{Integer}$  and  $k_1 < k_2$

In this case, solution corresponding to  $k_1$  and  $k_2$  may or may not be L.I. This leads to two possibilities:

(a) One of the coefficient becomes  $\infty$  for smaller indicial root  $k = k_1$

- The procedure is modified by putting  $a_0 = C_0(k - k_1)$ ,  $C_0 \neq 0$

$$\text{Then, } y_1 = (y)_{k=k_1}, \quad y_2 = \left( \frac{\partial y}{\partial k} \right)_{k=k_1}$$

The solution corresponding to second root  $k_2$  is usually multiple of  $y_1$  or a part of  $\left( \frac{\partial y}{\partial k} \right)_{k=k_1}$ .

Hence, it produces L.D. solution.

$$\text{The general solution is: } y = C_1(y)_{k=k_1} + C_2 \left( \frac{\partial y}{\partial k} \right)_{k=k_1}$$

(b) One of the coefficient becomes indeterminate for smaller indicial root  $k = k_1$ .

- This root produces the complete solution as it contains two arbitrary constants.

The second indicial root  $k_2$  produces L.D. solution.

### METHOD – 3: EXAMPLES ON FROBENIUS METHOD

H	1	$4xy'' + 2y' + y = 0.$ <b>Answer:</b> $A \left( 1 - \frac{x}{2} + \frac{x^2}{24} + \dots \right) + B\sqrt{x} \left( 1 - \frac{x}{6} + \frac{x^2}{120} - \dots \right); A = C_1 a_0,$ $B = C_2 a_0$	
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C	<b>2</b>	$x^2y'' + xy' - (2 - x)y = 0.$ <b>Answer:</b> $C_1a_0x^{\sqrt{2}} \left(1 - \frac{x}{2\sqrt{2} + 1} + \frac{x^2}{(1 + 2\sqrt{2})(4 + 2\sqrt{2})} + \dots\right)$ $+ C_2a_0x^{-\sqrt{2}} \left(1 - \frac{x}{1 - 2\sqrt{2}} + \frac{x^2}{(1 - 2\sqrt{2})(4 - 2\sqrt{2})} + \dots\right)$	
H	<b>3</b>	$8x^2y'' + 10xy' - (1 + x)y = 0.$ <b>Answer:</b> $Ax^{\frac{1}{4}} \left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \dots\right) + Bx^{-\frac{1}{2}} \left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \dots\right);$ $A = C_1a_0, B = C_2a_0$	
C	<b>4</b>	$2x^2y'' + x(2x + 1)y' - y = 0$ near $x = 0.$ <b>Answer:</b> $y = C_1x \left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \dots\right) + C_2x^{-\frac{1}{2}} \left(1 - x + \frac{1}{2}x^2 - \dots\right)$	
T	<b>5</b>	$x^2y'' + x^3y' + (x^2 - 2)y = 0$ about $x = 0.$ <b>Answer:</b> $y = \frac{A}{x} + Bx^2 \left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \dots\right); A = C_1a_0, B = C_2a_0$	
H	<b>6</b>	Find the series solution about $x = 0$ of $(x^2 - x)y'' - xy' + y = 0.$ <b>Answer:</b> $y = (A + B \log x)x + B(1 - 3x); A = C_1C_0, B = C_2C_0$	
T	<b>7</b>	$xy'' + 2y' + xy = 0.$ <b>Answer:</b> $y = \frac{a_0}{x} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)$	
C	<b>8</b>	Using Frobenius method, solve $x^2y'' + 4xy' + (x^2 + 2)y = 0.$ <b>Answer:</b> $y = \frac{1}{x^2} (A \cos x + B \sin x)$	W 2019 (7)
H	<b>9</b>	Find a Frobenius series solution of the differential equation $2x^2y'' + xy' - (x + 1)y = 0$ near a regular-singular point $x = 0.$ <b>Answer:</b> $y = a_0x \left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \dots\right) + a_1x^{-\frac{1}{2}} \left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 + \dots\right)$	S 2019 (7)





**LIST OF ASSIGNMENT**

ASSIGNMENT NO.	UNIT NO.	METHOD NO.
1	6B	Method – 1 (3, 4, 7) Method – 2 (4, 4, 8, 9, 12 )
2	1	Method – 2 (3, 8) Method – 3 (4, 6, 11) Method – 7 (2, 6, 10, 16, 21) Method – 9 (2, 5, 7, 10, 12, 16, 17, 19)
3	3	Method – 1 (3, 5)
4	4	Method – 2 (3, 4, 6, 15) Method – 4 (4, 5, 8, 9) Method – 5 (7, 10, 11)
5	5	Method – 2 (4, 7, 14, 17, 21, 23) Method – 5 (3, 8, 7, 9, 12, 14) Method – 6 (3, 5, 11)
6	2	Method – 4 (5, 9, 10) Method – 8 (3, 4, 8, 9) Method – 18 (2, 5, 7, 9, 10, 14) Method – 20 (3, 5, 7)





# GUJARAT TECHNOLOGICAL UNIVERSITY

Bachelor of Engineering

Subject Code: 3110015

SUBJECT NAME: Mathematics-2

1<sup>st</sup> Year

**Type of course:** Basic Science Course

**Prerequisite:** Calculus, fourier series

**Rationale:** To compute line integrals, solution techniques of higher order ordinary differential equations, fourier integral representation.

**Teaching and Examination Scheme:**

Teaching Scheme			Credits	Examination Marks				Total Marks
L	T	P		Theory Marks		Practical Marks		
				ESE (E)	PA (M)	ESE (V)	PA (I)	
3	2	0	5	70	30	0	0	100

Sr. No.	Content	Total Hrs	% Weightage
01	Vector Calculus: Parametrization of curves, Arc length of curve in space, Line Integrals, Vector fields and applications as Work, Circulation and Flux, Path independence, potential function, piecewise smooth, connected domain, simply connected domain, fundamental theorem of line integrals, Conservative fields, component test for conservative fields, exact differential forms, Div, Curl, Green's theorem in the plane (without proof).	9	20
02	Laplace Transform and inverse Laplace transform, Linearity, First Shifting Theorem (s-Shifting), Transforms of Derivatives and Integrals, ODEs, Unit Step Function (Heaviside Function), Second Shifting Theorem (t-Shifting), Laplace transform of periodic functions, Short Impulses, Dirac's Delta Function, Convolution, Integral Equations, Differentiation and Integration of Transforms, ODEs with Variable Coefficients, Systems of ODEs.	7	20
03	Fourier Integral, Fourier Cosine Integral and Fourier Sine Integral.	02	
04	First order ordinary differential equations, Exact, linear and Bernoulli's equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.	6	14
05	Ordinary differential equations of higher orders, Homogeneous Linear ODEs of Higher Order, Homogeneous Linear ODEs with Constant Coefficients, Euler-Cauchy Equations, Existence and Uniqueness of Solutions, Linear Dependence and Independence of Solutions, Wronskian, Nonhomogeneous ODEs, Method of Undetermined Coefficients, Solution by Variation of Parameters.	10	26
06	Series Solutions of ODEs, Special Functions, Power Series Method, Legendre's Equation, Legendre Polynomials, Frobenius Method, Bessel's Equation, Bessel functions of the first kind and their properties.	8	20



# GUJARAT TECHNOLOGICAL UNIVERSITY

Bachelor of Engineering

Subject Code: 3110015

Suggested Specification table with Marks (Theory):

Distribution of Theory Marks					
R Level	U Level	A Level	N Level	E Level	C Level
10	25	35	0	0	0

Legends: R: Remembrance; U: Understanding; A: Application, N: Analyze and E: Evaluate C: Create and above Levels (Revised Bloom's Taxonomy)

## Reference Books:

- (1) Erwin Kreyszig, Advanced Engineering Mathematics, 10th Edition, John Wiley and Sons.
- (2) Peter O'Neill, Advanced Engineering Mathematics, 7th Edition, Cengage.
- (3) Dennis G. Zill, 4th edition, Advanced Engineering Mathematics, 4th Edition, Jones and Bartlett Publishers.
- (4) Maurice D. Weir, Joel Hass, Thomas' Calculus, Early Transcendentals, 13e, Pearson, 2014.
- (5) Howard Anton, Irl Bivens, Stephens Davis, Calculus, 10e, Wiley, 2016.

## Course Outcomes:

The objective of this course is to familiarize the prospective engineers with techniques in vector calculus, ordinary differential equations, fourier integrals and laplace transform. It aims to equip the students to deal with advanced level of mathematics and applications that would be essential for their disciplines.

Sr. No.	Course Outcomes	Weightage in %
1	To apply mathematical tools needed in evaluating vector calculus and their usage like Work, Circulation and Flux.	20
2	To apply the laplace transform as tools which are used to solve differential equations and fourier integral representation.	20
3	To apply effective mathematical tools for the solutions of first order ordinary differential equations.	14
4	To apply effective mathematical methods for the solutions of higher order ordinary differential equations.	26
5	To use series solution methods and special functions like Bessels' functions.	20

## List of Open Source Software/learning website:

Scilab, MIT Opencourseware.

**GUJARAT TECHNOLOGICAL UNIVERSITY****BE - SEMESTER-I & II (NEW) EXAMINATION – SUMMER-2019****Subject Code: 3110015****Date: 01/06/2019****Subject Name: Mathematics –2****Time: 10:30 AM TO 01:30 PM****Total Marks: 70****Instructions:**

1. Attempt all questions.
2. Make suitable assumptions wherever necessary.
3. Figures to the right indicate full marks.

			Marks
Unit-3	Q.1	(a) Find the Fourier integral representation of $f(x) = \begin{cases} x & ; x \in (0, a) \\ 0 & ; x \in (a, \infty) \end{cases}$	03
Unit-2	(b)	Define: Unit step function. Use it to find the Laplace transform of $f(t) = \begin{cases} (t-1)^2 & ; t \in (0, 1] \\ 1 & ; t \in (1, \infty) \end{cases}$	04
Unit-5	(c)	Use the method of undetermined coefficients to solve the differential equation $y'' - 2y' + y = x^2 e^x$ .	07
Unit-1	Q.2	(a) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ ; where $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ and $C$ is the curve given by the parametric equation $C : r(t) = t^2\hat{i} + t\hat{j}; 0 \leq t \leq 2$ .	03
	(b)	Apply Green's theorem to find the outward flux of a vector field $\vec{F} = \frac{1}{xy}(x\hat{i} + y\hat{j})$ across the curve bounded by $y = \sqrt{x}$ , $2y = 1$ and $x = 1$ .	04
	(c)	Integrate $f(x, y, z) = x - yz^2$ over the curve $C = C_1 + C_2$ , where $C_1$ is the line segment joining (0,0,1) to (1,1,0) and $C_2$ is the curve $y=x^2$ joining (1,1,0) to (2,4,0).	07
<b>OR</b>			
	(c)	Check whether the vector field $\vec{F} = e^{y+2z}\hat{i} + x e^{y+2z}\hat{j} + 2x e^{y+2z}\hat{k}$ is conservative or not. If yes, find the scalar potential function $\varphi(x, y, z)$ such that $\vec{F} = \text{grad } \varphi$ .	07
Unit-4	Q.3	(a) Write a necessary and sufficient condition for the differential equation $M(x, y)dx + N(x, y)dy = 0$ to be exact differential equation. Hence check whether the differential equation $[(x+1)e^x - e^y]dx - xe^y dy = 0$ is exact or not.	03
Unit-4	(b)	Solve the differential equation $(1 + y^2)dx = (e^{-\tan^{-1}y} - x)dy$	04
Unit-2	(c)	By using Laplace transform solve a system of differential equations $\frac{dx}{dt} = 1 - y$ , $\frac{dy}{dt} = -x$ , where $x(0) = 1, y(0) = 0$ .	07
<b>OR</b>			
Unit-4	Q.3	(a) Solve the differential equation $(2x^3 + 4y)dx - xdy = 0$ .	03

<b>Unit-4</b>	(b)	Solve: $(x + 1) \frac{dy}{dx} - y = e^{3x}(x + 1)^2$ .	<b>04</b>
<b>Unit-2</b>	(c)	By using Laplace transform solve a differential equation $\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = e^{-t}$ , where $y(0) = 0$ , $y'(0) = -1$ .	<b>07</b>
<b>Unit-4</b>	<b>Q.4</b> (a)	Find the general solution of the differential equation $e^{-y} \frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2}$	<b>03</b>
<b>Unit-5</b>	(b)	Solve: $\frac{d^3y}{dx^3} - 7 \frac{dy}{dx} + 6y = e^x$	<b>04</b>
<b>Unit-6</b>	(c)	Find a power series solution of the differential equation $y'' - xy = 0$ near an ordinary point $x=0$ .	<b>07</b>
<b>OR</b>			
<b>Unit-4</b>	<b>Q.4</b> (a)	Find the general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} - \sqrt{y} = 0$ .	<b>03</b>
<b>Unit-5</b>	(b)	Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = x$	<b>04</b>
<b>Unit-6</b>	(c)	Find a Frobenius series solution of the differential equation $2x^2y'' + xy' - (x + 1)y = 0$ near a regular-singular point $x=0$ .	<b>07</b>
<b>Unit-6</b>	<b>Q.5</b> (a)	Write Legendre's polynomial $P_n(x)$ of degree- $n$ and hence obtain $P_1(x)$ and $P_2(x)$ in powers of $x$ .	<b>03</b>
<b>Unit-6</b>	(b)	Classify ordinary points, singular points, regular-singular points and irregular-singular points (if exist) of the differential equation $y'' + xy' = 0$ .	<b>04</b>
<b>Unit-5</b>	(c)	Solve the differential equation $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3 \cos x$ by using the method of variation of parameters.	<b>07</b>
<b>OR</b>			
<b>Unit-6</b>	<b>Q.5</b> (a)	Write Bessel's function $J_p(x)$ of the first kind of order- $p$ and hence show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ .	<b>03</b>
<b>Unit-6</b>	(b)	Classify ordinary points, singular points, regular-singular points and irregular-singular points (if exist) of the differential equation $xy'' + y' = 0$ .	<b>04</b>
<b>Unit-5</b>	(c)	Solve the differential equation $y'' + 25y = \sec 5x$ by using the method of variation of parameters.	<b>07</b>

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**GUJARAT TECHNOLOGICAL UNIVERSITY****BE - SEMESTER– I & II (NEW) EXAMINATION – WINTER 2019****Subject Code: 3110015****Date: 01/01/2020****Subject Name: Mathematics –2****Time: 10:30 AM TO 01:30 PM****Total Marks: 70****Instructions:**

1. Attempt all questions.
2. Make suitable assumptions wherever necessary.
3. Figures to the right indicate full marks.

		Marks
<b>Unit-1</b>	<b>Q.1 (a)</b> Find the length of curve of the portion of the circular helix $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$ from $t = 0$ to $t = \pi$	<b>03</b>
	<b>(b)</b> $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ is independent of path joining the points (1, 2) and (3,4). Hence, evaluate the integral.	<b>04</b>
	<b>(c)</b> Verify tangential form of Green's theorem for $\vec{F} = (x - \sin y)\hat{i} + (\cos y)\hat{j}$ , where C is the boundary of the region bounded by the lines $y = 0, x = \pi/2$ and $y = x$ .	<b>07</b>
<b>Unit-2</b>	<b>Q.2 (a)</b> Find the Laplace transform of $f(t)$ defined as $f(t) = \begin{cases} \frac{t}{k} & 0 < t < k \\ 1 & t > k \end{cases}$	<b>03</b>
<b>Unit-2</b>	<b>(b)</b> Find the inverse Laplace transform of $\frac{\epsilon^2}{(s^2 + a^2)(s^2 + b^2)}$	<b>04</b>
<b>Unit-1</b>	<b>(c)</b> (i) Calculate the curl of the vector $xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ (ii) The temperature at any point in space is given by $T = xy + yz + zx$ . Determine the derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point (1, 1, 1).	<b>07</b>
	<b>OR</b>	
<b>Unit-1</b>	<b>(c)</b> Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , $r =  \vec{r} $ , and $\vec{a}$ is a constant vector. Find the value of $\text{div} \left( \frac{\vec{a} \times \vec{r}}{r^2} \right)$	<b>07</b>
<b>Unit-1</b>	<b>Q.3 (a)</b> Find constants a, b and c such that $\vec{V} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational.	<b>03</b>
<b>Unit-3</b>	<b>(b)</b> Using Fourier cosine integral representation show that $\int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi e^{-kx}}{2k}$	<b>04</b>
<b>Unit-4</b>	<b>(c)</b> Solve the following differential equations: (i) $\cos(x + y) dy = dx$ (ii) $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$	<b>07</b>

**OR**

<b>Unit-2</b>	<b>Q.3 (a)</b>	Find the Laplace transform of (i) $\int_0^t \frac{\sin t}{t} dt$ (ii) $t^2 u(t-3)$	<b>03</b>
<b>Unit-2</b>	<b>(b)</b>	Using Convolution theorem obtain $L^{-1}\left(\frac{1}{s(s^2 + a^2)}\right)$	<b>04</b>
<b>Unit-6</b>	<b>(c)</b>	Find the power series solution of $\frac{d^2 y}{dx^2} + xy = 0$	<b>07</b>
<b>Unit-2</b>	<b>Q.4 (a)</b>	Find the Laplace transform of the waveform $f(t) = \left(\frac{2t}{3}\right), 0 \leq t \leq 3$	<b>03</b>
<b>Unit-2</b>	<b>(b)</b>	Using the Laplace transforms, find the solution of the initial value problem $y'' + 25y = 10 \cos 5t \quad y(0) = 2, y'(0) = 0$	<b>04</b>
<b>Unit-5</b>	<b>(c)</b>	Using variation of parameter method solve $(D^2 + 1)y = x \sin x$	<b>07</b>
<b>OR</b>			
<b>Unit-5</b>	<b>Q.4 (a)</b>	Solve $y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$	<b>03</b>
	<b>(b)</b>	Solve $y'' - 3y' + 3y - y = 4e^t$	<b>04</b>
	<b>(c)</b>	Solve $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$ using method of undetermined coefficients.	<b>07</b>
<b>Unit-6</b>	<b>Q.5 (a)</b>	Classify the singular points of the equation $x^3(x-2)y' + x^3y' + 6y = 0$	<b>03</b>
<b>Unit-5</b>	<b>(b)</b>	Solve $(D^2 + 4)y = \cos 2x$	<b>04</b>
<b>Unit-4</b>	<b>(c)</b>	Solve (i) $ye^x dx + (2y + e^x)dy = 0$ (ii) $\frac{dy}{dx} + 2y \tan x = \sin x$	<b>07</b>
<b>OR</b>			
<b>Unit-4</b>	<b>Q.5 (a)</b>	Solve $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$	<b>03</b>
<b>Unit-5</b>	<b>(b)</b>	If $y_1 = x$ is one of solution of $x^2 y' + xy' - y = 0$ find the second solution.	<b>04</b>
<b>Unit-6</b>	<b>(c)</b>	Using Frobenius method solve $x^2 y' + 4xy' + (x^2 + 2)y = 0$	<b>07</b>

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