Decisions in Economics and Finance
https://doi.org/10.1007/s10203-020-00295-7

Efficiency evaluation under uncertainty: a stochastic DEA approach

P. Beraldi[1] **· M. E. Bruni**[1]

Received: 4 September 2019 / Accepted: 16 July 2020

Abstract
In conventional data envelopment analysis (DEA) models, the efficiency measurement is carried out by using deterministic data typically referring to past observations. However, in many operative contexts, decision makers are called to predict the future performance for planning and control purposes. In these situations, ignoring the stochastic nature of data might lead to misleading results. The paper proposes a stochastic DEA approach based on the chance constrained paradigm and accounts for risk measured in terms of tail γ-mean. A deterministic equivalent reformulation is presented under the assumption of discrete distributions. The computational experiments are carried out on an empirical case study related to the evaluation of the credit risk. The results demonstrate the validity of the proposed approach as proactive evaluation technique.

Keywords Data envelopment analysis · Probabilistic constraints · Firm efficiency evaluation · Tail measures

JEL Classification C6

1 Introduction

Efficiency measurement is a critical issue for any type of business or organization. It allows to quantify the performance of a company in its creating-value activities in comparison with other competitors. Since its introduction by Charnes et al. (1978), the data envelopment analysis (DEA) has been recognized as an effective approach for

✉ P. Beraldi
patrizia.beraldi@unical.it

M. E. Bruni
mariaelena.bruni@unical.it

[1] Department of Mechanical, Energy and Management Engineering, University of Calabria, Cosenza, Rende, Italy

Published online: 13 August 2020

measuring the relative efficiency of a homogeneous set of production units (termed as decision making units—DMUs) in converting multiple resources, "inputs", into multiple outcomes, "outputs." The DMUs are ranked according to the efficiency level, by definition a value between 0 and 1, identifying the best performers (with score 1) and eventually suggesting the under-performers possible improvement strategies.

Conventional DEA models are designed to measure past performance, using historical observations of the inputs consumed and outputs produced. The evaluation carried out on deterministic data, assumed to be known in advance, might lead to misleading results, especially when future performance must be predicted for planning and control purposes. Indeed, new observations could be very different from the old ones, especially when DMUs operate in high volatile and competitive environments. In all these situations, ignoring data uncertainty could lead to biased estimates owing to the extreme point nature of the DEA technique which makes efficiency measurement sensible to data changes.

As classified by Olesen and Petersen (2016), there are two main streams in the DEA literature that address the issue of efficiency measurement with random data. While the former handles deviations from the production frontier as random (see, for example, the representative work of Rajiv and Banker 1993), the latter considers random data with specific distributions. In this second case, the corresponding DEA model becomes a stochastic programming problem (stochastic DEA–SDEA). The approach proposed in this paper belongs to the second stream.

Most of the contributions on the SDEA rely on the paradigm of chance constraints introduced by Charnes and Cooper (1959). In this setting, the violation of the stochastic constraints is allowed provided that it occurs with a low probability level.

Over the last decades, several stochastic formulations have been proposed differing for the starting deterministic DEA model (envelopment vs. multiplier form), for the nature of the chance constraints (separate vs. joint) and for the choice of the objective function (E-model vs. P-model). For example, Land et al. (1993) analyzed the case of deterministic inputs and random outputs and proposed a model where separate chance constraints are imposed on the DEA envelopment form. The model considers the maximization of the expected efficiency level (E-model). Later on, Cooper et al. (2004) proposed a chance constrained DEA E-model modified to address congestion. Olesen and Petersen (1995) developed an E-model where separate chance constraints are imposed on the DEA multiplier form. Other chance constrained DEA formulations consider the maximization of the probability (P-model) of stochastic events related to inputs and outputs. For example, Cooper et al. (1996) formulated a P-model with separate chance constraints on the efficiency level of each DMU. Additional aspects of the SDEA have been also investigated in Sengupta (1987), Sueyoshi (2000) and Chang et al. (2016) to name a few. We also mention the very recent contribution by Kao and Liu (2019) (see also the references therein), where the authors propose a stochastic model which is able to consider the correlation between the random input and output factors. They still assume that the random variables follow a normal distribution and use classical standardization techniques. The repeated solution of the model is integrated within a simulation approach so to derive the distribution of the random efficiency scores on which statistical elaborations can be carried out.

All the aforementioned contributions rely on the assumption that the random variables follow the Gaussian distribution. In the case of separate chance constraints, this hypothesis allows to derive a deterministic equivalent reformulation of the corresponding models by adopting a classical standardization technique. The resulting models present a standard deviation term accounting for variability and, thus, belong to the class of nonlinear problems.

The more involved case of joint chance constraints can be still deterministically reformulated in the independent case as shown in Cheng and Lisser (2012). Interested readers are referred to Chen and Zhu (2019) for a general discussion on the computational tractability of chance constrained DEA models.

In contrast to the case of continuous random variables, the discrete case has been less investigated in the scientific literature. Among the few contributions, we cite Bruni et al. (2009) where the authors propose a stochastic formulation of a DEA model in the envelopment form. Here, under the same assumption on the random variable, propose a joint chance constrained formulation of the DEA model in the *multiplier form*.

Besides this distinctive feature, the proposed model accounts for risk, a topic of critical importance in many application domains. We point out that only few papers, always designed for the case of Gaussian distributed random variables, explicitly address the risk issue. For example, following the traditional mean–variance approach (Markowitz 1952; Post 2001) proposed a stochastic formulation where the objective function includes a correction term, accounting for the variance, to be minimized. Wu and Olson (2010) presented a stochastic DEA model for vendors evaluation that integrates the value at risk (VaR) measure (Rockafellar and Uryasev 2000). Being a quantile, the VaR can be still represented as chance constraint and as such, rewritten in a deterministic equivalent form. Later on, Wei et al. (2014) presented a stochastic formulation with a reliability constraint that is related to the VaR. The model aims at determining the highest efficiency level that a given DMU can achieve for a fixed reliability value.

Although being intuitively appealing, the VaR suffers for many deficiencies, among which the lack of handling the losses that may be incurred beyond the threshold indicated by this measure. For compensating these drawbacks, the companion measure CVaR measure introduced by Rockafellar and Uryasev (2000) has been proposed: It allows to control the expected value of the losses exceeding the VaR with a given probability level.

In the proposed model, we do not consider losses, but, more naturally, we focus on the random efficiency levels. In particular, we look at the tail of the efficiency density function with the aim of controlling the worst-case efficiency outcomes. We consider a "safety" measure (to be maximized) to control the average performance that can be achieved in the $\gamma\%$ of worst case realizations (see, for example, Ogryczak 2014). We note that a preliminary attempt to introduce this measure in a scenario-based DEA formulation appeared in the conference paper (Beraldi and Bruni 2012).

To sum up, the paper contributes to the scientific literature on the SDEA by: (a) proposing a chance constrained formulation for the general case of discrete random variables, (b) explicitly dealing with risk by introducing the γ-tail mean safety measure.

The rest of the paper is organized as follows: Section 2 introduces the proposed stochastic model, which is detailed in Sect. 3 for the case of discrete random variables. Section 4 reports on the computational experiments carried out on a case study and discusses the use of the approach as proactive tool to assist the decision maker in the evaluation process. Some final considerations and remarks are presented in Sect. 5.

2 Model development

Let us assume to have L DMUs to evaluate and that each DMU l uses m types of inputs to produce n types of outputs. We denote by $x_{il}, i = 1, \ldots, m$, and by $y_{jl}, j = 1, \ldots, n$, the ith input and jth output of the lth DMU. For each DMU, the efficiency level is computed as the weighted sum of the outputs over the inputs. By using the model proposed by Charnes et al. (1978), the relative efficiency of a DMU k, denoted by φ_k, can be determined by solving the model:

$$\varphi_k = \max \frac{\sum_{j=1}^{n} y_{jk} u_j}{\sum_{i=1}^{m} x_{ik} w_i} \tag{1}$$

$$\frac{\sum_{j=1}^{n} y_{jl} u_j}{\sum_{i=1}^{m} x_{il} w_i} \leq 1 \quad l = 1, \ldots, L \tag{2}$$

$$w_i, u_j \geq \epsilon \quad i = 1, \ldots, m \;\; j = 1, \ldots, n \tag{3}$$

Here, w_i and u_j represent the weights associated with the input i and output j, respectively, and ϵ is a non-Archimedean number used to impose that all the input and output factors should be considered in the evaluation. The model is run for every DMU and on the basis of the optimal objective functions, DMUs are ranked. Efficient DMUs are those achieving a score equal to 1. The fractional model (1)–(3) can be transformed into a linear programming problem by applying a classical variable substitution technique:

$$\varphi_k = \max \sum_{j=1}^{n} y_{jk} u_j \tag{4}$$

$$\sum_{i=1}^{m} x_{ik} w_i = 1 \tag{5}$$

$$\sum_{j=1}^{n} y_{jl} u_j - \sum_{i=1}^{m} x_{il} w_i \leq 0 \quad l = 1, \ldots, L \tag{6}$$

$$w_i, u_j \geq \epsilon \quad i = 1, \ldots, m \;\; j = 1, \ldots, n \tag{7}$$

The injection of data uncertainty strengthens the efficacy of the DEA as performance evaluation approach and makes the problem more challenging. As common in other SDEA formulations, hereafter, we assume that the input data are deterministic, whereas outputs are uncertain and represented as random variables defined on a given probability space $(\Omega, \Im, \mathbb{P})$. In particular, we denote by $y_{jl}(\omega)$ the jth random output level of DMU l. The introduction of stochastic data modifies the nature of the problem and requires a proper redefinition of the feasibility and optimality conditions.

We handle the stochastic constraints by the paradigm of chance constraints. More formally (6) are replaced by

$$\mathbb{P}\left(\sum_{j=1}^{n} y_{jl}(\omega)u_j - \sum_{i=1}^{m} x_{il}w_i \le 0 \quad l = 1, \ldots, L\right) \ge \alpha \tag{8}$$

where α denotes a given reliability level on the constraints satisfaction.

As highlighted in Sect. 1, most of the contributions presented in the scientific literature deal with (8) under the assumptions of Gaussian distribution and of separate chance constraints individually imposed on each DMU l (eventually with a different reliability level α_l). In this case, a deterministic equivalent reformulation can be derived by applying the classical standardization technique:

$$\sum_{j=1}^{n} \overline{y}_{jl}u_j - \sum_{i=1}^{m} x_{il}w_i + \Phi^{-1}(\alpha_l)\sqrt{\sum_{j=1}^{n}\sum_{r=1}^{n} \sigma^l_{jr}u_ju_r} \le 0 \quad l = 1, \ldots, L \tag{9}$$

where for each DMU l, $\overline{y}_{jl}$ represents the expected value of jth output and σ^l_{jr} denotes the covariance between the outputs j and r. Finally, $\Phi^{-1}(\alpha_l)$ denotes the α_l-quantile of the standard normal distribution function.

In this paper, we assume that the random variables are discretely distributed. We note that discrete distributions arise in many real-world applications either naturally or as empirical approximation of continuous ones derived, for example, by taking Monte Carlo samples from general distributions. Moreover, under this assumption a deterministic equivalent reformulation can be derived for the more general case of joint probabilistic constraints that are not necessarily independent, as detailed in Sect. 3.

Besides the constraints, uncertainty should be handled into the objective function too. While traditionally, the expected value (E-model) or the most probable value (P-model) of the efficiency level has been used in the SDEA models, the proposed formulation includes a risk measure. In particular, we focus on the tail of the efficiency distribution with the aim of controlling the expected efficiency level that can be achieved for a specified size (quantile) γ of worst case realizations. More formally, the considered measure, known as tail γ-mean (Ogryczak and Ruszczynski 2002a), can be defined as

$$\mathbb{E}_\gamma[\varphi(\omega)] = \mathbb{E}[\varphi(\omega)|\varphi(\omega) \le F^{(-1)}_\varphi(\gamma)] \tag{10}$$

where for easy of notation, we omit the subscript k by $\varphi(\omega)$. In (10), $\mathbb{E}[\cdot]$ denotes the expected value operator and $F^{(-1)}_\varphi(\gamma)$ is the γ-quantile of the efficiency cumulative distribution function, i.e., $F_\varphi(\eta) = \mathbb{P}(\varphi(\omega) \le \eta)$.

We note that in contrast to the deviation measures that are minimized with the aim of controlling the loss, the tail γ-mean considered here should be maximized.

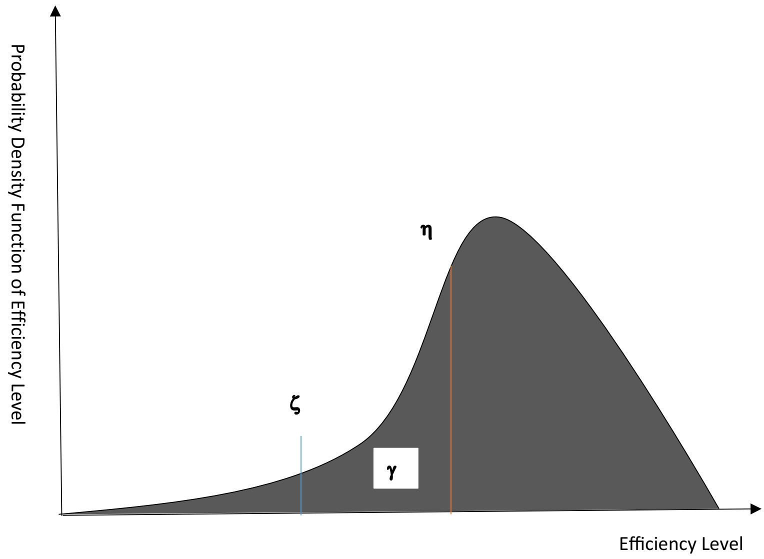

Fig. 1 Probability density function of the efficiency level

Under this respect (10) can be viewed as a safety measure. Ogryczak and Ruszczynski (2002b) showed that the tail mean maximization is consistent with the second-degree stochastic dominance and it is coherent in the terminology of Artzner et al. (1999). Following these results, it is possible to show that (10) is related to the second quantile function and that

$$\max \mathbb{E}_{\gamma}[\varphi(\omega)] = \max_{\eta \in R} \left(\eta - \mathbb{E}\left[\eta - \frac{1}{\gamma}\varphi(\omega) \right]^{+} \right) \tag{11}$$

where $[\eta - \frac{1}{\gamma}\varphi(\omega)]^{+}$ denotes the $\max\{0, \eta - \frac{1}{\gamma}\varphi(\omega)\}$ and η is a real variable taking the value of the γ-quantile at the optimum.

Figure 1 shows the probability density function of the efficiency level of a given DMU and the values of the γ-quantile (η) and γ-mean (ζ) for a given reliability value γ.

The figure is helpful to highlight that depending on the choice of γ, the considered measure can be used to represent a broad spectrum of preferences, from the most risk averse, for γ equal to 0 to the most neutral, for γ equal to 1. Indeed, it is evident that:

$$\lim_{\gamma \to 1} \mathbb{E}_{\gamma}[\varphi(\omega)] = \mathbb{E}[\varphi(\omega)]$$
$$\lim_{\gamma \to 0} \mathbb{E}_{\gamma}[\varphi(\omega)] = \inf[\varphi(\omega)]$$

The proposed SDEA model for the efficiency evaluation is defined by the objective function (11) and by the constraints (5), (7), (8). The model contains two probability values α and γ that should be properly selected by the decision maker. We point out that the choice of α reflects the reliability level imposed on the satisfaction of the stochastic constraints that in the proposed model refer to the efficiency level of all the DMUs under investigation. In the extreme case of α equal to 1, the solution provided by the model will result feasible under all the circumstances that might occur. The choice of the value γ used in the objective function is related to that of α since the objective function considers itself the efficiency level (because of constraint (5)). In order to make the definition of the stochastic efficiency consistent with its deterministic counterpart, the value of γ should be less than (equal) to the α one, since the efficiency levels associated with "violated scenarios" should be discarded. We denote by $\zeta_k^* = 1$ the optimal value of proposed model corresponding to the γ-mean and by η_k^* the quantile. We introduce the following definition.

Definition Given α and γ, with $\gamma \leq \alpha$, a DMU k under evaluation is called:

1. *strong* stochastic efficient at level γ if and only if $\zeta_k^* = 1$;
2. *weak* stochastic efficient at level γ if and only if $\eta_k^* = 1$;
3. stochastic inefficient if $\eta_k^* < 1$.

Strong efficiency occurs when $\zeta_k^* = 1$ and, thus, the efficiency level achieved for all the γ-quantile of worst realizations is 1. When the ζ_k^* is less than 1, but $\eta_k^* = 1$ the DMU under investigation is deemed as weak efficient. In this case, the best among the γ worst realizations will take value 1 and if $\gamma < \alpha$, the DMU will be weak efficient with probability equal to γ. On the contrary, when none of the DMUs is stochastic efficient, on the basis of the ζ^* values associated with the different DMUs, we may identify the pseudo-efficient DMUs as those for which the ζ^* is highest.

In the following section, we show how the proposed stochastic model can reformulated in the case of discrete distributions.

3 The proposed formulation

We assume that the random variables $y_{jl}(\omega)$ can take a finite number S of realizations (scenarios), denoted by y_{jl}^s, each occurring with probability $\pi_s \geq 0$ and such that $\sum_{s=1}^S \pi_s = 1$. Under this assumption, the probabilistic constraints (8) can be rewritten as

$$\sum_{s=1}^{S} \pi_s \chi \left(\sum_{j=1}^{n} y_{jl}^s u_j - \sum_{i=1}^{m} x_{il} w_i \right) \geq \alpha \tag{12}$$

where $\chi(.)$ represents the characteristic function that takes the value 1 if

$$\sum_{j=1}^{n} y_{jl}^s u_j - \sum_{i=1}^{m} x_{il} w_i \leq 0 \quad \forall l$$

and 0 otherwise. A natural approach to rewrite (12) is to consider "big-M" ' constraints and use supporting binary variables z_s associated with every scenario s. More specifically,

$$\sum_{j=1}^{n} y_{jl}^{s} u_j - \sum_{i=1}^{m} x_{il} w_i - M z_s \leq 0 \quad l = 1, \ldots, L \;\; s = 1, \ldots, S \tag{13}$$

$$\sum_{s=1}^{S} \pi_s z_s \leq (1 - \alpha) \tag{14}$$

$$z^s \in \{0, 1\} \quad s = 1, \ldots, S \tag{15}$$

Here, M is a real number large enough to ensure that when $z_s = 1$, constraints (13) are not active, the opposite when z_s is 0. Constraint (14) is a binary knapsack restriction that limits to $(1 - \alpha)$ the violation of the stochastic constraints.

Under the considered assumption, the objective function (11) can be rewritten as:

$$\max \; \mathbb{E}_{\gamma}[\varphi_k(\omega)] = \max \; \eta - \frac{1}{\gamma} \sum_{s=1}^{S} \pi_s [\eta_k - \varphi_k^s]^+ \tag{16}$$

where $\varphi_k^s = \sum_{j=1}^{n} y_{jk}^s u_j$ denotes the efficiency level of the DMU k under the scenario s. The nonlinear term in (16) can be easily linearized by including S auxiliary variables and S constraints. In particular, for each scenario s, a non-negative decision variable δ_s is introduced to measure the positive deviation between the efficiency level of the DMU k and the γ-quantile η_k:

$$\delta_s \geq \eta_k - \sum_{j=1}^{n} y_{jk}^s u_j \tag{17}$$

The overall model can be stated as follows:

$$\zeta_k = \max \quad \eta_k - \frac{1}{\gamma} \sum_{s=1}^{S} \pi_s \delta_s \tag{18}$$

$$\sum_{i=1}^{m} x_{ik} w_i = 1 \tag{19}$$

$$\sum_{j=1}^{n} y_{jl}^{s} u_j - \sum_{i=1}^{m} x_{il} w_i - M z_s \leq 0 \quad l = 1, \ldots, L \quad s = 1, \ldots, S \tag{20}$$

$$\sum_{s=1}^{S} \pi_s z_s \leq (1 - \alpha) \tag{21}$$

$$\delta_s \geq \eta_k - \sum_{j=1}^{n} y_{jk}^s u_j \quad s = 1, \ldots, S \tag{22}$$

$$\delta_s \geq 0 \quad s = 1, \ldots, S \tag{23}$$

$$w_i, u_j \geq \epsilon \quad i = 1, \ldots, m \quad j = 1, \ldots, n \tag{24}$$

$$z_s \in \{0, 1\} \quad s = 1, \ldots, S \tag{25}$$

Model (18)–(25) belongs to the class of mixed-integer linear problems, and depending on the number of considered scenarios, its solution can be computationally demanding. Over the last decades, different approaches (both exact and heuristic) have been proposed to solve problems under probabilistic constraints involving discrete distributions (see, for example, Beraldi and Bruni 2010; Beraldi et al. 2013). However, for the test cases considered in the paper, the computational time is still limited and off-the-shelf solvers can be used.

4 Computational study

This section reports on the computational experiments carried out to assess the proposed approach. The lack of available test cases (only few are available for the case of continuous random variables) has challenged us to design a test case that could be meaningful from an applicative standpoint. The illustrative example and the associated numerical results are introduced and discussed in the next subsections.

4.1 The illustrative example

The illustrative example was inspired by the flourishing literature on the application of the DEA approach for the evaluation and management of the credit risk (see, for example, Beraldi et al. 2014; Iazzolino et al. 2013; Paradi et al. 2004; Premachandra et al. 2011, and the references therein). For example, the DEA score can be used in peer group analysis, as an input in the credit rating models, as benchmarking for credit ratings that meet the Basel regulation. In addition, the efficiency level could be interpreted as "early warning signal" to indicate expected corporate credit failure or a "business opportunity signal" to identify where large profits can be made or indicate where to focus the business in the future. In the following, we shall consider the case of bank operator who wants to evaluate a set of companies with the aim of deciding whether to grant a loan. In addition to other traditional techniques, the decision maker may wish to apply the DEA approach with the aim of deriving a sort of scale merit by comparing similar companies that operate in the same sector. The results provided by the problem solution can be thus interpreted as "warning" signals to indicate expected credit failure.

As in any DEA application, the choice of the input and output parameters represents an important issue, even more critical, in this contest, for the necessity of taking into account their uncertain nature influenced by the continuing turmoil of the financial markets.

In designing our test case, we have considered one input, associated with the total liabilities which is assumed to be known, and two random outputs, i.e., earnings before interest, taxes, depreciation and amortization and cash flow. The first output parameter concerns the ability to make profit by the firm at an operating level, whereas the second one is the simplest form of cash flow. We have considered a set of 18 DMUs corresponding to medium enterprises belonging to the Italian leather manufacturing and wholesale industry. The data required to perform the comparative evaluation have been elaborated from the Balance Sheets, written according to the Italian Civil Code, for the time horizon 2001–2008. For the considered firms, all the parameters take non-negative values. The uncertain output values (for the incoming year 2009) have been represented by a finite set of possible realizations (scenarios) that have been generated by adopting the two-phase approach proposed in Beraldi et al. (2010). In particular, in the first phase, the realizations of the uncertain parameters are generated by the Monte Carlo simulation technique. In the second phase, an optimization model is solved with the aim of the determining the optimal probability values that minimize the distance between the moments of the distribution associated with the generated scenarios and some target moments defined by the end-user on the basis of the historical data and/or his preferences. Details can be found in Beraldi and Bruni (2012) (see also Beraldi and Bruni 2014). The results presented in the next subsection have been collected by considering a number of scenarios equal to 1000. We point out that we report the average efficiency scores collected by considering 10 different scenario sets of the same cardinality. Nevertheless, we have empirically found that the determine solutions are stable, i.e., regardless the considered scenario set, similar optimal solutions are determined.

4.2 Numerical results

The proposed model has been implemented in the General Algebraic System Gams[1] and solved by using ILOG CPLEX 12.6.2.[2] The computational experiments have been carried out on a computer equipped with an Intel Core CPU at 3.40 GHz and 8 GByte RAM. For all the solved instances, the computational times are very short (of the order of few minutes) and are not reported. In what follows, we shall analyze the results focusing on specific issues.

4.2.1 The effect of uncertainty

The first set of experiments has been carried out with the aim of examining the potential benefits of explicitly including uncertainty in the DEA-based evaluation process. As basis of comparison, we have considered two classical approaches. The first one relies on the solution of a deterministic DEA model as (4)–(7) for every scenario s. Table 1 reports, for every DMU, the average, the standard deviation, the minimum, the maximum values, respectively, computed on the basis of the collected results. As evident, the gap between the minimum and the maximum values can be very large. For example, for the second DMU the minimum value is around 0.09, whereas the maximum is 1. This large variability range, observed especially for some DMUs, underlines the sensitivity of the solutions to data variation.

The second approach relies on the solution of a deterministic DEA model where the uncertain parameters are replaced by the expected values computed starting from the generated scenarios. Figure 2 reports the average error (in percentage) made by applying this simplified approach. More specifically, assuming that a given scenario

[1] www.gams.com.

[2] www.ilog.com/products/cplex.

Table 1 Efficiency levels

DMU	Average	STD	Min	Max
1	0.472	0.181	0.130	1.000
2	0.346	0.132	0.095	1.000
3	0.311	0.098	0.074	0.727
4	0.936	0.112	0.434	1.000
5	0.751	0.176	0.224	1.000
6	0.347	0.088	0.174	0.656
7	0.376	0.106	0.184	0.757
8	0.484	0.137	0.211	1.000
9	0.588	0.172	0.203	1.000
10	0.228	0.081	0.047	0.553
11	0.203	0.048	0.110	0.390
12	0.324	0.250	0.021	1.000
13	0.553	0.218	0.109	1.000
14	0.122	0.044	0.038	0.332
15	0.972	0.072	0.543	1.000
16	0.118	0.028	0.054	0.229
17	0.175	0.061	0.060	0.411
18	0.545	0.116	0.282	0.912

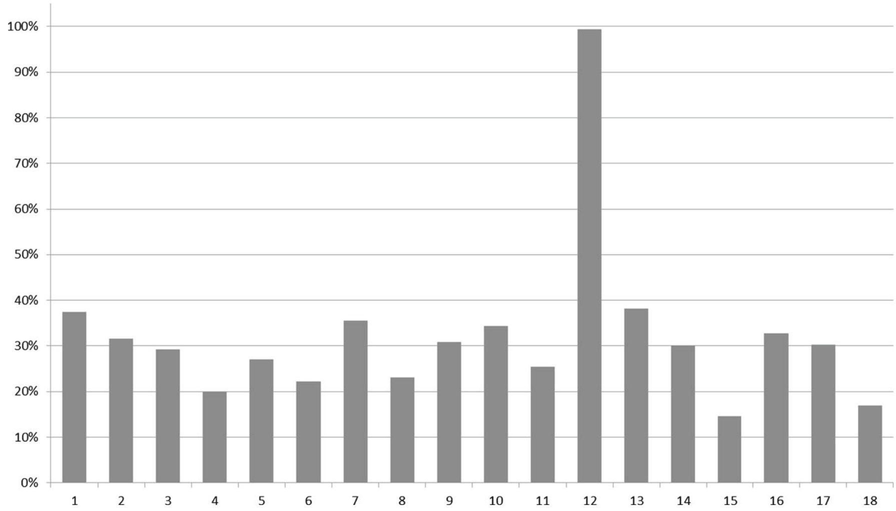

Fig. 2 Average error in percentage of the expected value approach

occurs, the reported values indicate the error made by using the deterministic evaluation instead of the correct scenario one. As evident, the evaluation errors can be huge: For example, for the DMU 12 the average error is around 98%, whereas the average error is around 32%. In the section devoted to the out-of-sample analysis, we shall report the more meaningful comparison based on the "real" score.

Table 2 Stochastic efficiency levels as function of γ for $\alpha = 1$

DMU	$\gamma = 1$		$\gamma = 0.95$		$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$
	ζ	η	ζ	η	ζ	η	ζ	η	η
1	0.258	0.482	0.252	0.340	0.248	0.310	0.245	0.294	0.046
2	0.144	0.334	0.138	0.243	0.133	0.219	0.129	0.204	0.027
3	0.161	0.294	0.157	0.213	0.154	0.196	0.152	0.184	0.038
4	0.596	1.000	0.587	0.714	0.581	0.666	0.576	0.661	0.182
5	0.297	0.678	0.284	0.511	0.272	0.476	0.261	0.454	0.142
6	0.167	0.328	0.162	0.244	0.158	0.229	0.154	0.219	0.087
7	0.228	0.312	0.226	0.281	0.222	0.281	0.219	0.281	0.082
8	0.204	0.427	0.196	0.333	0.189	0.311	0.182	0.299	0.112
9	0.292	0.553	0.284	0.417	0.277	0.379	0.272	0.354	0.113
10	0.108	0.226	0.104	0.162	0.101	0.146	0.099	0.135	0.026
11	0.116	0.144	0.115	0.125	0.115	0.125	0.114	0.125	0.078
12	0.108	0.549	0.092	0.343	0.081	0.270	0.071	0.201	0.008
13	0.214	0.831	0.198	0.458	0.185	0.412	0.173	0.366	0.051
14	0.060	0.128	0.058	0.084	0.057	0.076	0.056	0.072	0.009
15	0.497	1.000	0.479	0.767	0.464	0.716	0.451	0.682	0.217
16	0.071	0.092	0.071	0.083	0.070	0.083	0.069	0.083	0.032
17	0.077	0.150	0.074	0.116	0.072	0.105	0.070	0.098	0.013
18	0.257	0.398	0.251	0.348	0.246	0.330	0.242	0.320	0.195

4.2.2 Sensitivity of the evaluation process as function of the probability levels

In what follows, we discuss how the DMU performance varies as function of the probability levels α and γ. In particular, we have considered for α the values $1, 0.95, 0.9, 0.85$. We remind that the γ values are always lower than the α ones.

Tables 2, 3, 4 and 5 report, for a fixed α, the values of ζ and η as a function of γ. We recall that ζ measures the expected efficiency score obtained when considering the γ quantile of worst-case realizations, whereas η provides the corresponding best achievable performance. Thus, $\zeta \leq \eta$ and the two values coincide for γ equal to 0.

The analysis of the results shows that for a fixed value of α, the efficiency levels decrease with γ. This behavior can be explained by observing that by reducing γ we focus on a smaller number of worst-case realizations. For example, if we consider the DMU 4, for α equal to 1, the ζ value ranges from 0.596 to 0.182 when varying γ from 1 to 0. A different behavior is observed when we keep γ fixed and we vary α. The results show that lower efficiency levels are obtained as α increases. Figure 3 shows the efficiency levels of the different DMUs by setting γ equal to 0.85 and varying α. For example, for the DMU 4 the efficiency level passes from 0.576 for $\alpha = 1$ to 0.796 for $\alpha = 0.85$.

The variation of the efficiency level is related to the nature of the probabilistic constraint. For α equal to 1, the satisfaction of the stochastic constraints is required for all the possible realizations of the random parameters. In general, the higher is α, the

Table 3 Stochastic efficiency levels as function of γ for $\alpha = 0.95$

DMU	$\gamma = 0.95$		$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$
	ζ	η	ζ	η	ζ	η	η
1	0.317	0.434	0.312	0.397	0.310	0.369	0.058
2	0.177	0.310	0.170	0.281	0.164	0.261	0.034
3	0.202	0.275	0.198	0.253	0.195	0.273	0.050
4	0.739	0.898	0.731	0.841	0.727	0.835	0.244
5	0.364	0.654	0.349	0.612	0.334	0.581	0.182
6	0.205	0.307	0.199	0.285	0.196	0.276	0.111
7	0.291	0.362	0.287	0.362	0.283	0.362	0.104
8	0.251	0.425	0.242	0.400	0.234	0.379	0.142
9	0.362	0.508	0.358	0.477	0.349	0.434	0.146
10	0.134	0.210	0.130	0.188	0.127	0.175	0.033
11	0.148	0.162	0.146	0.159	0.146	0.159	0.103
12	0.116	0.414	0.103	0.335	0.092	0.252	0.011
13	0.249	0.572	0.231	0.520	0.219	0.450	0.067
14	0.075	0.113	0.074	0.099	0.072	0.095	0.012
15	0.611	0.967	0.589	0.905	0.576	0.867	0.279
16	0.091	0.112	0.090	0.110	0.089	0.109	0.042
17	0.096	0.153	0.093	0.136	0.090	0.133	0.017
18	0.325	0.449	0.316	0.425	0.315	0.417	0.250

worse is the objective function value. The choice of the α value is up to the decision maker who could select the proper value on the basis of his/her attitude toward risk.

4.2.3 The stochastic ranking

In what follows, we comment on the use of the proposed model for ranking purposes. We observe that in our case, ranking is also affected by the chosen probability values. Thus, to investigate this influence, we have ranked the different DMUs as function of the probability values. The whole set of results has been reported in "Appendix", whereas in the following we report aggregate values. The analysis of the results reveals a slight influence of the probability value on the ranking. In effect, the major difference is registered when setting the γ value equal to 0. Figure 4 summarizes the aggregated results for some DUMs. For example, DMU 4 occupies the first position in the 71% of cases and the third one in the 29% of cases. The less efficient DMU seems to be 14 that is in the last position in 71% of cases and the second to last position in the 29% of cases. In an intermediate position, we find the DMU 8 that occupies the eighth position in the 57% of cases, position sixth in the 29% and ninth in 14% of cases. Looking at the overall results, we may conclude that the ranking is quite stable and, as such, it provides a valuable tool to assist the decision maker in the evaluation process.

Table 4 Stochastic efficiency levels as function of γ for $\alpha = 0.90$

DMU	$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$
	ζ	η	ζ	η	η
1	0.333	0.425	0.329	0.392	0.061
2	0.183	0.305	0.177	0.282	0.035
3	0.209	0.267	0.206	0.250	0.052
4	0.778	0.895	0.768	0.884	0.258
5	0.370	0.645	0.355	0.619	0.191
6	0.212	0.309	0.206	0.291	0.118
7	0.306	0.386	0.301	0.385	0.111
8	0.257	0.426	0.248	0.403	0.149
9	0.375	0.485	0.372	0.459	0.156
10	0.139	0.200	0.135	0.184	0.034
11	0.157	0.169	0.157	0.171	0.111
12	0.109	0.337	0.098	0.265	0.011
13	0.246	0.545	0.232	0.470	0.069
14	0.079	0.108	0.077	0.099	0.013
15	0.622	0.956	0.608	0.914	0.290
16	0.095	0.112	0.095	0.114	0.044
17	0.100	0.146	0.096	0.132	0.017
18	0.339	0.456	0.331	0.438	0.265

Table 5 Stochastic efficiency levels as function of γ for $\alpha = 0.85$

DMU	$\gamma = 0.85$		γ=0
	ζ	η	η
1	0.340	0.407	0.063
2	0.187	0.298	0.038
3	0.213	0.258	0.054
4	0.796	0.912	0.266
5	0.362	0.631	0.201
6	0.212	0.301	0.124
7	0.318	0.407	0.116
8	0.259	0.423	0.157
9	0.386	0.423	0.162
10	0.140	0.192	0.036
11	0.166	0.178	0.118
12	0.101	0.276	0.012
13	0.238	0.487	0.072
14	0.080	0.103	0.013
15	0.628	0.947	0.305
16	0.099	0.118	0.046
17	0.102	0.141	0.018
18	0.349	0.464	0.276

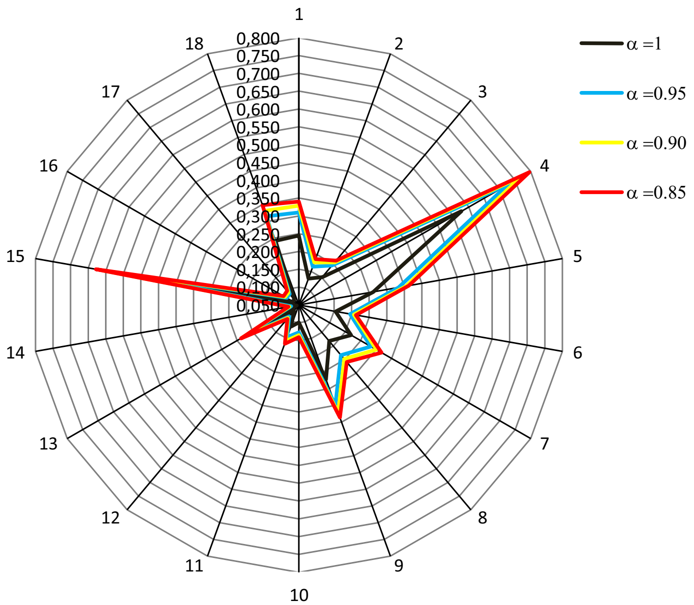

Fig. 3 Efficiency levels as function of α for γ fixed to 0.85

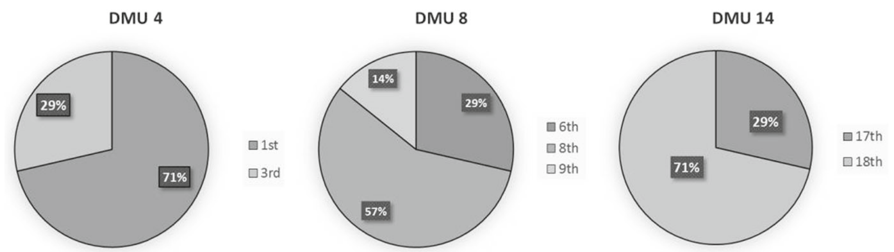

Fig. 4 Ranking of selected DMUs

4.3 The out-of-sample analysis

This subsection illustrates the managerial insights deriving from the application of the proposed approach as a proactive tool for evaluating the future performance for the considered case study. We point out that by varying the values of α and γ, the decision maker can carry the evaluation process from different risk perspectives. The choice of the reliability values is always controversial since from one side, the current financial crisis might suggest to adopt more conservative positions, but from the other, in the spirit of promoting the financial recovery more risky behavior could be adopted.

The elaboration of the collected results allows to derive additional measures. In particular, an "out-of-sample" analysis has been carried out with the aim of verifying to what extent the scores provided by the proposed approach are more reliable than the deterministic ones. The basis of comparison is represented by the true evaluation,

Table 6 Classification of the DMUs as function of the cutoff points

Cutoff	Real (%)		Deterministic (%)		Stochastic (%)	
	H	I	H	I	H	I
0.45	27.78	72.22	38.89	61.11	11.12	88.88
0.50	11.11	88.89	22.23	77.77	5.55	94.45

obtained by solving a deterministic model on the real data referring to a subsequent investigation year (2009).

On the basis of the average efficiency levels obtained by considering the DEA scores computed by using the past input/output observations, we have fixed a reasonable cutoff point and we have computed the percentages of correct identification for the considered case study. In particular, we have classified as healthy (H), the companies with a DEA score greater than the threshold and as ill (I) the others.

Table 6 reports the results obtained by the real, deterministic and stochastic evaluation. For the stochastic case, we report the results obtained by fixing $\alpha = 1$, since this configuration corresponds to a more risk averse position.

As evident from the results, the stochastic evaluation never classifies as healthy an ill company. Thus, it never happens that the predicted score is above the cutoff, but the real score results below. For example, by fixing the cutoff level equal to 0.5, we may observe that according to the real score, two companies are classified as healthy, whereas according to the stochastic score, only one is considered healthy. The situation is different for the deterministic evaluation: In this case, four companies seem to be eligible for loan. Obviously, this classification error could be dangerous because it can lead to severe losses (related to the size of the loan) resulting from an overestimation of the repaying capability of the firm. The stochastic score can, on the contrary, lead to a situation where the predicted score is below the cutoff, whereas the real score is above. In this case, we classify as ill a company that in effect is not. This second classification error, always limited for our tests, is obviously less dangerous.

5 Concluding remarks

One of the main weaknesses of the conventional DEA models comes from the assumption of perfect information of input and output parameters used in the evaluation process. In this paper, we have proposed a new way to overcome this major drawback by presenting a new stochastic formulation where uncertain data referring to future time are represented by random variables and risk is properly controlled by the tail γ-mean measure. The resulting risk-based DEA model provides the decision maker with a tool to control a specified size (quantile) of worst performance realizations. A deterministic equivalent formulation of the stochastic model has been derived under the assumption of representing the uncertain parameters as discrete random variables. The proposed model has been tested on a meaningful case study related to the evaluation of the future performance of a sample of firms operating in the Italian industry. Our findings have shown the validity of the proposed approach as proactive evaluation technique which provides the decision maker with useful insights according on his/her risk aversion degree.

Appendix A

The following tables report the ranking of the different DMUs computed on the basis of the ζ values as function of the probability levels α and γ (Tables 7, 8, 9 10).

Table 7 Ranking as function of γ for $\alpha = 1$

DMU	$\gamma = 1$		$\gamma = 0.95$		$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$	
	ζ	Rank	ζ	Rank	ζ	Rank	ζ	Rank	η	Rank
1	0.258	5	0.252	5	0.248	5	0.245	5	0.046	11
2	0.144	12	0.138	12	0.133	12	0.129	12	0.027	14
3	0.161	11	0.157	11	0.154	11	0.152	11	0.038	12
4	0.596	1	0.587	1	0.581	1	0.576	1	0.182	3
5	0.297	3	0.284	3	0.272	4	0.261	4	0.142	4
6	0.167	10	0.162	10	0.158	10	0.154	10	0.087	7
7	0.228	7	0.226	7	0.222	7	0.219	7	0.082	8
8	0.204	9	0.196	9	0.189	8	0.182	8	0.112	6
9	0.292	4	0.284	4	0.277	3	0.272	3	0.113	5
10	0.108	15	0.104	14	0.101	14	0.099	14	0.026	15
11	0.116	13	0.115	13	0.115	13	0.114	13	0.078	9
12	0.108	14	0.092	15	0.081	15	0.071	15	0.008	18
13	0.214	8	0.198	8	0.185	9	0.173	9	0.051	10
14	0.060	18	0.058	18	0.057	18	0.056	18	0.009	17
15	0.497	2	0.479	2	0.464	2	0.451	2	0.217	1
16	0.071	17	0.071	17	0.070	17	0.069	17	0.032	13
17	0.077	16	0.074	16	0.072	16	0.070	16	0.013	16
18	0.257	6	0.251	6	0.246	6	0.242	6	0.195	2

Table 8 Ranking as function of γ for $\alpha = 0.95$

DMU	$\gamma = 0.95$		$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$	
	ζ	Rank	ζ	Rank	ζ	Rank	η	*Rank*
1	0.317	6	0.312	6	0.310	6	0.058	11
2	0.177	12	0.170	12	0.164	12	0.034	14
3	0.202	11	0.198	11	0.195	11	0.050	12
4	0.739	1	0.731	1	0.727	1	0.244	3
5	0.364	3	0.349	4	0.334	4	0.182	4
6	0.205	10	0.199	10	0.196	10	0.111	7
7	0.291	7	0.287	7	0.283	7	0.104	8
8	0.251	8	0.242	8	0.234	8	0.142	6
9	0.362	4	0.358	3	0.349	3	0.146	5
10	0.134	14	0.130	14	0.127	14	0.033	15
11	0.148	13	0.146	13	0.146	13	0.103	9
12	0.116	15	0.103	15	0.092	15	0.011	18
13	0.249	9	0.231	9	0.219	9	0.067	10
14	0.075	18	0.074	18	0.072	18	0.012	17
15	0.611	2	0.589	2	0.576	1	0.279	1
16	0.091	17	0.090	17	0.089	17	0.042	13
17	0.096	16	0.093	16	0.090	16	0.017	16
18	0.325	5	0.316	5	0.315	5	0.250	2

Table 9 Ranking as function of γ for $\alpha = 0.90$

DMU	$\gamma = 0.90$		$\gamma = 0.85$		$\gamma = 0$	
	ζ	Rank	ζ	Rank	η	*Rank*
1	0.333	6	0.329	6	0.061	11
2	0.183	12	0.177	12	0.035	14
3	0.209	11	0.206	10	0.052	10
4	0.778	1	0.768	1	0.258	3
5	0.370	4	0.355	4	0.191	4
6	0.212	10	0.206	11	0.118	7
7	0.306	7	0.301	7	0.111	8
8	0.257	8	0.248	8	0.149	6
9	0.375	3	0.372	3	0.156	5
10	0.139	14	0.135	14	0.034	15
11	0.157	13	0.157	13	0.111	9
12	0.109	15	0.098	15	0.011	18
13	0.246	9	0.232	9	0.069	10
14	0.079	18	0.077	18	0.013	17
15	0.622	2	0.608	2	0.290	1
16	0.095	17	0.095	17	0.044	13
17	0.100	16	0.096	16	0.017	16
18	0.339	5	0.331	5	0.265	2

Table 10 Ranking as function of γ for $\alpha = 0.85$

DMU	$\gamma = 0.85$		$\gamma = 0$	
	ζ	Rank	η	Rank
1	0.340	6	0.063	11
2	0.187	12	0.038	14
3	0.213	10	0.054	12
4	0.796	1	0.266	3
5	0.362	4	0.201	4
6	0.212	11	0.124	7
7	0.318	7	0.116	9
8	0.259	8	0.157	6
9	0.386	3	0.162	5
10	0.140	14	0.036	15
11	0.166	13	0.118	8
12	0.101	16	0.012	18
13	0.238	9	0.072	10
14	0.080	18	0.013	17
15	0.628	2	0.305	1
16	0.099	17	0.046	13
17	0.102	15	0.018	16
18	0.349	5	0.276	2

References

Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. Math. Finance **9**, 203–228 (1999)

Banker, R.D.: Maximum likelihood, consistency and data envelopment analysis: a statistical foundation. Manag. Sci. **39**(10), 1265–1273 (1993)

Beraldi, P., Bruni, M.E.: An exact approach for solving integer problems under probabilistic constraints with random technology matrix. Ann. Oper. Res. **177**(1), 127–137 (2010)

Beraldi, P., Bruni, M.E.: Data envelopment analysis under uncertainty and risk. WASET **66**, 837–842 (2012)

Beraldi, P., Bruni, M.E.: A clustering approach for scenario tree reduction: an application to a stochastic programming portfolio optimization problem. TOP **22**, 934–949 (2014)

Beraldi, P., De Simone, F., Violi, A.: Generating scenario trees: a parallel integrated simulation–optimization approach. J. Comput. Appl. Math. **23**(9), 2322–2331 (2010)

Beraldi, P., Bruni, M.E., Laganá, D.: The express heuristic for probabilistically constrained integer problems. J. Heurist. **19**(3), 423–441 (2013)

Beraldi, P., Bruni, M.E., Iazzolino, G.: Lending decision under uncertainty: a DEA approach. Int. J. Prod. Res. **52**(3), 766–775 (2014)

Bruni, M.E., Conforti, D., Beraldi, P., Tundis, E.: Probabilistically constrained models for efficiency and dominance in DEA. Int. J. Prod. Econ. **177**(1), 219–228 (2009)

Chang, T.S., Tone, K., Wu, C.-H.: DEA models incorporating uncertain future performance. Eur. J. Oper. Res. **254**(2), 532–549 (2016)

Charnes, A., Cooper, W.W.: Chance constrained programming. Manag. Sci. **5**(1), 73–79 (1959)

Charnes, A., Cooper, W.W., Rhodes, E.: Measuring the efficiency of decision making units. Eur. J. Oper. Res. **6**(2), 429–444 (1978)

Chen, K., Zhu, J.: Computational tractability of chance constrained data envelopment analysis. Eur. J. Oper. Res. **274**(3), 1037–1046 (2019)

Cheng, J., Lisser, A.: A second-order cone programming approach for linear programs with joint probabilistic constraints. Oper. Res. Lett. **40**(5), 325–328 (2012)

Cooper, W.W., Huang, Z., Li, S.: Satisficng DEA model under chance constraints. Ann. Oper. Res. **66**(5), 79–295 (1996)
Cooper, W.W., Deng, H., Huang, Li S: Chance constrained programming approaches to congestion in stochastic data envelopment analysis. Eur. J. Oper. Res. **155**(2), 487–501 (2004)
Iazzolino, G., Bruni, M.E., Beraldi, P.: Using DEA and financial ratios for credit risk evaluation: an empirical analysis. Appl. Econ. Lett. **20**(14), 1310–1317 (2013)
Kao, C., Liu, S.-T.: Stochastic efficiency measures for production units with correlated data. Eur. J. Oper. Res. **273**(1), 278–287 (2019)
Land, K.C., Lovell, C.A.K., Thore, S.: Chance-constrained data envelopment analysis. Manag. Decis. Econ. **14**, 541–554 (1993)
Markowitz, H.M.: Portfolio selection. J. Finance **7**, 77–91 (1952)
Ogryczak, W.: Tail mean and related robust solution concept. Int. J. Syst. Sci. **45**, 29–38 (2014)
Ogryczak, W., Ruszczynski, A.: Dual stochastic dominance and related mean-risk models. SIAM J. Optim. **13**, 60–78 (2002a)
Ogryczak, W., Ruszczynski, A.: Dual stochastic dominance and quantile risk measures. Int. Trans. Oper. Res. **9**, 661–680 (2002b)
Olesen, O.B., Petersen, N.C.: Chance constrained efficiency evaluation. Manag. Sci. **41**, 442–457 (1995)
Olesen, O.B., Petersen, N.C.: Stochastic data envelopment analysis: a review. Eur. J. Oper. Res. **251**(1), 2–21 (2016)
Paradi, J.C., Asmild, M., Simak, P.: Using DEA and worst practice DEA in credit risk evaluation. J. Prod. Anal. **21**, 153–165 (2004)
Post, T.: Performance evaluation in stochastic environments using mean–variance data envelopment analysis. Oper. Res. **49**(2), 281–292 (2001)
Premachandra, I.M., Chen, Y., Watson, J.: DEA as a tool for predicting corporate failure and success: a case of bankruptcy assessment. Omega **39**, 620–626 (2011)
Rockafellar, R.T., Uryasev, S.: Optimization of conditional value-at-risk. J. Risk **2**, 21–41 (2000)
Sengupta, J.K.: Data envelopment analysis for efficiency measurement in the stochastic case. Comput. Oper. Res. **14**, 117–129 (1987)
Sueyoshi, T.: Stochastic DEA for restructure strategy: an application to a Japanese petroleum company. Omega **28**, 385–398 (2000)
Wei, G., Chen, J., Wang, J.: Stochastic efficiency analysis with a reliability consideration. Omega **48**, 1–9 (2014)
Wu, D., Olson, D.: Enterprise risk management: a DEA VaR approach in vendor selection. Int. J. Prod. Res. **40**(6), 4919–4932 (2010)