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DEA production games with fuzzy output prices

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Abstract In DEA production models the technology is assumed to be implicit in the input-output data given by a set of recorded observations. DEA production games assess the benefits to different firms of pooling their resources and sharing their technology. The crisp version of this type of problems has been studied in the literature and methods to obtain stable solutions have been proposed. However, no solution approach exists when there is uncertainty in the unit output prices, a situation that can clearly occur in practice. This paper extends DEA production games to the case of fuzzy unit output prices. In that scenario the total revenue is uncertain and therefore the corresponding allocation among the players is also necessarily uncertain. A core-like solution concept is introduced for these fuzzy games, the Preference Least Core. The computational burden of obtaining allocations of the fuzzy total profit reached through cooperation that belong to the preference least core is high. However, the results presented in the paper permit us to compute the fuzzy total revenue obtained by the grand coalition and a fuzzy allocation in the preference least core by solving a single linear programming model. The application of the proposed approach is illustrated with the analysis of two cooperative production situations originated by data sets from the literature.

Keywords DEA production games · Fuzzy unit output prices · Fuzzy revenue · Fuzzy cooperative games · Fuzzy allocation

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1 Introduction

Owen (1995) considered linear production programming problems in which multiple decision-makers pool their resources in a common production process, and analyzed these situations by using cooperative game theory. Recently, Lozano (2013) has looked at the model under a new perspective in which a set of recorded observations of the production process is used to plan the production process on the basis of a technology inspired in Data Envelopment Analysis (DEA). The new linear (in essence) production model is called DEA production model and the corresponding cooperative game is called a DEA production game.

In DEA production models, the objective function represents the total revenue obtained from selling certain kinds of products, and the problem is formulated as a linear programming problem in which the revenue is maximized in the production possibility set induced by the set of recorded observations. However, very often, in real world-situations, the assumption of certainty with respect to the nature of the parameters involved in the model is unrealistic and in many applications, the use of fuzzy logic (Zadeh 1965) has proved to be advantageous to deal with the imprecise nature of the data. Particularly, in the analysis of efficiency by using DEA models, imprecision in the data is a main drawback and their representation as fuzzy numbers enables a more realistic assessment of the efficiency of the decision making units [see for instance, Lertworasirikul et al. (2003); Hatami-Marbini et al. (2011) and Lozano (2014)].

In this paper we address the cooperative model arising from DEA production problems with uncertain parameters. Specifically, we consider the scenario of fuzzy output prices. Such uncertainty is present in our daily life. The prices of gas, fresh food, flight tickets, accommodation, etc. are often difficult to predict. The same happens to many companies, mainly, but not only, those operating in competitive markets. A typical example, illustrated numerically in this paper, is the case of electric utilities which face uncertainty in the wholesale price of the electricity they generate.

Since this price uncertainty may become a hurdle for companies willing to enter into collaboration agreements, it is important to develop tools and methodologies to help companies assess the benefits of cooperation even in the presence of price uncertainty. The introduction of price uncertainty into the cooperative model raises new and interesting issues, since coalitions can form prior to the resolution of uncertainty and they must discuss divisions of the uncertain revenue by taking into account their potential worths which may also be uncertain. We assume that the lack of precision in the parameters of the linear production problem is modeled via fuzzy logic, that is, some of the parameters involved in the objective function and/or in the constraints of the production game are represented by fuzzy numbers. The proposed approach extends the corresponding crisp method (to which it reduces in case the data uncertainty is eliminated) at the expense of an increase in the complexity of the analysis and in the size of the optimization models to be solved. An interesting feature of the proposed approach is that it can be used more than once so that as time passes and the uncertainty is reduced, the fuzzy allocation computed by the proposed approach can be revised and refined.

Several cooperative models involving fuzziness can be found in the literature. The line initiated in Aubin (1981) studies games with fuzzy coalitions, where the agents may consider different levels of participation in cooperation. Recent work in this line is, for instance, Wu (2012) and Li and Zhang (2009). The present investigation deals with models in which the fuzziness concerns the values that the coalitions can achieve. Nishizaki and Sakawa (2001) precede us in investigating these games. For fuzzy cooperative games arising from linear production programming problems with fuzzy parameters they proposed an infinite family of cores, each of which consists of a set of non-fuzzy payoff vectors. Recently, in Hinojosa et al. (2013) and Monroy et al. (2013), a different approach has been presented to analyse the solutions of cooperative games with fuzzy payoffs and applied to the cases of fuzzy linear production games and fuzzy assignment games.

The game arising from the production situation, when the pool of resources is controlled by several agents in the fuzzy context considered in this paper, is a set-valued game in which each element of the set is a fuzzy number. Lozano et al. (2015) have investigated vector-valued DEA production games and their results serve as a basis for the analysis in a fuzzy environment which is developed in this paper. The main goal in these cooperative situations is to determine which will be the allocations of the total fuzzy revenues that the agents will be willing to accept. In this situation, since there is not a total order among the payoffs, the comparisons between the fuzzy payoffs obtained by the players and by the coalitions are not straightforward as in scalar games and, therefore, classic solution concepts are not applicable. Previous literature has addressed this difficulty by establishing a utility function in order to induce a scalar game and to obtain allocations of the associated total revenue based on different rationality principles. However, this approach seldom helps towards an accurate analysis of the situation, since it generates allocations consisting of non-fuzzy payoffs for the agents.

This paper carries out an *ex-ante* analysis of the production situation and proposes a solution concept for the DEA production game with fuzzy prices, namely the preference least core, which is applicable before the fuzziness is resolved. In this solution the fuzzy nature of the allocations is preserved, and therefore, the quantity finally assigned to each agent is a fuzzy number. The preference least core has recently been introduced in Lozano et al. (2015) for set-valued DEA production games, and is based on the same idea as the least core in standard TU games. The main advance of the present paper with respect to the results presented in Lozano et al. (2015) is the setting of a general framework that more realistically accommodates situations in which there is uncertainty in the parameters of the model. Its main drawback, derived from the complexity of the fuzzy environment, is the difficulty involved in the effective computation of the fuzzy allocations. However, the paper shows how, from a computational point of view, only linear programming tools are needed.

We adopt standard fuzzy orders in the set of fuzzy numbers [see González and Vila (1991), González and Vila (1992) and Ramík and Římánek (1985)], and define the excess of the coalitions accordingly. The main contributions in the paper are the definition of the preference least core in this fuzzy environment and the proposal of a procedure to compute allocations in this set. The procedure requires solving a single linear programming model, which at the same time yields the efficient fuzzy revenue obtained by cooperation and the allocation to the agents of this fuzzy quantity.

We also show how our approach is applied in two applications, for which allocations in the preference least core are obtained, both for the case where uncertain prices are represented by triangular fuzzy numbers and for trapezoidal fuzzy numbers.

The rest of the paper is as follows. In Sect. 2 we describe fuzzy DEA production problems. In Sect. 3, the fuzzy DEA production game is introduced and the concepts of preference core and the preference least core of the fuzzy game are defined. Results are provided which permit the computation of allocations of the fuzzy revenues obtained by cooperation. Section 4 contains two illustrative examples, where the effectiveness of the proposed approach is shown. It is illustrated how in both cases the level of ambiguity of the fuzzy numbers which represent per-unit profits affects the precision of the fuzzy quantities which are finally allocated to the agents.

2 Fuzzy DEA production problems

A DEA production problem is described as follows: P kinds of products have to be produced by using M different resources. The availability of each resource is given by a vector $b \in \mathbb{R}^M$ and the positive per-unit income, c_p, is obtained by selling product p, $p = 1, 2, \ldots, P$. In DEA production models the technology is assumed to be implicit in the input-output data given by a set of D recorded observations. Denote by $X \in \mathbb{R}^{M \times D}$ the matrix whose columns are the observed inputs, and by $Y \in \mathbb{R}^{P \times D}$ the matrix whose columns are the observed outputs. The goal is to plan the input and output levels $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^P$ which provide the maximum revenue. That is, to select feasible input and output levels in the production possibility set $\{(x, y) \mid X\lambda \leq x,\ Y\lambda \geq y,\ x \leq b,\ \lambda \in \Lambda\}$, which is induced by the recorded observations and by the limitation of resources. The components of $\lambda \in \mathbb{R}^D$ are the parameters of the DEA envelopment model, and the set Λ is the domain of these parameters which depend on the assumptions about the returns to scale of the model. The production possibility set in DEA represents the set of feasible operation points among which the agent can select the one that best fits its objective (i.e. revenue maximization). It is a key feature of DEA, the one that gives it its non-parametric character, that the production possibility set is inferred from the observed data. This is done making some basic assumptions such as free disposability of inputs and outputs (i.e. given a feasible operation point it is always feasible to consume more input and produce less output), convexity (i.e. if two operation points are feasible any convex linear combination of them is also feasible) or scalability (i.e. given a feasible operation point any scaled version of it is also feasible). This type of assumptions is the reason why the production possibility set is formed using linear combinations of the observed data, with the λ variables determining the participation of each observation on the definition of the target operation point. Depending on the type of scalability axiom (full, downward-only or none) different constraints are imposed on the λ variables. Thus, when full scalability is assumed (which corresponds to assuming constant returns to scale, CRS) then the only constraints are the non-negativity of the λ variables. If downward-only scalability is assumed (which corresponds to assuming non-increasing returns to scale, NIRS) then the sum of the λ variables must be less than unity and, finally, if no scalability is assumed (which corresponds to variable returns to scale, VRS) then the λ variables must add to unity. This is the standard way of inferring the DEA technology from the observed data [e.g. Cooper et al. (2000)]. However, there is one additional constraint in the possibility set defined above and it is that the amount of resources available is given. This $x \leq b$ constraint is specifically imposed in this production planning scenario to reasonably restrict the feasibility of the operation points to be considered.

Formally, the following linear programming problem has to be solved.

$$\begin{aligned} \max\ & c^t y \\ s.t.\ & X\lambda \leq x; \\ & Y\lambda \geq y; \\ & x \leq b; \\ & \lambda \in \Lambda, \end{aligned} \tag{1}$$

where $c^t = (c_1, c_2, \ldots, c_P)$.

Throughout this paper we consider that the technology exhibits variable returns to scale, and therefore, $\Lambda = \{\lambda \in \mathbb{R}^D : \lambda \geq 0,\ \sum_{l=1}^{D} \lambda_l = 1\}$. A similar reasoning could be developed for constant returns to scale.

Jahanshahloo et al. (2007) show that in any optimal solution of the above model, the constraints represented by $Y\lambda \geq y$, hold as equalities. On the other hand, there is an optimal solution of these models for which the constraints that define the target inputs $X\lambda \leq x$, also hold as equalities. As a consequence, the DEA production problem (1) can equivalently be formulated as:

$$\begin{aligned} \max\ & c^t Y\lambda \\ s.t.\ & X\lambda \leq b; \\ & \lambda \in \Lambda. \end{aligned}$$

In a real-world production problem, the parameters of the above model may only be imprecisely known, and therefore these parameters admit a representation as fuzzy numbers[1]. For instance, fuzzy incomes reflect the ambiguity or fuzzy understanding of the nature of prices. For simplicity, in this paper we only consider fuzzy incomes, although problems with a whole set of fuzzy parameters could also be studied.

The fuzzy DEA production model is formulated here as a problem with a fuzzy objective function under the assumption that the decision-maker maximizes the total revenue in the production possibility set induced by the set of recorded observations, by selling the products without limitation of their demand. Formally,

[1] A fuzzy number, which we denote by $\widetilde{z}$, is a fuzzy set on the space of real numbers $\mathbb{R}$, whose membership function $\mu_{\widetilde{z}} : \mathbb{R} \rightarrow [0, 1]$ satisfies (i) there is a real number z, such that $\mu_{\widetilde{z}}(z) = 1$, (ii) $\mu_{\widetilde{z}}$ is upper semicontinuous, (iii) $\mu_{\widetilde{z}}$ is quasi-concave and (iv) $supp(\widetilde{z})$ is compact, where $supp(\widetilde{z})$ denotes the support of $\widetilde{z}$. We denote the set of all fuzzy numbers by $\mathbb{N}(\mathbb{R})$.

$$\begin{array}{ll} \max_{\succeq} & \tilde{c}^t y \\ s.t. & X\lambda \leq x; \\ & Y\lambda \geq y; \qquad [\tilde{P}_{\succeq}] \\ & x \leq b; \\ & \lambda \in \Lambda, \end{array}$$

where $\max_{\succeq}$ must be understood as the search for the set of efficient solutions with respect to the binary relation $\succeq$, which represents a partial order defined on the set of fuzzy numbers $\mathbb{N}(\mathbb{R})$, and vector $\tilde{c} \in \mathbb{N}(\mathbb{R})^P$ is the vector of fuzzy incomes per unit.

The set of efficient output levels (with respect to the partial order considered) is adopted as the solution for this maximization problem. If Q denotes the set of feasible production vectors, and $\succ$ is the asymmetric part of the partial order $\succeq$, then the set of efficient production vectors for the problem $[\tilde{P}_{\succeq}]$ is:

$$\mathcal{E}(\tilde{P}_{\succeq}) = \{y \in Q : \nexists\ y' \in Q,\ \tilde{c}^t y' \succ \tilde{c}^t y\}.$$

Different efficient production vectors provide different objective fuzzy values. As a consequence, there is a set of efficient fuzzy values, $Ef(\tilde{P}_{\succeq})$, each corresponding to an efficient production vector, $Ef(\tilde{P}_{\succeq}) = \{\tilde{c}^t y : y \in \mathcal{E}(\tilde{P}_{\succeq})\}$. Note that $Ef(\tilde{P}_{\succeq})$ depends on the partial order considered in the set of fuzzy numbers.

Similarly to the case of the crisp income problem, under mild assumptions on the partial order considered[2], the fuzzy DEA production problem, $[\tilde{P}_{\succeq}]$, can equivalently be formulated as:

$$\begin{array}{ll} \max_{\succeq} & \tilde{c}^t Y\lambda \\ s.t. & X\lambda \leq b; \\ & \lambda \in \Lambda. \end{array}$$

Note that for each observation, $l \in D$, $\tilde{c}^t Y_l$ represents the fuzzy evaluation of the corresponding output, and therefore the fuzzy objective function, $\tilde{c}^t Y\lambda = \sum_{l=1}^{D} \lambda_l \tilde{c}^t Y_l$, represents the overall fuzzy evaluation of the observed outputs with weights in Λ. That is, the problem can be seen as that of finding feasible combinations of the evaluations of the observed outputs that yield maximal values.

3 Fuzzy DEA production games

The game theoretical aspects in the production model arise when the pool of resources is controlled by n different agents (players). Let $N = \{1, 2, \ldots n\}$ be the set of players and let us assume that player $i \in N$ holds a resource vector $b^i = (b^i_1, b^i_2, \ldots, b^i_M)^t$, $i = 1, 2, \ldots, n$. If coalition S forms, it will control a bundle of resources $b(S) = \sum_{i \in S} b^i$. In addition to pooling the resources, the members of S share their best practices (technology-sharing model) [see Lozano (2013)] by taking into account, when planning the production of each member of the coalition, the whole set of recorded observations corresponding to all the producers in the coalition. In this situation, a

[2] It suffices that for any $\tilde{c} \in \mathbb{N}(\mathbb{R})^P$ and $y, z \in \mathbb{R}^P$, such that $y \geq z$, $\tilde{c}y \succeq \tilde{c}z$ holds.

cooperative game can be considered in order to determine how the total outcome should be allocated among the players.

For a given coalition, $S \subseteq N$, consider the set of recorded observations of the production process corresponding to all the members of the coalition. If the number of recorded observations of agent i is $D(i)$, then $D(S) = \sum_{i \in S} D(i)$ is the number of available observations for coalition S. Consider the matrix whose columns are the observed inputs, $X(S) \in \mathbb{R}^{M \times D(S)}$, and the matrix whose columns are the observed outputs, $Y(S) \in \mathbb{R}^{P \times D(S)}$.

When these data are incorporated to the model, the vector of resources $b(S)$ enables the coalition S to produce goods according to the following problem:

$$\begin{array}{rl} \max_{\succeq} & \tilde{c}^t y(S) = \tilde{c}^t \sum_{i \in S} y^i \\ s.t. & X(S)\lambda(i) \leq x^i, \ \forall i \in S; \\ & Y(S)\lambda(i) \geq y^i, \ \forall i \in S; \\ & \sum_{i \in S} x^i \leq b(S); \\ & \lambda(i) \in \Lambda^S, \ \forall i \in S, \end{array}$$

which is equivalent to the fuzzy optimization problem:

$$\begin{array}{rl} \max_{\succeq} & \displaystyle\sum_{i \in S} \tilde{c}^t Y(S)\lambda(i) \\ s.t. & \displaystyle\sum_{i \in S} X(S)\lambda(i) \leq b(S); \qquad [\tilde{P}(S)] \\ & \lambda(i) \in \Lambda^S, \ \forall i \in S, \end{array}$$

where $\Lambda^S = \{\lambda \in \mathbb{R}^{D(S)} : \lambda \geq 0, \sum_{l=1}^{D(S)} \lambda_l = 1\}$.

Notice that coalitions $S \subseteq N$ may exist for which there is no x in the convex hull of the set of observed inputs such that $x \leq b(S)$, and therefore $[\tilde{P}(S)]$ is unfeasible. We restrict attention to fuzzy production problems such that the problem of the grand coalition, $[\tilde{P}(N)]$, is feasible, and distinguish between feasible and unfeasible coalitions, depending on whether $[\tilde{P}(S)]$ is feasible or not. The set of feasible coalitions is denoted by F, and the set of unfeasible coalitions is denoted by $\bar{F} = 2^N \backslash F$.

The efficient solutions of $[\tilde{P}(S)]$ provide a set of fuzzy values for coalition S, $Ef(\tilde{P}(S))$. The result of cooperation $Ef(\tilde{P}(N))$ is also represented by a set of fuzzy numbers. Our concern is to design procedures to allocate the fuzzy benefits between the agents involved. These procedures should take into account the fuzzy nature of the worth of the coalition by loosing as least as possible, the information contained in the original model. Note that solving problem $[\tilde{P}(S)]$ for the different coalitions $S \subseteq N$ involves a significant computational burden. However, in the approach presented in this paper, $[\tilde{P}(S)]$ need not be solved for intermediate allocations since the solution to a single linear problem associated to the grand coalition will provide the allocations of the fuzzy benefits.

In this framework we can naturally introduce the fuzzy DEA production game. A fuzzy DEA production game is a pair, $(N, \tilde{V})$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $\tilde{V}$ is a map that assigns a subset of fuzzy numbers, $\tilde{V}(S)$, to each coalition $S \subseteq N$. The sets $\tilde{V}(S)$ are called characteristic sets and are given by:

$$\tilde{V}(\emptyset) = \tilde{0}$$
$$\tilde{V}(S) = Ef(\tilde{P}(S)) \quad \forall S \subseteq N.$$

Each fuzzy number in $\tilde{V}(S)$ represents a fuzzy value that the players of coalition S can guarantee by themselves.

3.1 Fuzzy allocations

Since the amounts assigned to the coalitions by $\tilde{V}$ are sets of fuzzy numbers, the situations that we study present impreciseness and vagueness with respect to the payoffs. We present here an *ex-ante* analysis, that is, we propose a solution that apply before fuzziness is resolved.

The main target in the fuzzy DEA production game is to determine how an achievable fuzzy payoff in $\tilde{V}(N)$ should be divided among the players. The extension of the idea of allocation in scalar games to fuzzy DEA production games consists in using a fuzzy allocation $\tilde{Z} = (\tilde{z}(1), \ldots, \tilde{z}(n))$ where $\tilde{z}(i) \in \mathbb{N}(\mathbb{R})$, $i = 1, \ldots, n$, stands for the payoff of the i-th player. The sum $\tilde{Z}(S) = \sum_{i \in S} \tilde{z}(i)$ is the overall fuzzy payoff obtained by coalition S. For each fuzzy DEA production game $(N, \tilde{V})$, a fuzzy allocation is a vector $\tilde{Z} \in \mathbb{N}(\mathbb{R})^n$ such that $\tilde{Z}(N) \in \tilde{V}(N)$. Denote by $I^*(N, \tilde{V})$ the set of all the allocations of the fuzzy production game $(N, \tilde{V})$.

3.2 Preference core and preference least core

In scalar games an allocation dominates another with respect to a coalition if the former gives the members of the coalition a greater amount than the latter. In our context we have to compare fuzzy allocations through coalitions. Furthermore, in the class of games we are analyzing, the fuzzy value given by an allocation to the coalition has to be compared with the set of fuzzy values of the coalition. A way to compare the fuzzy value given by an allocation to the coalition with the set of fuzzy values of the coalition is to establish that the allocation $\tilde{Z}$ is as least as preferred as the set $\tilde{V}(S)$ whenever the fuzzy aggregated payoffs given by allocation $\tilde{Z}$ to coalition S, $\tilde{Z}(S)$, is as least as preferred as any of the fuzzy numbers in $\tilde{V}(S)$.

This relation leads us to the preference core for set-valued cooperative fuzzy games, as defined in Hinojosa et al. (2013).

Definition 1 The preference core of the fuzzy DEA production game $(N, \tilde{V})$ is the set of allocations, $\tilde{Z} \in I^*(N, \tilde{V})$, such that $\tilde{Z}(S) \succeq \tilde{w}$, $\forall \tilde{w} \in \tilde{V}(S)$, $\forall S \subset N$. We denote this set as $C(N, \tilde{V})$.

With allocations in the preference core, the coalitions obtain a fuzzy value which is at least as good as any of the fuzzy values in the characteristic set.

At this point we need to specify which partial order will be adopted to compare fuzzy numbers. Fuzzy numbers are usually represented by a finite number of α-cuts[3].

[3] The α-cut of a fuzzy number, $\tilde{z} \in \mathbb{N}(\mathbb{R})$, is the real closed interval $\tilde{z}_\alpha = \{z \in \mathbb{R} \mid \mu_{\tilde{z}}(z) \geq \alpha\}$, where $\mu_{\tilde{z}}$ is the membership function for $\tilde{z}$ (for $\alpha = 0$ we set $\tilde{z}_0 = cl\{z \in \mathbb{R} \mid \mu_{\tilde{z}}(z) > 0\}$, where cl denotes the closure of sets).

In general this representation implies a loss of information, however, in most of the cases considered in the literature this approximation is exact. Indeed, if the membership function is given by a piecewise linear function, only a finite number of different α-cut sets is needed to exactly describe the corresponding fuzzy number. Therefore, fuzzy numbers can be compared by only comparing a finite number of α-cut sets. In addition, since piecewise quasi-concave linear functions are dense in the set of quasi-concave functions, this approach can be used to approximate fuzzy numbers within any given accuracy.

Given a generic set of cuts, $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{r-1} < \alpha_r = 1$, for $j = 1, 2$, $\tilde{z}^j_{\alpha_k}$, denotes respectively the lower and upper extreme point of the interval $\tilde{z}_{\alpha_k}$. Note that for each $k = 1, \ldots, r$, $\tilde{z}^1_{\alpha_k} \leq \tilde{z}^2_{\alpha_k}$ and, for all $k = 1, \ldots, r-1$, $\tilde{z}^1_{\alpha_{k+1}} \geq \tilde{z}^1_{\alpha_k}$ and $\tilde{z}^2_{\alpha_{k+1}} \leq \tilde{z}^2_{\alpha_k}$. Note also that for $\tilde{c} \in \mathbb{N}(\mathbb{R})^P$ and $y \in \mathbb{R}^P$,

$$(\tilde{c}^t y)^j_{\alpha_k} = (\tilde{c}_1)^j_{\alpha_k} y_1 + \cdots + (\tilde{c}_P)^j_{\alpha_k} y_P.$$

In this paper we consider *standard fuzzy orders*, which are defined by considering a finite number of α-cuts. Given a generic set of cuts, the corresponding standard fuzzy order is: For $\tilde{z}, \tilde{w} \in \mathbb{N}(\mathbb{R})$

$$\tilde{z} \succeq \tilde{w} \Leftrightarrow \tilde{z}^j_{\alpha_k} \geq \tilde{w}^j_{\alpha_k}, \quad \text{for all } k = 1, \ldots, r,\ j = 1, 2.$$

If a standard fuzzy order is considered, a necessary and sufficient condition for an allocation to belong to the preference core of the fuzzy game can be established in terms of the maximum values of the α-cuts corresponding to the worth of the coalitions. Formally, let $w^*(S) \in \mathbb{R}^{r\times 2}$ be an $r \times 2$ matrix whose components are, for each $j = 1, 2$ and each $k = 1, \ldots, r$, $w^*(S)^j_k = \max_{\tilde{w}\in\tilde{V}(S)}\{\tilde{w}^j_{\alpha_k}\}$.

With this notation, an allocation of the fuzzy DEA production game $(N, \tilde{V})$ belongs to the preference core, $\tilde{Z} \in C(N, \tilde{V})$, if and only if for all $S \subset N$, $\tilde{Z}(S)^j_{\alpha_k} \geq w^*(S)^j_k$ for all $k = 1, \ldots, r,\ j = 1, 2$.

Given an allocation $\tilde{Z} \in I^*(N, \tilde{V})$ and a coalition $S \subset N$, we define the *preference excess* of coalition S with respect to the allocation $\tilde{Z}$ as:

$$E(S, \tilde{Z}) = \max_{j=1,2;\, k=1,2,\ldots,r} \left\{ w^*(S)^j_k - \tilde{Z}(S)^j_{\alpha_k} \right\}.$$

This excess quantifies the dissatisfaction of the coalition when the fuzzy allocation is $\tilde{Z}$.

Note that the preference core of the fuzzy DEA production game $(N, \tilde{V})$ can be written as:

$$C(N, \tilde{V}) = \{\tilde{Z} \in I^*(N, \tilde{V}) :\ E(S, \tilde{Z}) \leq 0,\ \ \forall S \subset N\}.$$

Depending on the game, the condition underlying this definition of preference core may yield a set with an infinite number of fuzzy allocations, a unique allocation, or may generate an empty set. In order to realistically solve the problem, and come up with a single allocation (or a reduced set of allocations) which captures the rationality notion underlying the preference core, this idea can be extended. Following the same idea as with the least core for set-valued crisp games, we are interested in what we call preference least core allocations (allocations such that the preference excess is minimum), that is, those allocations, $\tilde{Z}$, for which the most dissatisfied coalitions cannot be better off. These allocations belong to the preference core when it is non-empty. Moreover, they could be considered the best selection inside this set. In the case in which the preference core is empty these allocations are the closest to belonging to the preference core, and in this sense they can also be considered the best selection. Formally, we introduce the concept of preference least core for the fuzzy game as follows:

Definition 2 The preference least core of the fuzzy DEA production game $(N, \tilde{V})$ is

$$LC(N, \tilde{V}) = \{\tilde{Z} \in I^*(N, \tilde{V}) : \ E(S, \tilde{Z}) \leq \varepsilon^*, \ \forall S \subset N\},$$

where $\varepsilon^* = \min_{\tilde{Z} \in I^*(N, \tilde{V})} \max_{S \subset N} E(S, \tilde{Z})$.

The allocations in the preference least core are those which minimize the maximum preference excess of the coalitions in the set of all the allocations of production vectors in $\tilde{V}(N)$. Therefore, the set only contains allocations of those production vectors in which this minimum is attained.

In what follows, we provide results to compute allocations in the preference least core.

For $S \subset N$ and $\tilde{Z} \in I^*(N, V)$, denote $t_k^j(S, \tilde{Z}) = w^*(S)_k^j - \tilde{Z}(S)_{\alpha_k}^j$, for $j = 1, 2$, $k = 1, \ldots, r$. Thus, $E(S, \tilde{Z}) = \max_{j=1,2;\, k=1,2,\ldots,r} \left\{ t_k^j(S, \tilde{Z}) \right\}$.

Observe that, since $w^*(S)_k^j = \max_{\tilde{w} \in \tilde{V}(S)} \{\tilde{w}_{\alpha_k}^j\}$, the deviation $t_k^j(S, \tilde{Z}) = -\tilde{Z}(S)_{\alpha_k}^j$ if $S \in \bar{F}$, and, if $S \in F$ then the deviation can be obtained by solving the linear programming problem:

$$\begin{aligned} t_k^j(S, \tilde{Z}) = max & \sum_{i \in S} (\tilde{c}Y(S))_{\alpha_k}^j \lambda(i) - \tilde{Z}(S)_{\alpha_k}^j; \\ s.t. & \sum_{i \in S} X(S)\lambda(i) \leq b(S); \qquad [P_k^j(S, \tilde{Z})] \\ & \lambda(i) \geq 0, \ \textstyle\sum_{l=1}^{D(S)} \lambda_l(i) = 1, \ \forall i \in S. \end{aligned}$$

In which the variables are $\lambda(i) \in \mathbb{R}_+^{D(S)}$, for all $i \in S$.

The following result provides a single linear programming model to compute a fuzzy allocation that belongs to the Fuzzy Preference Least Core of the fuzzy DEA production game and the corresponding fuzzy total revenue that the grand coalition can obtain with this allocation.

Theorem 1 *Allocations in the preference least core, $LC(N, \tilde{V})$, can be computed by solving the following linear programming problem:*

$$
\begin{aligned}
&min\ \varepsilon\\
&s.t.\ \ b^t(S)\mu_S(j,k) + \textstyle\sum_{i\in S}\xi_S(i;j,k) - \tilde{Z}(S)^j_{\alpha_k} \le \varepsilon,\ \ \forall j=1,2,\ k=1\ldots,r,\ \ \forall S\in F;\\
&\quad -\tilde{Z}(S)^j_{\alpha_k} \le \varepsilon,\ \ \forall j=1,2,\ k=1,\ldots,r,\ \ \forall S\in \bar{F};\\
&\quad \mu^t_S(j,k)X(S) - (\tilde{c}Y(S))^j_{\alpha_k} + \xi_S(i;j,k)\mathbf{e}^t_{D(S)} \ge 0,\ \forall j=1,2;\ \forall k=1,\ldots,r,\ \forall i\in S,\ \forall S\in F;\\
&\quad \mu_S(j,k)\ge 0,\ \ \forall j=1,2,\ \forall k=1,\ldots,r,\ \ \forall S\in F;\\
&\quad \sum_{i\in N} X(N)\lambda(i) \le b(N);\\
&\quad \sum_{i\in N} (\tilde{c}Y(N))^j_{\alpha_k}\lambda(i) = (\tilde{Z}(N))^j_{\alpha_k},\ \forall j=1,2;\ \forall k=1,\ldots,r;\\
&\quad (\tilde{z}(i))^1_{\alpha_k} \le (\tilde{z}(i))^1_{\alpha_{k+1}},\ \forall k=1,\ldots r-1,\ \ \forall i\in N;\\
&\quad (\tilde{z}(i))^2_{\alpha_k} \ge (\tilde{z}(i))^2_{\alpha_{k+1}},\ \forall k=1,\ldots r-1,\ \ \forall i\in N;\\
&\quad (\tilde{z}(i))^1_{\alpha_k} \le (\tilde{z}(i))^2_{\alpha_k},\ \forall k=1,\ldots r,\ \forall i\in N;\\
&\quad \lambda(i)\in\Lambda^N,\ \forall i\in N;\\
&\quad \mu_S\ge 0,\ \forall S\in F;
\end{aligned}
$$

where F and $\bar{F}$ are respectively the set of feasible and unfeasible coalitions of the DEA production problem.

Proof Recall the notation $(\tilde{Z}(S))^j_{\alpha_k} = \sum_{i\in S}(\tilde{z}(i))^j_{\alpha_k}$ for $S \subseteq N$.

It follows from Definition 2, that the allocations in the preference least core for the fuzzy production game $(N, \tilde{V})$, are the solutions to

$$
\begin{aligned}
&min\ \varepsilon\\
&s.t.\ \ t^j_k(S,\tilde{Z}) \le \varepsilon,\ \forall j=1,2,\ k=1\ldots,r,\ \ \forall S\in N; \qquad [PLC]\\
&\qquad \tilde{Z}\in I(N,\tilde{V})
\end{aligned}
$$

As a consequence of duality in linear optimization problems, the value of $t^j_k(S,\tilde{Z})$ represented by means of a linear maximization problem in $[P^j_k(S,\tilde{Z})]$, can alternatively be computed as the solution of a linear minimization problem. That is to say, given a fuzzy allocation, $\tilde{Z}\in I^*(N,V)$, and a feasible coalition, $S\in F$, for $j=1,2$ and $k=1,\ldots,r$, $t^j_k(S,\tilde{Z})$ can be computed as

$$
\begin{aligned}
t^j_k(S,\tilde{Z}) = min\ & b^t(S)\mu_S(j,k) + \textstyle\sum_{i\in S}\xi_S(i;j,k) - \tilde{Z}(S)^j_{\alpha_k}\\
s.t.\ \ & \mu^t_S(j,k)X(S) - \left(\tilde{c}^tY(S)\right)^j_{\alpha_k} + \xi_S(i;j,k)\mathbf{e}^t_{D(S)} \ge 0,\ \forall i\in S;\\
& \mu_S(j,k)\ge 0,
\end{aligned}
$$

where $\mathbf{e}_{D(S)}$ is a $D(S)$-dimensional vector with all the components equal to one.

Note that for each $k = 1, \ldots, r$ and each $j = 1, 2$, the variables in this representation of $t_k^j(S, \tilde{Z})$ are $\mu_S(j,k) \in \mathbb{R}_+^M$ and $\xi_S(i; j, k) \in \mathbb{R}$, for $i \in S$. For a fixed S and a fixed $\tilde{Z}$, these variables do not depend on the variables in the constraints which assure $\tilde{Z} \in I(N, \tilde{V})$, $(\tilde{z}(i))^j_{\alpha_k}$, which are the extreme points of the intervals corresponding to the α-cuts of the fuzzy quantities $\tilde{z}(i)$. The other variables in the problem are the DEA parameters $\lambda(i)$, $i \in N$. Thus, the variables involved in the minimization problem to obtain the quantities $t_k^j(S, \tilde{Z})$ are independent of the variables which generate the efficient production vectors. Therefore, for each S and each $\tilde{Z}$, the expression of $t_k^j(S, \tilde{Z})$ as a minimization problem can be integrated into the minimization problem $[PLC]$, generating a single linear minimization problem containing all the constraints and variables.

By solving this linear programming problem, fuzzy allocations are obtained which represent the benefit obtained with an efficient production vector when the agents cooperate. Moreover, this efficient production vector, $\sum_{i \in N} Y(N)\lambda^*(i)$, yields the minimum preference excess of the coalitions. $\square$

It is worth noting that the representation of the fuzzy quantities resulting from the linear model in terms of a number of α-cuts is in accordance with the nature of the fuzzy quantities representing the prices of the products.

4 Numerical examples

4.1 The Färe and Zelenyuk data set

The data for the following numerical example are a fuzzy version of those in Färe and Zelenyuk (2003). Consider four firms, labelled 1, 2, 3 and 4, $N = \{1, 2, 3, 4\}$, that produce two different outputs by using two inputs. Five recorded observations are available for each firm. The vector of available resources and the recorded observations of the production process corresponding to the four producers are shown in Table 1

It is worth noting that in this case all the coalitions are feasible. That is, for all $S \subseteq N$, there exist x in the convex hull of the set of observed inputs such that $x \leq b(S)$.

Table 1 Available resources and recorded observations

Firm i	Available resources $\mathbf{b}^i$	Recorded observations $X(i)$	Recorded observations $Y(i)$
1	$\begin{pmatrix} 40 \\ 50 \end{pmatrix}$	$\begin{pmatrix} 39 & 37 & 35 & 34 & 33 \\ 49 & 45 & 55 & 63.97 & 53 \end{pmatrix}$	$\begin{pmatrix} 12 & 19 & 17.29 & 25 & 28 \\ 17.53 & 22 & 17 & 12.97 & 18.72 \end{pmatrix}$
2	$\begin{pmatrix} 60 \\ 80 \end{pmatrix}$	$\begin{pmatrix} 70 & 45 & 60 & 30 & 40 \\ 50 & 55.56 & 62.38 & 83.33 & 90 \end{pmatrix}$	$\begin{pmatrix} 35 & 25 & 45 & 75 & 34 \\ 43 & 0 & 37.42 & 64.03 & 59.27 \end{pmatrix}$
3	$\begin{pmatrix} 80 \\ 110 \end{pmatrix}$	$\begin{pmatrix} 75 & 45 & 60 & 87 & 85 \\ 75 & 125 & 93.75 & 53.57 & 66.18 \end{pmatrix}$	$\begin{pmatrix} 82 & 78 & 35 & 75 & 85 \\ 75 & 101.7 & 93.54 & 120 & 111 \end{pmatrix}$
4	$\begin{pmatrix} 100 \\ 140 \end{pmatrix}$	$\begin{pmatrix} 91 & 115.2 & 86.4 & 247 & 240 \\ 99 & 169 & 240 & 189 & 180 \end{pmatrix}$	$\begin{pmatrix} 100 & 115 & 80 & 230 & 247 \\ 171 & 212 & 151 & 347 & 359 \end{pmatrix}$

Table 2 Fuzzy prices

α-level	Fuzzy prices			
	$\tilde{c}_1$		$\tilde{c}_2$	
	Lower	Upper	Lower	Upper
1	1		0.1	
0	0.9	1.1	0	0.2

Table 3 A fuzzy triangular allocation

α-level	$\tilde{z}(1)$		$\tilde{z}(2)$		$\tilde{z}(3)$		$\tilde{z}(4)$		$\tilde{Z}(N) \in \tilde{V}(N)$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
1	42.957		115.669		127.046		169.278		454.95	
0	38,5	51.46	90.873	134.312	104.357	151.911	134.179	204.308	367.909	541.991

To compute the revenue, we first consider triangular fuzzy prices. Only two α-cuts are needed to represent these prizes. They are shown in Table 2.

By solving the problem in Theorem 1, $\varepsilon^* = -16.337$ is obtained. As a consequence, the preference core is non-empty, that is, fuzzy allocations exist in which all the agents and all the coalitions obtain fuzzy revenues at least as good as any of the revenues they can guarantee by themselves. A fuzzy allocation in the preference least core is also obtained. The corresponding α-cuts are given in Table 3, and the membership functions are represented in Fig. 1.

This means that the triangular fuzzy number $\tilde{z}(N)$ is the fuzzy revenue associated to an efficient production vector which provides the minimum excess of the coalitions. An allocation of this fuzzy revenue in the preference least core consists of a triangular fuzzy quantity for each one of the agents.

We want to check the effect on the allocations of the preference least core of a reduction of the uncertainty in prices. For this purpose we consider triangular fuzzy prices with a narrower support as shown in Table 4.

The minimum excess of the coalitions is now $\varepsilon^* = -18.3$. A fuzzy allocation in the preference least core and the corresponding α-cuts are given in Table 5, and the membership functions are represented in Fig. 2.

The reduction of uncertainty in the fuzzy numbers representing the output prices has affected the results. Note the relationship between the total fuzzy value allocated and the fuzzy quantities allocated to each agent in both cases. The total quantity to allocate is very similar at level α_1 with less uncertainty at level α_0 in the second case. There is also less uncertainty in the quantities allocated to each of the agents. The value of the maximum excess has also decreased in the second case, that is, the level of satisfaction of the coalitions with respect to the allocation has increased.

In the limit case in which the prices are crisp, $c_1 = 1$ and $c_2 = 0.1$, the value of the minimum excess is $\varepsilon^* = -20.365$. Thus, the core of the scalar TU game is non-empty. This is a general result for the class of DEA production games [see Lozano

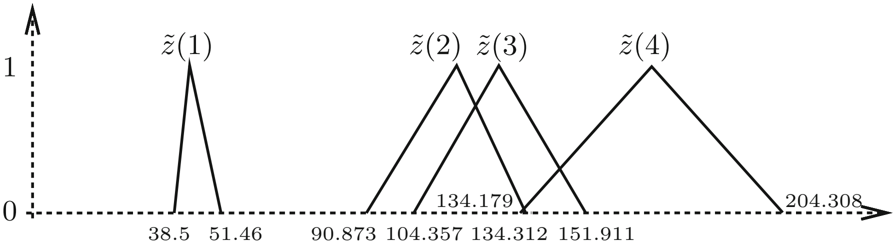

Fig. 1 An allocation in the preference least core

Table 4 More precise fuzzy prices

α-level	Fuzzy prices			
	$\tilde{c}_1$		$\tilde{c}_2$	
	Lower	Upper	Lower	Upper
1	1		0.1	
0	0.95	1.05	0.05	0.15

Table 5 A fuzzy triangular allocation

α-level	$\tilde{z}(1)$		$\tilde{z}(2)$		$\tilde{z}(3)$		$\tilde{z}(4)$		$\tilde{Z}(N) \in \tilde{V}(N)$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
1	44.92		115.669		127.046		167.851		455.486	
0	42.691	47.149	102,07	126,372	116,868	140,12	149,966	185,736	411.595	499.376

(2013)]. A core allocation in the least core consists of $z(1) = 46.985$, $z(2) = 115.669$, $z(3) = 127.046$, and $z(4) = 165.786$, and the level of satisfaction of the coalitions has increased with respect to the cases of uncertainty. Finally, we consider the case in which prices are given by trapezoidal fuzzy numbers, showing uncertainty at level α_0, and also at level α_1, as shown in Table 6.

A fuzzy allocation in the preference least core and the corresponding α-cuts are given in Table 7, and the membership functions are represented in Fig. 3.

In this case, the total fuzzy quantity to allocate is similar to that of the first case at level α_0, but with uncertainty at level α_1. The quantity allocated to each agent in most cases also shows uncertainty at level α_1. The maximum excess in this case coincides with that of the triangular fuzzy prices's case: $\varepsilon^* = -16.337$.

4.2 The Welch and Barnum data set

This section presents the application of the proposed approach in an electricity generation context. Specifically, the dataset comes from Welch and Barnum (2009) and corresponds to 40 power plants consuming two different inputs, namely coal and gas [both measured in millions of British Thermal Units (MBTU)] to produce a sin-

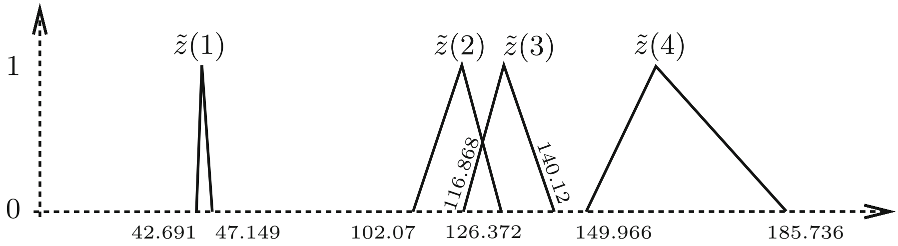

Fig. 2 An allocation in the preference least core

Table 6 Fuzzy prices

α-level	Fuzzy prices			
	$\tilde{c}_1$		$\tilde{c}_2$	
	Lower	Upper	Lower	Upper
1	0.95	1.05	0.05	0.15
0	0.9	1.1	0	0.2

Table 7 A fuzzy trapezoidal allocation

α-level	$\tilde{z}(1)$		$\tilde{z}(2)$		$\tilde{z}(3)$		$\tilde{z}(4)$		Total	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
1	40.729	46.164	104.521	126.372	114.416	139.142	151.763	186.793	411.429	498.471
0	38.5	47.415	92.332	126.372	102.892	163.897	134.179	204.308	367.909	541.991

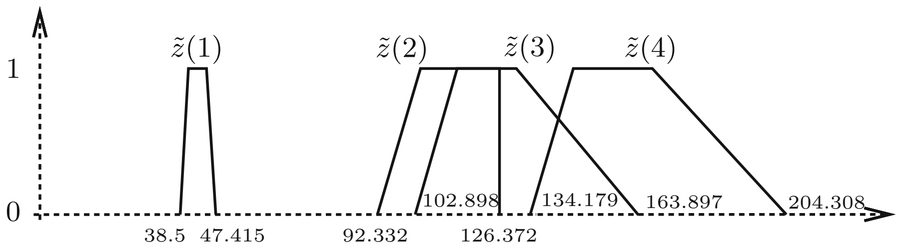

Fig. 3 An allocation in the preference least core

gle output, electricity [measured in Megawatts hour (MWh)]. It is well known that the wholesale price of electricity varies greatly depending on the demand, on wind generation, hydroelectric generation, etc. Therefore, in this case, it makes sense to model this uncertainty by considering a fuzzy unit price for electricity. We assume that there are four companies that are planning to cooperate and that each company has a technology that is defined by ten of the forty observed input/outputs vectors. The recorded observations are shown in Table 8. The available resources are assumed to be the same for the four companies and correspond to 25,000,000 MBTU of coal and 2,000,000 MBTU of gas.

Table 8 Recorded observations that define the technology of each firm

Firm	Observation	Coal (MBTU)	Gas (MBTU)	Electricity (MWh)
1	1	30, 294, 398	790, 320	3, 113, 275
	2	27, 926, 967	2, 550, 636	2, 867, 964
	3	21, 986, 752	1, 410, 674	2, 327, 664
	4	20, 974, 899	508, 244	2, 137, 471
	5	2, 502, 618	244, 302	187, 144
	6	15, 856, 662	3, 725, 422	1, 941, 594
	7	5, 325, 384	951, 687	452, 211
	8	3, 536, 185	163, 373	266, 080
	9	78, 253, 149	8, 748, 091	8, 819, 872
	10	33, 231, 640	2, 909, 784	3, 558, 851
2	1	13, 606, 948	2, 770, 980	1, 463, 766
	2	35, 122, 303	2, 509, 806	3, 793, 263
	3	3, 966, 695	93, 407	296, 085
	4	40, 586, 506	2, 569, 911	4, 189, 024
	5	6, 854, 478	4, 753, 060	1, 068, 182
	6	37, 629, 418	1, 666, 149	3, 953, 298
	7	12, 728, 829	987, 059	1, 206, 994
	8	20, 492, 696	14, 586, 950	3, 988, 193
	9	5, 487, 513	132, 764	472, 943
	10	10, 851, 116	1, 101, 141	1, 050, 721
3	1	105, 792, 226	2, 044, 087	10, 068, 396
	2	8, 194, 842	600, 839	755, 527
	3	68, 427, 176	23, 656, 034	9, 387, 075
	4	7, 672, 278	5, 699, 362	1, 268, 612
	5	8, 669, 126	272, 008	730, 010
	6	13, 640, 274	1, 885, 875	1, 518, 371
	7	9, 429, 930	218, 790	844, 955
	8	37, 642, 747	18, 584, 254	5, 438, 674
	9	21, 833, 538	540, 271	2, 148, 128
	10	25, 605, 350	749, 231	2, 687, 304
4	1	8, 354, 412	349, 435	690, 258
	2	33, 651, 611	8, 964, 868	3, 849, 802
	3	2, 498, 037	367, 034	197, 010
	4	3, 297, 863	150, 411	285, 503
	5	10, 158, 074	306, 901	854, 383
	6	39, 376, 529	1, 396, 389	4, 044, 295
	7	5, 719, 220	4, 435, 170	1, 130, 380
	8	45, 426, 084	6, 839, 605	5, 153, 723
	9	35, 898, 099	648, 762	3, 399, 125
	10	61, 873, 256	2, 203, 385	6, 076, 761

Table 9 Allocation computed by the proposed approach in each scenario (in million dollars)

Firm	Output unit prices				
	Crisp	Triangular fuzzy number		Trapezoidal fuzzy number	
1	247.9415	$\alpha = 1.0$	248.2304	$\alpha = 1.0$	[221.0411, 276.8304]
		$\alpha = 0.0$	[220.3924, 275.3749]	$\alpha = 0.0$	[165.2943, 332.6197]
2	253.7036	$\alpha = 1.0$	249.9866	$\alpha = 1.0$	[221.8184, 276.744]
		$\alpha = 0.0$	[222.5119, 278.8452]	$\alpha = 0.0$	[166.7872, 331.6696]
3	254.7764	$\alpha = 1.0$	258.0145	$\alpha = 1.0$	[230.013, 285.664]
		$\alpha = 0.0$	[229.4704, 285.7237]	$\alpha = 0.0$	[172.1995, 342.3736]
4	248.0405	$\alpha = 1.0$	248.2304	$\alpha = 1.0$	[219.9826, 276.8304]
		$\alpha = 0.0$	[220.4804, 276.1251]	$\alpha = 0.0$	[165.3603, 332.6197]
Total revenue	1,004.462	$\alpha = 1.0$	1, 004.462	$\alpha = 1.0$	[892.8551, 1, 116.069]
		$\alpha = 0.0$	[892.8551, 1, 116.069]	$\alpha = 0.0$	[669.6413, 1, 339.283]

We will consider three scenarios with increasing uncertainty. Thus, let us first assume a crisp unit price of 90 dollars/ MWh. The total revenue that the grand coalition could obtain in that scenario is 1,004.462 million dollars. In a second and in a third scenario, the unit price is represented by a triangular fuzzy number, (80, 90, 100), and by a trapezoidal fuzzy number, (60, 80, 100, 120), respectively. Accordingly, the total revenue of the grand coalition is represented by the triangular and the trapezoidal fuzzy numbers indicated in the last row of Table 9.

In the three scenarios an allocation in the preference least core has been computed by using the proposed approach. The associated excesses are $\varepsilon^* = -3.174049$, $\varepsilon^* = -2.821377$, and $\varepsilon^* = -2.116033$ (million dollars) for the crisp, triangular and trapezoidal fuzzy output unit price, respectively. Therefore, the core of the DEA production game with crisp unit prices is nonempty, and the preference cores of the DEA production games with fuzzy output prices are also nonempty. Moreover, the allocations obtained belong to the corresponding preference least cores. That is, with these allocations all the agents and all the coalitions obtain fuzzy revenues at least as good as any of the revenues they can guarantee by themselves. The corresponding α-cuts are given in Table 9. Note that the larger the uncertainty in the output unit price, the larger the uncertainty in the total revenue and in the computed fuzzy allocation.

5 Concluding remarks

In this paper the case of DEA production games with fuzzy unit output prices has been studied. The uncertainty in the output price translates into fuzzy total revenues for the grand coalition and accordingly to fuzzy allocations among the players. In order to deal with this situation a standard fuzzy order derived from a finite number of α-cuts is used to transform fuzzy DEA production games into a constrained version of Set-valued DEA production games, for which a Preference Least Core allocation can be computed. The constraints refer to the fact that the lower and upper limits of the different α-cuts (of both the total revenue and of the share of it allocated to each player) must be ordered, as a consequence of the nested character of the α-cuts.

Therefore, the proposed solution to this type of fuzzy DEA production games uses a linear programming model to compute the fuzzy total revenue and a corresponding fuzzy allocation which is guaranteed to lie within the Preference Least Core of the associated Set-valued DEA production game. As a consequence, this fuzzy allocation is as stable as it can be, in the sense that no other feasible fuzzy allocation can lead to a lower value of the maximum excess (computed for all coalitions). In other words, there are no fuzzy allocations that can outperform the solution obtained in terms of the satisfaction of the least satisfied coalition of players. This stability criterion is extensively used in cooperative games as it does guarantee that the grand coalition does not break, since there is no gain in doing that for any player.

The proposed approach has been illustrated with two different datasets from the literature. Each of these illustrative examples has considered different scenarios, corresponding to different degrees of uncertainty in the unit output prices. As expected, the larger the uncertainty in the output prices, the larger also the uncertainty in the total revenue and in its allocation to the players. In every case, the results show the effectiveness and usefulness of the proposed approach.

As challenging topics for further research, the first that comes to one's mind is considering other sources of uncertainty in addition to unit output prices. Thus, the same as there are many fuzzy DEA approaches that deal with fuzzy input and output data, DEA production games can also be considered with uncertainty in inputs and outputs of the recorded observations. This leads to a fuzzy production possibility set that clearly complicates the problem with respect to the case studied in this paper, in which the production possibility set is crisp, and only the revenue associated to the different operating points is uncertain.

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