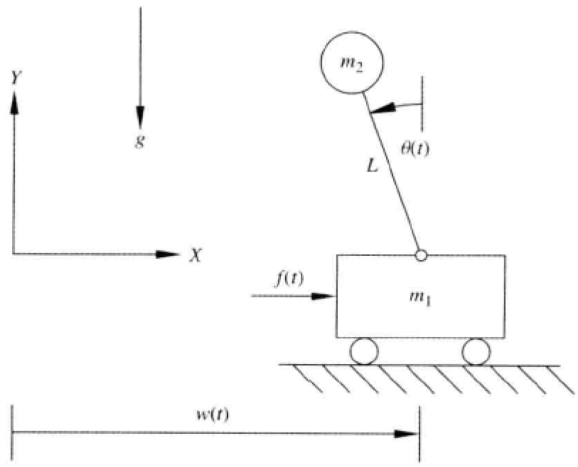
一阶倒立摆最优控制 Invert pendulum Optimal Control

考虑一阶倒立摆简化模型如下图，如图所示为非线性不稳定的倒立摆，目标是通过传感器测量$\theta(t)$构成反馈控制器来产生输入力$f(t)$，以保持倒立摆角度$\theta(t)$ = 0。小车的质量为$m1$，倒立摆质点质量为$m2$，假设倒立摆杆没有质量，同时地面光滑。



推导过程

系统状态方程求解 using Euler-Lagrange Equation

该系统的Euler-Lagrange Equation：

$$L = T - V$$

设车质量M，球质量m，杆长L，车x轴方向的位置为P

<u>车动能</u>:

$$T_M = \frac{1}{2}M\dot{P}^2$$

<u>球的动能</u>:

先表示出球的位置:

$$x_m = P + Lsin(\theta), y_m = Lcos(\theta)$$

则球的动能:

$$\begin{aligned} T_m &= \tfrac{1}{2}m({\dot{x}_m}^2 + {\dot{y}_m}^2) \\ &= \tfrac{1}{2}m\dot{P}^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 + m\dot{P}Lcos(\theta)\dot{\theta} \end{aligned}$$

<u>球的势能</u>:

$$V = mgLcos(\theta)$$

则:

$$L = T - V = \frac{1}{2}(m+M)\dot{P}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + m\dot{P}Lcos(\theta)\dot{\theta} - mgLcos(\theta)$$

写出欧拉方程：

$$\begin{cases} \frac{d}{dt} \cdot \left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = F \\ \frac{d}{dt} \cdot \left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \theta} = 0 \end{cases}$$

代入化简得：

$$\begin{cases} (m+M)\ddot{P} + (mL)\ddot{\theta} = F \\ (mLcos(\theta))\ddot{P} + (mL^2)\ddot{\theta} = mgLsin(\theta) \end{cases}$$

线性化近似得：

$$\begin{cases} (m+M)\ddot{P} + (mL)\ddot{\theta} = F \\ \ddot{P} + L^2\ddot{\theta} = gL\theta \end{cases}$$

解上述方程：

$$\begin{cases} \ddot{P} = (F - mg\theta)/M \\ \ddot{\theta} = ((M+m)g\theta - F)/(ML) \end{cases}$$

设状态变量$x = [P, \dot{P}, \theta, \dot{\theta}]$，由上述关系可得：

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{ML} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(ML) \end{bmatrix}$$

设计状态反馈

此为LQR问题，Cost Function为：

$$J = \int_0^\infty \left(x^TQx + u^TRu\right) dt$$

其最优控制为：

$$u = -Kx$$

其中，$K = R^{-1}B^TP$.

此问题是无限时长情况，P为代数$Riccati$方程的解：

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

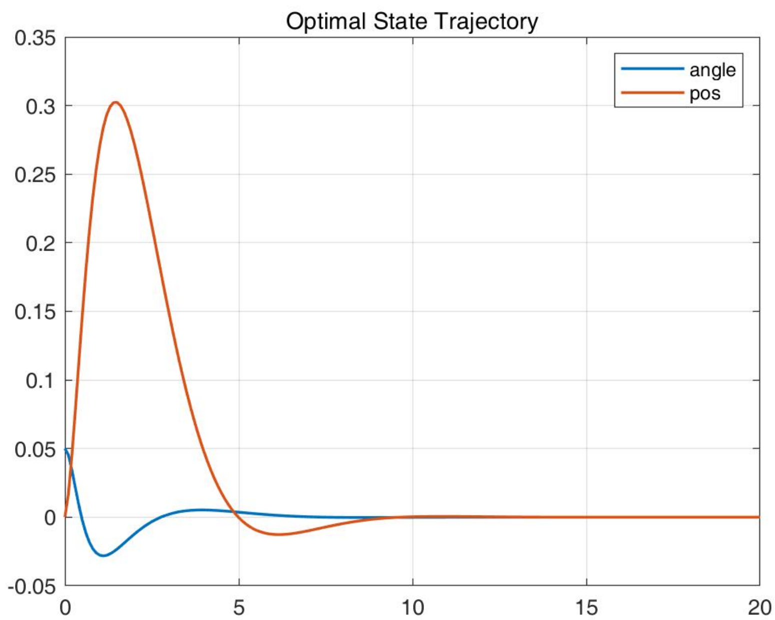
利用MATLAB中的`dlqr`函数将此系统作为离散系统求解，代码如下：

```
%% State-Space Model
A1 = [0,1,0,0;
    0,0,-m*g/M,0;
    0,0,0,1;
    0,0,(M+m)*g/M*L,0];

B1 = [0;1/M;0;-1/M*L];

C = [0 0 1 0;
```

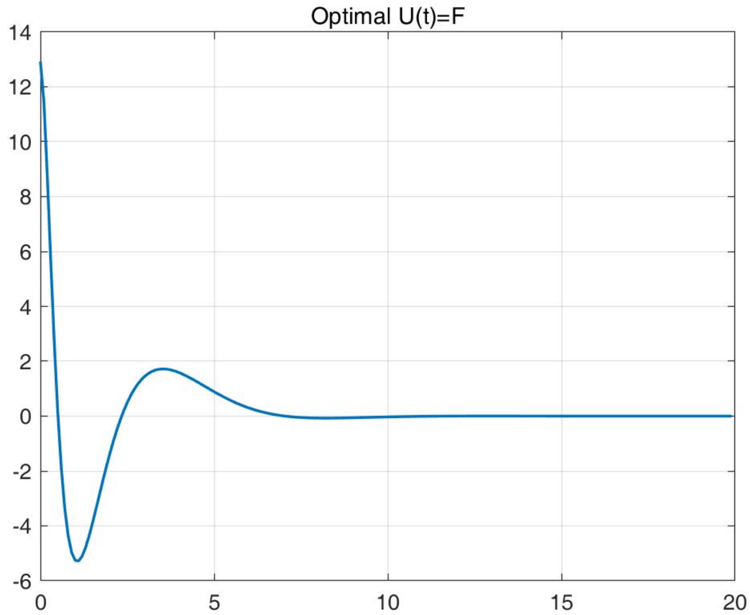
```
    1,0,0,0];

%% Cost-Fnc wight matrix init
Q = [100,0,0,0;
    0,0,0,0;
    0,0,10,0;
    0,0,0,0];
R = 1;

%% Generate sys
S1 = ss(A1,B1,C,0); % define the sys
Ts = 0.1; % sample time

Sd = c2d(S1,Ts); % transfer to disperse sys
[Ad,Bd,Cd,Dd,TS] = ssdata(Sd); % get disperse-sys state-space matrix

%% LQR
[K,S,e] = dlqr(Ad,Bd,Q,R);

%% Generate new sys with state-feedback
tS = ss(Ad-Bd*K,Bd,Cd,Dd,Ts); % get new sys with state-feedback

%% Given initial state & Plot the result
x0 = [0,0.1,0.05,0]'; % init state: P'=0.1;theta=0.05
t=[0:0.1:20]; % timespan
[Y,X] = initial(tS,x0,t); % calculates the response of sys
```

Output

最优状态轨线:



最优控制:



由上图可知，求得了使$P=0,\theta=0$的最优控制.