CS 70 Discrete Mathematics and Probability Theory
Summer 2023 Huang, Suzani, and Tausik **DIS 4C**

1 Probability Potpourri

Note 13
Note 14

Provide brief justification for each part.

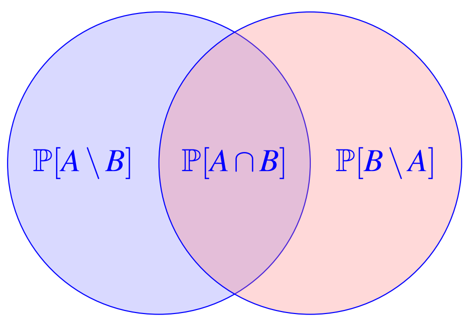
(a) For two events A and B in any probability space, show that $\mathbb{P}[A \setminus B] \geq \mathbb{P}[A] - \mathbb{P}[B]$.

(b) Suppose $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(c) If A and B are disjoint, does that imply they're independent?

Solution:

(a) It can be helpful to first draw out a Venn diagram:



We can see here that $\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \setminus B]$, and that $\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[B \setminus A]$. Looking at the RHS, we have

$$\begin{aligned}
\mathbb{P}[A] - \mathbb{P}[B] &= (\mathbb{P}[A \cap B] + \mathbb{P}[A \setminus B]) - (\mathbb{P}[A \cap B] + \mathbb{P}[B \setminus A]) \\
&= \mathbb{P}[A \setminus B] - \mathbb{P}[B \setminus A] \\
&\leq \mathbb{P}[A \setminus B]
\end{aligned}$$

(b) Using the total probability rule, we have

$$\mathbb{P}[D] = \mathbb{P}[D \cap C] + \mathbb{P}[D \cap \overline{C}] = \mathbb{P}[D \mid C] \cdot \mathbb{P}[C] + \mathbb{P}[D \mid \overline{C}] \cdot \mathbb{P}[\overline{C}].$$

But we know that $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, so this simplifies to

$$\mathbb{P}[D] = \mathbb{P}[D \mid C] \cdot (\mathbb{P}[C] + \mathbb{P}[\overline{C}]) = \mathbb{P}[D \mid C] \cdot 1 = \mathbb{P}[D \mid C],$$

which defines independence.

(c) No; if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}[A \cap B] = 0$. But these events are not independent: $\mathbb{P}[B \mid A] = 0$, but $\mathbb{P}[B] = 1/6$.

Since disjoint events have $\mathbb{P}[A \cap B] = 0$, we can see that the only time when disjoint A and B are independent is when either $\mathbb{P}[A] = 0$ or $\mathbb{P}[B] = 0$.

2 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

Solution:

(a) X_1 and X_2 are not independent. Intuitively, the fact that X_1 happened gives some information about the first coin that was chosen; this provides some information about the second coin that was chosen (since the first and second coins can't be the same coin), which directly affects whether X_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[X_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[X_2] = \mathbb{P}[X_1]$, so

$$\mathbb{P}[X_1]\,\mathbb{P}[X_2] = \frac{1}{4}.$$

But if we consider the probability that both X_1 and X_2 happen, we have

$$\begin{aligned}\mathbb{P}[X_1 \cap X_2] &= \frac{1}{6}\Big[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \\ &\qquad \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\Big] \\ &= \frac{22}{96} = \frac{11}{48}\end{aligned}$$

which is not equal to $1/4$, violating the definition of independence.

(b) Y_1 and Y_2 are not independent. Intuitively, the fact that Y_1 happens gives some information about the coin that was picked, which directly influences whether Y_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[Y_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[Y_2] = \mathbb{P}[Y_1]$, so

$$\mathbb{P}[Y_1]\,\mathbb{P}[Y_2] = \frac{1}{4}$$

But if we consider the probability that both Y_1 and Y_2 happen, we have

$$\mathbb{P}[Y_1 \cap Y_2] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24}$$

which is not equal to $1/4$, violating the definition of independence.

(c) Let A be the coin with bias $1/4$, B be the fair coin, and C be the coin with bias $3/4$. There are six orderings, each with probability $1/6$: ABC, ACB, BAC, BCA, CAB, and CBA. Thus

$$\begin{aligned}
&\mathbb{P}[\text{Third coin shows heads} \mid \text{First two coins show heads}] \\
&= \frac{\mathbb{P}[\text{All three coins show heads}]}{\mathbb{P}[\text{First two coins show heads}]} \\
&= \frac{(\frac{1}{4})(\frac{1}{2})(\frac{3}{4})}{\sum_{\text{Orderings}} \mathbb{P}[\text{First two coins show heads} \mid \text{Ordering}]\,\mathbb{P}[\text{Ordering}]} \\
&= \frac{(\frac{1}{4})(\frac{1}{2})(\frac{3}{4})}{(\frac{1}{6})\sum_{\text{Orderings}} \mathbb{P}[\text{First two coins show heads} \mid \text{Ordering}]} \\
&= \frac{(\frac{1}{4})(\frac{1}{2})(\frac{3}{4})}{(\frac{1}{6})\left((\frac{1}{4})(\frac{1}{2}) + (\frac{1}{4})(\frac{3}{4}) + (\frac{1}{2})(\frac{1}{4}) + (\frac{1}{2})(\frac{3}{4}) + (\frac{3}{4})(\frac{1}{4}) + (\frac{3}{4})(\frac{1}{2})\right)} \\
&= \frac{3/32}{11/48} = \frac{9}{22}.
\end{aligned}$$

3 Aces

Note 13
Note 14

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:

(a) Inclusion-Exclusion Principle: $\frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$.

(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52} + \frac{13}{52} - 2 \cdot \frac{1}{52} = \frac{15}{52}$.

(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.

(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$.

(e) Complement to getting no aces: $\mathbb{P}[\text{at least one ace}] = 1 - \mathbb{P}[\text{zero aces}] = 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$.

(f) Complement to getting no aces and no hearts: $\mathbb{P}[\text{at least one ace OR at least one heart}] = 1 - \mathbb{P}[\text{zero aces AND zero hearts}] = 1 - \frac{\binom{36}{5}}{\binom{52}{5}}$. This is because $52 - 13 - 3 = 36$, where 13 is the number of hearts and 3 is the number of non-heart aces.

4 Balls and Bins

Note 14

Suppose you throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Let A be the event that at least k bins are empty. Let m be the number of subsets of k bins out of the total n bins. If we assume A_i is the event that the ith set of k bins is empty. Then we can write A as the union of A_i's:

$$A = \bigcup_{i=1}^{m} A_i.$$

Compute m, and use the union bound to give an upper bound on the probability $\mathbb{P}[A]$.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution: Since the balls are thrown one at a time, there is an ordering, and so we are sampling with replacement where order matters rather than where it doesn't (which would correspond to each configuration in the stars and bars setup being equally likely).

(a) Note that this is a uniform sample space, with outcomes representing all possible ways to throw each ball individually into the bins. Here, $|\Omega| = n^n$, as each of the n balls has n possible bins to fall into, and out of these possibilities, $(n-1)^n$ of them leave the first bin empty—each ball would then have $n-1$ possible bins to fall into. This gives us an overall probability $\left(\frac{n-1}{n}\right)^n$ that the first bin is empty.

Equivalently, we can note that each throw is independent of all of the other throws. Since the probability that ball i does not land in the first bin is $\frac{n-1}{n}$, the probability that all of the balls do not land in the first bin is $\left(\frac{n-1}{n}\right)^n$.

(b) Similar to the previous part, we have the same uniform sample space of size n^n. Now, there are a total of $(n-k)^n$ possible ways to throw the balls into bins such that the first k bins are empty—each ball has $n-k$ possible bins to fall into.

Alternatively, we can similarly make use of independence. Since the probability that ball i does not land in the first k bins is $\frac{n-k}{n}$, the probability that all of the balls do not land in the first k bins is $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Then

$$\mathbb{P}[A] = \mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} \mathbb{P}[A_i].$$

We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}[A_i] = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}[A] \leq m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using Bayes' Rule:

$$\begin{aligned}
\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\
&= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\
&= \left(\frac{n-2}{n-1}\right)^n
\end{aligned}$$

Alternate solution: We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $\frac{n-2}{n-1}$. For n total balls, this probability is $\left(\frac{n-2}{n-1}\right)^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $\left(\frac{n-2}{n-1}\right)^n$. The probability that the second bin is empty (without any prior information) is $\left(\frac{n-1}{n}\right)^n$. Since these probabilities are not equal, the events are dependent.