# 1 Continuous Intro

Note 21

(a) Is

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate the PDF $f_X(x)$, along with $\mathbb{E}[X]$ and $\text{Var}(X)$ if the CDF of $X$ is

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{\ell}, & 0 \le x \le \ell, \\ 1, & x \ge \ell \end{cases}$$

(c) Suppose $X$ and $Y$ are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

What is their joint distribution? (Hint: for parts (c) and (d), we can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the $X$ and $Y$ in part (c).

**Solution:**

(a) Yes, it is a valid density function; it is non-negative and integrates to 1.

No, it is not a valid CDF; a CDF should go to 1 as $x$ goes to infinity and be non-decreasing.

(b) We have

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} \frac{1}{\ell}, & 0 \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_{x=0}^{\ell} x \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell}{2}$$

$$\mathbb{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell^2}{3}$$

$$\text{Var}(X) = \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{\ell^2}{12}$$

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted Uniform$[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{P}[X \in [x,x+\mathrm{d}x], Y \in [y,y+\mathrm{d}y]] \\ &= \mathbb{P}[X \in [x,x+\mathrm{d}x]]\mathbb{P}[Y \in [y,y+\mathrm{d}y]] \\ &\approx f_X(x)f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint distribution is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) We have

$$\mathbb{E}[XY] = \int_{x=0}^{1}\int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = \frac{1}{3}.$$

Alternatively, since $X$ and $Y$ are independent, we can compute $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that

$$\mathbb{E}[X] = \int_{0}^{1} x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\bigg|_0^1 = \frac{2}{3},$$

and $\mathbb{E}[Y] = \frac{1}{2}$ since the density of $Y$ is symmetric around $\frac{1}{2}$. Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3}.$$

# 2 Uniform Distribution

Note 21

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let $X$ be the position of the first spinner's mark and $Y$ be the position of the second spinner's mark, what is the probability that $X \ge 5$, given that $Y \ge X$?

**Solution:**

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \ge 5 \mid Y \ge X] = \frac{\mathbb{P}[Y \ge X \cap X \ge 5]}{\mathbb{P}[Y \ge X]}.$$

$\mathbb{P}[Y \ge X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \ge X \cap X \ge 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

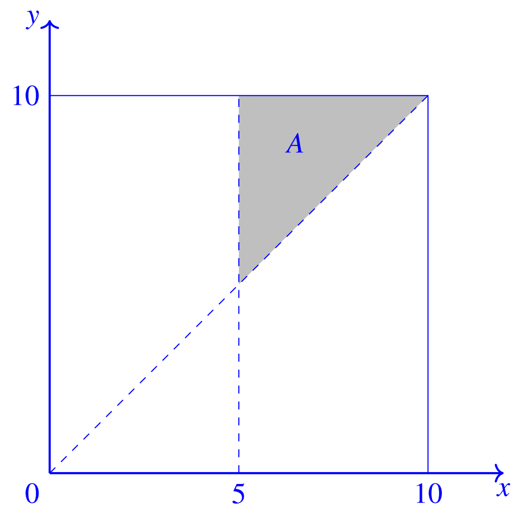


Figure 1: Joint probability density for the spinner.

Looking at the picture in Figure 1, we have

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8},$$

so $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

# 3 Darts

Yiming is playing darts. Her aim follows an exponential distribution with parameter 1; that is, the probability density that the dart is $x$ distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Yiming made it on the board. What is the probability she is within 1 unit from the center?

(c) If Yiming is within 1 unit from the center, she scores 4 points, if she is within 2 units, she scores 3, etc. In other words, Yiming scores $\lfloor 5 - x \rfloor$, where $x$ is the distance from the center. (This implies that Yimin scores 0 points if she throws it off the board). What is Yiming's expected score after one throw?

**Solution:**

(a) The CDF of an exponential is $\mathbb{P}[X \leq x] = 1 - \exp(-x)$. Therefore,

$$\mathbb{P}[X \leq 4] = 1 - \exp(-4).$$

(b) We are given that the dart must be within the board, which means that the dart is at least 4 units away from the center. We can use the definition of conditional probability:

$$\mathbb{P}[X \le 1 \mid X \le 4] = \frac{\mathbb{P}[X \le 1 \cap X \le 4]}{\mathbb{P}[X \le 4]} = \frac{\mathbb{P}[X \le 1]}{\mathbb{P}[X \le 4]} = \frac{1-\exp(-1)}{1-\exp(-4)}.$$

(c)

$$\begin{aligned}\mathbb{E}[\text{score}] &= \int_0^1 4\exp(-x)\,\mathrm{d}x + \int_1^2 3\exp(-x)\,\mathrm{d}x + \int_2^3 2\exp(-x)\,\mathrm{d}x + \int_3^4 \exp(-x)\,\mathrm{d}x \\ &= 4\big(-\exp(-1)+1\big) + 3\big(-\exp(-2)+\exp(-1)\big) + 2\big(-\exp(-3)+\exp(-2)\big) \\ &\quad + \big(-\exp(-4)+\exp(-3)\big) \\ &= 4 - \exp(-1) - \exp(-2) - \exp(-3) - \exp(-4).\end{aligned}$$

# 4 Darts Again

Note 21

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let $X$ be the distance of Edward's dart from the center, and $Y$ be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* $X$ is not uniform over $[0,10]$. Solve for the distribution of $X$ by first computing the CDF of $X$, $\mathbb{P}[X < x]$.]

**Solution:** We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of $X$ by solving for the CDF of $X$, $\mathbb{P}[X < x]$. To get this, we'll consider the ratio of the area where the distance to the center is less than $x$, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X < x] = \frac{\pi x^2}{\pi 10^2} = \frac{x^2}{100}.$$

Differentiating gives us the PDF of $X$, which is given by $f_X(x) = \frac{x}{50}$. Now, we solve for $\mathbb{P}[X < Y]$ with total probability:

$$\begin{aligned}\mathbb{P}[X < Y] &= \int_0^{10} \mathbb{P}[Y > X \mid X = x] f_X(x)\,\mathrm{d}x \\ &= \int_0^{10} \mathbb{P}[Y > x] f_X(x)\,\mathrm{d}x \\ &= \int_0^{10} \frac{x}{50} e^{-0.5x}\,\mathrm{d}x\end{aligned}$$

Evaluating this integral gives us $\mathbb{P}[X < Y] \approx 0.0767$.