1 Counting, Counting, and More Counting

Note 10 The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

i. How many different 13-card bridge hands are there?

ii. How many different 13-card bridge hands are there that contain no aces?

iii. How many different 13-card bridge hands are there that contain all four aces?

iv. How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

i. How many different anagrams of ALABAMA are there?

ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

i. C is the left neighbor of E

ii. C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 \cdot 2^{18}$

(c) i. We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

ii. We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

iii. We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

iv. We need our hand to contain 4 out of the 13 spade cards, and 9 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{4}\binom{39}{9}$ ways to make up the hand.

(d) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $\frac{104!}{2^{52}}$.

(e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.

This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99} \binom{99}{k} = \sum_{k=50}^{99} \binom{99}{99-k} = \sum_{l=0}^{49} \binom{99}{l}$$

by substituting $l = 99-k$.

Now $\sum_{k=50}^{99} \binom{99}{k} + \sum_{l=0}^{49} \binom{99}{l} = \sum_{m=0}^{99} \binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99} \binom{99}{k} = \frac{1}{2} \cdot 2^{99} = 2^{98}$.

Answer 2: Symmetry Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = \frac{1}{2} \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(f) ALABAMA: The number of ways of rearranging 7 distinct letters and is 7!. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of 4! (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want 1/4! out of the total rearrangements. Hence, there are $\frac{7!}{4!}$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 A's among themselves and another factor of 2! for the number of ways of permuting the 2 N's among themselves). Hence, there are $\frac{7!}{(2!)^2}$ different anagrams.

(g) i. We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is 5!.

ii. Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \cup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = \frac{6!}{2}$.

(h) Each ball has a choice of which bin it should go to. So each ball has 25 choices and the 8 balls can make their choices separately. Hence, there are 25^8 ways.

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 8$ and $n = 25$, so there are $\binom{32}{8}$ ways.

(j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. There are 2 cases to consider:

Case 1: The 2 balls land in the same bin. This gives 6 ways.

Case 2: The 2 balls land in different bins. This gives $\binom{6}{2}$ ways of choosing 2 out of the 6 bins for the balls to land in. Note that it is *not* 6×5 since the balls are identical and so there is no order on them.

Summing up the number of ways from both cases, we get $6 + \binom{6}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. From class (see note 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 6$, we get $\binom{7}{2}$ ways to do this.

EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $\frac{20!}{10! \cdot 2^{10}}$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\cdots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(l) $\binom{n+k}{k}$. This is just n indistinguishable balls into $k+1$ distinguishable bins (stars and bars). There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has. Note that this is the exact same answer as part (a) — make sure you understand why!

(m) $n-1$. It's easiest just to enumerate the solutions here. x_0 can take values $1,2,\ldots,n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(n) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. This is just $n-(k+1)$ indistinguishable balls into distinguishable $k+1$ bins. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Shipping Crates

Note 10

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks.

(a) How many ways are there to distribute the crates among the loading docks?

(b) Now, assume that any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away. (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock). We will now consider two configurations to be identical if, for every loading dock, the two configurations have the same number of leftover crates in that dock. How would your answer in the previous part compare to the number of outcomes given the new setup? Do not compute the actual value in this part, we will doing that in parts (c) - (e). We are looking for a qualitative answer (greater than part (a), equal to part (a), less than or equal to part (a), etc.) Justify your answer.

(c) We will now attempt to count the number of configurations of crates. First, we look at the case where crates are removed from the dock. How many ways are there to distribute the crates such that some crate gets removed from the dock?

(d) How many ways are there to distribute the crates such that no crates are removed from the dock; i.e. no dock receives at least 5 crates?

(e) Putting it together now, what are the total number of possible configurations for crates in the modified scenario? *Hint:* Observe that, regardless of which dock receives the 5 crates, we end up in the same situation. After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there?

Solution:

(a) This can be solved using stars and bars. We are simply distributing 8 indistinguishable balls into 4 distinguishable bins; the total amount of ways to count this is $\binom{11}{3}$.

(b) There are less possible outcomes in the new setup. The effect of the truck is that of a function, mapping the configurations that we counted in the previous part to a new set of outcomes; although the function may map two distinct configurations to the same outcome, it will certainly not map the same configuration to two different new outcomes. Thus, $\binom{11}{3}$ is a valid upper bound, which indicates that the number of outcomes is still finite (which means we can count it)!

(c) You may notice that it's only possible for one truck to receive five crates; we will leverage this fact in order to simplify our counting. We will count the number of ways to distribute the crate such that some dock receives 5 crates. Observe that, regardless of which dock receives the 5 crates, the scenario reduces to the same thing: we are simply distributing the leftover 3 crates among 4 docks. There are 4 ways to choose which dock has ≥ 5 crates, and $\binom{6}{3}$ ways to distribute the leftover 3 crates. Thus, there are $4\binom{6}{3}$ ways to distribute the crates such that some dock receives 5 crates.

(d) You can put all outcomes into one of two categories: outcomes in which some dock receives more than five crates, and outcomes in which no dock receives more than five crates. Note that these categories are mutually exclusive. As a result, we can take the complement of the previous part; the number of outcomes in which no crates are removed from the dock is simply the total number of outcomes, subtracted by the number of outcomes in which some dock receives more than five crates (found in the previous part). Putting it together, this gives us $\binom{11}{3} - 4\binom{6}{3}$.

(e) Regardless of which dock receives the 5 crates, we are left distributing 3 indistinguishable crates amount 4 docks. Using stars and bars, the total number of outcomes is $\binom{6}{3}$. Thus, to obtain the total amount of configurations, we count the amount of outcomes without removal of the crates, and add this to the amount of outcomes after the removal of the crates. Putting it together, we have that the total number of outcomes is

$$\binom{11}{3} - 4\binom{6}{3} + \binom{6}{3} = \binom{11}{3} - 3\binom{6}{3} = 105.$$

3 Fizzbuzz

Fizzbuzz is a classic software engineering interview question. You are given a natural number n, and for each integer i from 1 to n you have to print either "fizzbuzz" if i is divisible by 15, "fizz" if i is divisible by 3 but not 15, "buzz" if i is divisible by 5 but not 15, or the integer itself if i is not divisible by 3 or 5. Assume that n is a multiple of 15.

(a) How many printed lines will contain "fizzbuzz"?

(b) How many printed lines wlil contain "fizz"?

(c) How many printed lines will contain "buzz"?

(d) How many printed lines will contain an integer?

Solution:

(a) Every 15 numbers, fizzbuzz will be printed, and n is a multiple of 15, so fizzbuzz will be printed $\frac{n}{15}$ times.

(b) Every 15 numbers, there will be 5 numbers that are divisible by 3, 1 of which is also divisible by 15. Therefore, out of every 15 numbers, 4 will be divisible by 3 but not 15. Fizz will be printed $\frac{4n}{15}$ times.

(c) Every 15 numbers, there will be 3 numbers that are divisible by 5, 1 of which is also divisible by 15. Therefore, out of every 15 numbers, 2 will be divisible by 5 but not 15. Buzz will be printed $\frac{2n}{15}$ times.

(d) All the remaining numbers will be printed as themselves, so there are $n - \frac{n}{15} - \frac{4n}{15} - \frac{2n}{15} = \frac{8n}{15}$ integers that are printed.

4 Is This CS 61A?

Note 10

Define a Scheme bracket sequence to be a sequence of **n** opening and **n** closing brackets. A bracket sequence is considered **valid** when each opening bracket has a corresponding closing bracket. That is, for $n = 3$, $(())()$ is a valid bracket sequence whereas $())(()$ is not. Notice that in a valid sequence, if you read from left to right, the number of opening brackets seen so far must always be greater than or equal to the number of closing brackets.

(a) Compute the total number of bracket sequences in terms of n. Don't worry about whether the bracket sequence is valid yet!

(b) Compute the total number of **invalid** bracket sequences in terms of n. (Hint: Suppose $s = s_1 s_2 \dots s_{2n}$ is an invalid bracket sequence. Consider the first prefix $s_1 s_2 \dots s_i$ where there are more closed brackets than open brackets. What happens if we flip (change each open bracket to a closed bracket and vice versa) the rest of the bracket sequence $s_{i+1} \dots s_{2n}$?)

(c) Compute the total number of valid bracket sequences in terms of n. You may find your answers to part (a) and part (b) to be helpful.

Solution:

(a) $\binom{2n}{n}$ as there are total $2n$ characters, and n of which is (and the rest is).

(b) We show that the invalid bracket sequences can be mapped to a bracket sequence with $n-1$ open brackets and $n+1$ closed brackets. Consider an invalid bracket sequence $s = s_1 s_2 \dots s_{2n}$, and the prefix $s_1 s_2 \dots s_i$ is the first prefix with more closed brackets than open brackets. Then, there must be x open brackets and $x+1$ closed brackets in $s_1 s_2 \dots s_i$. Then, $s_{i+1} \dots s_{2n}$ has $n-x$ open brackets and $n-x-1$ closed brackets, so if we apply the "flip" operation described in the hint, then $\overline{s_{i+1}} \dots \overline{s_{2n}}$ has $n-x-1$ open brackets, $n-x$ closed brackets, so $\overline{s}$ has $n-1$ open brackets and $n+1$ closed brackets. To show that a bracket sequence with $n-1$ open brackets and $n+1$ closed brackets, $t = t_1 t_2 \dots d_{2n}$ can be mapped to an invalid bracket sequence, we first note that since there are more closed brackets than open brackets in t, there exists a prefix $t_1 t_2 \dots t_i$ with more closed brackets than open brackets. Suppose $t_1 t_2 \dots t_i$ be the first prefix with this property. Then, it must be that x open brackets and $x+1$ closed brackets in this prefix for some $x \in \mathbb{N}$. Then, $t_{i+1} \dots t_{2n}$ must have $n-1-x$ open brackets and $n-x$ closed brackets, so if we flip $t_{i+1} \dots t_{2n}$, then there must be $n-x$ open brackets and $n-1-x$ closed brackets in $\overline{t_{i+1}} \dots \overline{t_{2n}}$, so $\overline{t}$ has n open brackets and n closed brackets. Thus, the number of invalid bracket sequences is the same as the number of a bracket sequence with $n-1$ open brackets and $n+1$ closed brackets, so $\binom{2n}{n-1}$.

(c) The number of length $2n$ valid bracket sequences is the total number of length $2n$ bracket sequences minus the number of length $2n$ invalid bracket sequences, $\binom{2n}{n} - \binom{2n}{n-1}$.

5 Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c = m$.)

Solution:

(a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k, hence we get the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n-1$ people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k, we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

(b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can be do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, the picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a+b+c=m$) in $\binom{n}{a} \cdot \binom{n}{b} \cdot nc$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a,b,c), which gives us the expression

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

6 Is This EECS 126?

Note 13
Note 14

Youngmin loves birdwatching. There are three sites he birdwatches at: Site A, B, and C. He goes to Site A 60% of the time, Site B 35% of the time, and Site C 5% of the time. The probability of seeing a nightingale at each site is $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{2}{5}$, respectively. Using the information above, answer the following questions.

(a) What is the total probability that Youngmin sees a nightingale?

(b) Given that Youngmin sees a nightingale, what is the probability that Youngmin went to site B?

(c) Given that Youngmin didn't go to site B, what is the probability that Youngmin doesn't see a nightingale?

Solution:

(a) Let E be the event Youngmin sees a nightingale, A be the event that he goes to site A, B be the event that he goes to site B, and C be the event that he goes to site C. Then, we can calculate $\mathbb{P}[E]$ by total probability:

$$\begin{aligned}\mathbb{P}[E] &= \mathbb{P}[E|A]\mathbb{P}[A] + \mathbb{P}[E|B]\mathbb{P}[B] + \mathbb{P}[E|C]\mathbb{P}[C] \\ &= \frac{1}{10}\cdot\frac{3}{5} + \frac{3}{10}\cdot\frac{7}{20} + \frac{2}{5}\cdot\frac{1}{20} \\ &= \frac{3}{50} + \frac{21}{200} + \frac{1}{50} \\ &= \frac{37}{200}\end{aligned}$$

(b) Using Bayes' Rule,

$$\mathbb{P}[B|E] = \frac{\mathbb{P}[E|B]\mathbb{P}[B]}{\mathbb{P}[E]} = \frac{\frac{3}{10}\cdot\frac{7}{20}}{\frac{37}{200}} = \frac{21}{37}$$

(c) Noting that $\overline{B} = A \cup C$, we can write $\mathbb{P}[\overline{E}|\overline{B}]$ as

$$\mathbb{P}[\overline{E}|\overline{B}] = \mathbb{P}[\overline{E}|A\cup C] = 1 - \mathbb{P}[E|A\cup C]$$

Calculating $\mathbb{P}[E|A\cup C]$ can be done as the following:

$$\begin{aligned}\mathbb{P}[E|A\cup C] &= \frac{\mathbb{P}[E\cap A]}{\mathbb{P}[A\cup C]} + \frac{\mathbb{P}[E\cap C]}{\mathbb{P}[A\cup C]} \\ &= \frac{\mathbb{P}[E|A]\mathbb{P}[A]}{\mathbb{P}[A\cup C]} + \frac{\mathbb{P}[E|C]\mathbb{P}[C]}{\mathbb{P}[A\cup C]} \\ &= \frac{\frac{1}{10}\cdot\frac{3}{5}}{\frac{3}{5}+\frac{1}{20}} + \frac{\frac{2}{5}\cdot\frac{1}{20}}{\frac{3}{5}+\frac{1}{20}} \\ &= \frac{6}{65} + \frac{2}{65} = \frac{8}{65}\end{aligned}$$

7 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}[\omega]$.

iii. Compute $\mathbb{P}[E_1]$.

iv. Compute $\mathbb{P}[E_2]$.

(a) Fix a prime $q > 2$, and uniformly sample twice with replacement from $\{0,\dots,q-1\}$ (assume we have two $\{0,\dots,q-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other modulo q.

Let $E_1 =$ The resulting product is 0.

Let $E_2 =$ The product is $(q-1)/2$.

(b) Make a graph on v vertices by sampling uniformly at random from all possible edges. Here, assume for each edge we flip a fair coin; if it comes up heads, we include the edge in the graph, and otherwise we exclude that edge from the graph.

Let $E_1 =$ The graph is complete.

Let $E_2 =$ vertex v_1 has degree d.

(c) Create a random stable matching instance by having each person's preference list be a uniformly random permutation of the opposite entity's list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity's list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable).

Let $E_1 =$ All jobs have distinct favorite candidates.

Let $E_2 =$ The resulting pairing is the candidate-optimal stable pairing.

Solution:

(a) i. This is essentially the same as throwing two $\{0,\dots,q-1\}$-sided dice, so one appropriate sample space is $\Omega = \{(i,j)\ :\ i,j \in \mathrm{GF}(q)\}$.

ii. Since there are exactly q^2 such pairs, the probability of sampling each one is $\mathbb{P}[(i,j)] = 1/q^2$.

iii. Now in order for the product $i \cdot j$ to be zero, at least one of them has to be zero. There are exactly $2q-1$ such pairs, and so $\mathbb{P}[E_1] = \frac{2q-1}{q^2}$.

iv. For $i \cdot j$ to equal $(q-1)/2$ it doesn't matter what i is as long as $i \neq 0$ and $j \equiv i^{-1}(q-1)/2 \pmod q$. Thus $|E_2| = \left|\left\{(i,j)\ :\ j \equiv i^{-1}(q-1)/2\right\}\right| = q-1$, and whence $\mathbb{P}[E_2] = \frac{q-1}{q^2}$.
Alternative Solution for $\mathbb{P}[E_2]$*:* The previous reasoning showed that $(q-1)/2$ is in no

way special, and the probability that $i \cdot j = (q-1)/2$ is the same as $\mathbb{P}[i \cdot j = k]$ for any $k \in \mathrm{GF}(q)$. But

$$\begin{aligned} 1 &= \sum_{k=0}^{q-1} \mathbb{P}[i \cdot j = k] \\ &= \mathbb{P}[i \cdot j = 0] + (q-1)\mathbb{P}[i \cdot j = (q-1)/2] \\ &= \frac{2q-1}{q^2} + (q-1)\mathbb{P}[i \cdot j = (q-1)/2] \end{aligned}$$

and so $\mathbb{P}[E_2] = \left(1 - \frac{2q-1}{q^2}\right)/(q-1) = \frac{q-1}{q^2}$ as desired.

(b) i. Since any v-vertex graph can be sampled, Ω is the set of all graphs on v vertices.

ii. As there are $N = 2^{\binom{v}{2}}$ such graphs, the probability of each individual one g is $\mathbb{P}[g] = 1/N$ (by the same reasoning that every sequence of fair coin flips is equally likely!).

iii. There is only one complete graph on v vertices, and so $\mathbb{P}[E_1] = 1/N$.

iv. For vertex v_1 to have degree d, exactly d of its $v-1$ possible adjacent edges must be present. There are $\binom{v-1}{d}$ choices for such edges, and for any fixed choice, there are $2^{\binom{v}{2}-(v-1)}$ graphs with this choice. So $\mathbb{P}[E_2] = \frac{\binom{v-1}{d} 2^{\binom{v}{2}-(v-1)}}{2^{\binom{v}{2}}} = \binom{v-1}{d}\left(\frac{1}{2}\right)^{v-1}$.

(c) i. Here there are two random things we need to keep track of: The random preference lists and the random pairing. A person i's preference list can be represented as a permutation σ_i of $\{1,\ldots,n\}$, and the pairing itself is encoded in another permutation ρ of the same set (indicating that job i is paired with candidate $\rho(i)$). So $\Omega = \{(\sigma_1,\ldots,\sigma_{2n},\rho) : \sigma_i, \rho \in S_n\}$, where S_n is the set of permutations of $\{1,\ldots,n\}$.

ii. $|\Omega| = (n!)^{2n+1}$, and so $\mathbb{P}[\mathscr{P}] = 1/|\Omega|$ for each $\mathscr{P} \in \Omega$.

iii. For E_1, we observe that there are $n!$ possible configurations of all jobs having distinct favourite candidates, and that each job has $(n-1)!$ ways of ordering their non-favourite candidates, so

$$|E_1| = \underbrace{n!}_{\text{distinct favourites}} \cdot \underbrace{[(n-1)!]^n}_{\substack{\text{ordering of} \\ \text{non-favourites}}} \cdot \underbrace{(n!)^n}_{\substack{\text{candidate's} \\ \text{preferences}}} \cdot \underbrace{n!}_{\rho}.$$

Consequently, $\mathbb{P}[E_1] = n!\left(\frac{(n-1)!}{n!}\right)^n = \frac{n!}{n^n}$.

iv. No matter what $\sigma_1,\ldots,\sigma_{2n}$ are, there is exactly one candidate-optimal pairing, and so $\mathbb{P}[E_2] = \frac{(n!)^{2n}}{(n!)^{2n+1}} = \frac{1}{n!}$.