1 Set Operations

Note 0

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \dots\}$

(a) Given a set $A = \{1, 2, 3, 4\}$, what is $\mathscr{P}(A)$ (Power Set)?

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What is $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

Solution:

(a)

$$\mathscr{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

(b) $\mathscr{P}(B) = \{T \mid T \subseteq B\}$

(c) $\{\}$ or $\varnothing$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

(f) The set of irrational numbers

(g) $\varnothing$

2 Preserving Set Operations

Note 0
Note 2

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(b) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(c) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(d) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.

3 Inverses and Bijections

Note 0
Note 11

Recall that a function $f : A \to B$ is a bijection if it is an injection and a surjection, and it is invertible if there is a function $g : B \to A$ so that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, where $\mathrm{id}_A : A \to A$ and $\mathrm{id}_B : B \to B$ are the identity functions.

(a) Prove that if $f : A \to B$ is invertible then it is a bijection.

(b) Prove that if $f : A \to B$ is a bijection then it is invertible.

(c) Let $g : B \to A$ be the inverse function for some bijection f. Is g necessarily a bijection?

Solution:

(a) Suppose $g : B \to A$ is the inverse of f. First, we show f is injective. Suppose $f(x) = f(y)$ for some $x, y \in A$. Then, $g(f(x)) = g(f(y))$. Since g is the inverse of f, $g \circ f = \mathrm{id}_A$, so we get $x = y$. Thus, f is injective. Next, we show f is surjective. Consider any $b \in B$. Then, $g(b) \in A$ is such that $f(g(b)) = b$ because $f \circ g = \mathrm{id}_B$. So, f is surjective. Thus, f is a bijection.

(b) Since f is surjective, every element $b \in B$ is mapped to by something (in other words, the preimage $f^{-1}(\{b\})$ is nonempty). Since f is injective, every element $b \in B$ is mapped to by at most one thing (in other words, the preimage $f^{-1}(\{b\})$ has cardinality at most 1). Combining these facts, for each $b \in B$, $f^{-1}(\{b\}) = \{a\}$ for some $a \in A$. Define $g : B \to A$ so that $g(b)$ is the unique element in $f^{-1}(\{b\})$ for each $b \in B$. We claim g is the inverse of f.

First, consider $g \circ f : A \to B$. For any $a \in A$, $g(f(a))$ is the unique element in $f^{-1}(\{f(a)\})$ which must be a since f maps a to $f(a)$, so $g(f(a)) = a$ and thus $g \circ f = \mathrm{id}_A$. Now, consider $f \circ g : B \to A$. For any $b \in B$, $g(b)$ is the unique element in $f^{-1}(\{b\})$, which means f maps it to b. Thus, $f(g(b)) = b$ and so $f \circ g = \mathrm{id}_B$.

(c) Yes! The condition for being an inverse is symmetric, so f is the inverse of g. Therefore, g is invertible and hence a bijection by part (a).

4 Rationals and Irrationals

Note 2

Prove that the product of a non-zero rational number and an irrational number is irrational.

Solution: We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where $p, q, r,$ and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational.