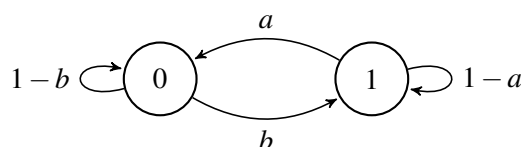


1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

Note 22

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i , the chain can transition to any other state j , possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i, i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathcal{X}$. If $d(i) = 1 \forall i \in \mathcal{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



- (a) For what values of a and b is the above Markov chain irreducible? Reducible?
- (b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.
- (c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.
- (d) Construct a transition probability matrix using the above Markov chain.
- (e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

- (a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one of a and b is 0.
- (b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2, 4, 6, \dots\} = 2.$$

This is because if we start at a state X then we can get back to it after taking an even number of steps only (2, 4, 6, 8, etc.), not by taking an odd number of steps (1, 3, 5, 7, etc.). Thus, the chain is periodic with period 2.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1, 2, 3, \dots\} = 1.$$

Thus, the chain is aperiodic. Notice that the self-loops allow us to stay at the same node, thereby letting us get to any other node in an odd *or* even number of steps.

(d)

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e)

$$\begin{aligned} \pi(0) &= (1-b)\pi(0) + a\pi(1), \\ \pi(1) &= b\pi(0) + (1-a)\pi(1). \end{aligned}$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

$$\pi = \frac{1}{a+b} \begin{bmatrix} a & b \end{bmatrix}.$$

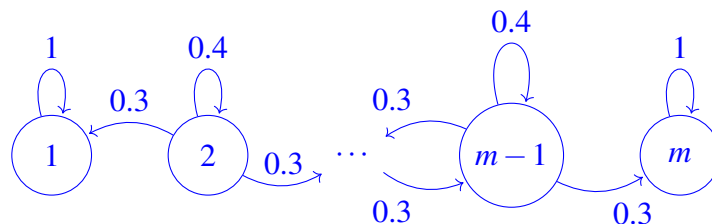
2 Can it be a Markov Chain?

Note 22

- (a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that the fly starts between positions 1 and m , model this process as a Markov Chain.
- (b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?

Solution:

(a) We can draw the Markov chain as such:



(b) No, because the memoryless property is violated.

For example, say $\mathbb{P}[X_0 = 2] = \mathbb{P}[X_0 = 3] = 1/2$ and $\mathbb{P}[X_0 = 1] = \mathbb{P}[X_0 = 4] = 0$. Then

$$\begin{aligned}\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 0] &= \mathbb{P}[X_2 \in \{1, 2\} \mid X_1 = 3, X_0 = 2] \\ &= \mathbb{P}[X_2 = 2 \mid X_1 = 3] = 0.3\end{aligned}$$

$$\begin{aligned}\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 1] &= \mathbb{P}[Y_2 = 0, Y_1 = 1, Y_0 = 1] / \mathbb{P}[Y_1 = 1, Y_0 = 1] \\ &= \mathbb{P}[X_2 = 2, X_1 = 3, X_0 = 3] / (\mathbb{P}[X_1 = 3, X_0 = 3] + \mathbb{P}[X_1 = 4, X_0 = 3]) \\ &= \frac{0.5 \cdot 0.4 \cdot 0.3}{0.5 \cdot 0.4 + 0.5 \cdot 0.3} = \frac{6}{35}\end{aligned}$$

If Y was Markov, then $\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 0] = \mathbb{P}[Y_2 = 0 \mid Y_1 = 1] = \mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 1]$. However, $0.3 > 6/35$, and so Y cannot be Markov.

3 Allen's Umbrella Setup

Note 22

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring exactly one umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p .

- Model this as a Markov chain. What is \mathcal{X} ? Write down the transition matrix.
- What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

- Let state i represent the situation that Allen has i umbrellas at his current location, for $i = 0, 1$, or 2 .

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

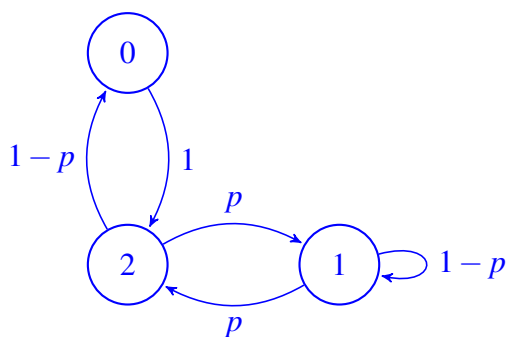
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p , it rains and Allen brings the umbrella, arriving at state 2. With probability $1 - p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \quad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1 - p$$

Suppose Allen is in state 2. With probability p , it rains and Allen brings the umbrella, arriving at state 1. With probability $1 - p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \quad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1 - p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

- (b) The transition matrices would be expressed as P^2 and P^n . Below we find the stationary distribution.

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P - I) = 0$. This yields the balance equations

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = [0 \quad 0 \quad 0].$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

Now solve for the distribution:

$$[\pi(0) \quad \pi(1) \quad \pi(2)] = \frac{1}{3-p} [1-p \quad 1 \quad 1]$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.