CS 70 Discrete Mathematics and Probability Theory
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DIS 5B

1 Such High Expectations

Note 19

Suppose X and Y are independently drawn from a Geometric distribution with parameter p.

(a) Compute $\mathbb{E}[\min(X,Y)]$.

(b) Compute $\mathbb{E}[\max(X,Y)]$.

Solution:

(a) By independence,

$$\mathbb{P}[\min(X,Y) \geq t] = \mathbb{P}[X \geq t]\mathbb{P}[Y \geq t] = (1-p)^{2(t-1)}.$$

By Tail Sum,

$$\mathbb{E}[\min(X,Y)] = \sum_{t=1}^{\infty} \mathbb{P}[\min(X,Y) \geq t] = \sum_{t=1}^{\infty} (1-p)^{2(t-1)} = \frac{1}{1-(1-p)^2}.$$

(b) We see that

$$\begin{aligned}
\mathbb{P}[\max(X,Y) \geq t] = 1 - \mathbb{P}[\max(X,Y) < t] &= 1 - \mathbb{P}[X < t]\mathbb{P}[Y < t] \\
&= 1 - (1 - \mathbb{P}[X \geq t])(1 - \mathbb{P}[Y \geq t]) \\
&= 1 - \left(1 - (1-p)^{t-1}\right)\left(1 - (1-p)^{t-1}\right) \\
&= 1 - \left(1 - 2(1-p)^{t-1} + (1-p)^{2(t-1)}\right) \\
&= 2(1-p)^{t-1} - (1-p)^{2(t-1)}.
\end{aligned}$$

Using the result from part (a),

$$\begin{aligned}
\mathbb{E}[\max(X,Y)] &= \sum_{t=1}^{\infty} \mathbb{P}[\max(X,Y) \geq t] \\
&= \sum_{t=1}^{\infty} 2(1-p)^{t-1} - (1-p)^{2(t-1)} \\
&= \sum_{t=1}^{\infty} 2(1-p)^{t-1} - \sum_{t=1}^{\infty} (1-p)^{2(t-1)} \\
&= \frac{2}{p} - \frac{1}{1-(1-p)^2}.
\end{aligned}$$

Alternate Solution: An extremely elegant one-liner with linearity:

$$\mathbb{E}[\max(X,Y)] = \mathbb{E}[X+Y-\max(X,Y)] = \mathbb{E}[X]+\mathbb{E}[Y]-\mathbb{E}[\min(X,Y)] = \frac{2}{p} - \frac{1}{1-(1-p)^2}.$$

2 Number Game

Note 20 Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0,100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V \mid S = s]$, where s is any constant such that $0 \le s \le 100$?

(c) What is $\mathbb{E}[V]$?

Solution:

(a) S is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.

(b) If $S = s$, we know that V will be uniformly distributed between s and 100. Similar to the previous part, this gives us that $\mathbb{E}[V \mid S = s] = \frac{s+100}{2}$.

(c) With the law of total expectation, we have that

$$\begin{aligned}
\mathbb{E}[V] &= \sum_{s=0}^{100} \mathbb{E}[V \mid S = s] \cdot \mathbb{P}[S = s] \\
&= \sum_{s=0}^{100} \frac{s+100}{2} \cdot \frac{1}{101} \\
&= \frac{1}{202} \left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100 \right)
\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

3 Number of Ones

Note 20 In this problem, we will revisit dice-rolling, except with conditional expectation. (*Hint*: for both of these subparts, the law of total expectation may be helpful.)

(a) If we roll a die until we see a 6, how many ones should we expect to see?

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

Solution:

(a) Let Y be the number of ones we see. Let X be the number of rolls we take until we get a 6.

Let us first compute $\mathbb{E}[Y \mid X = k]$. We know that in each of our $k-1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $1/5$ chance of getting a one in each of these $k-1$ previous rolls, giving

$$\mathbb{E}[Y \mid X = k] = \frac{1}{5}(k-1).$$

If this is confusing, we can write Y as a sum of indicator variables, $Y = Y_1 + Y_2 + \cdots + Y_k$, where Y_i is 1 if we see a one on the ith roll. This means that by linearity of expectation,

$$\mathbb{E}[Y \mid X = k] = \mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_k \mid X = k].$$

We know that on the kth roll, we must roll a 6, so $\mathbb{E}[Y_k] = 0$. Further, by symmetry, each term in this summation has the same value; this means that we have

$$\begin{aligned}\mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_{k-1} \mid X = k] &= (k-1)\,\mathbb{E}[Y_1 \mid X = k] \\ &= (k-1)\mathbb{P}[Y_1 = 1 \mid X = k] \\ &= (k-1)\frac{1}{5}.\end{aligned}$$

Using the law of total expectation, we now have

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{k=1}^{\infty} \mathbb{E}[Y \mid X = k]\mathbb{P}[X = k] && \text{(total expectation)} \\ &= \sum_{k=1}^{\infty} \frac{1}{5}(k-1)\mathbb{P}[X = k]\end{aligned}$$

Here, we can see that this is an application of LOTUS for $f(X) = \frac{1}{5}(X-1)$, so we can simplify this to

$$\begin{aligned}&= \mathbb{E}\left[\frac{1}{5}(X-1)\right] && \text{(LOTUS)} \\ &= \frac{1}{5}(\mathbb{E}[X] - 1) && \text{(linearity)}\end{aligned}$$

Since $X \sim \text{Geometric}(\frac{1}{6})$, the expected number of rolls until we roll a 6 is $\mathbb{E}[X] = 6$:

$$= \frac{1}{5}(6-1) = 1$$

Alternatively, we can use iterated expectation, along with the fact that $\mathbb{E}[Y \mid X] = \frac{1}{5}(X-1)$, to give

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\
&= \mathbb{E}\left[\frac{1}{5}(X-1)\right] \\
&= \frac{1}{5}(\mathbb{E}[X]-1) \\
&= \frac{1}{5}(6-1) = 1
\end{aligned}$$

(b) We use the same logic as the first part, except now each of the first $k-1$ rolls can only be 1, 2, or 3, so

$$\mathbb{E}[Y \mid X = k] = \frac{1}{3}(k-1).$$

Using the law of total expectation, we have

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{k=1}^{\infty} \mathbb{E}[Y \mid X = k]\mathbb{P}[X = k] && \text{(total expectation)} \\
&= \sum_{k=1}^{\infty} \frac{1}{3}(k-1)\mathbb{P}[X = k] \\
&= \mathbb{E}\left[\frac{1}{3}(X-1)\right] && \text{(LOTUS)} \\
&= \frac{1}{3}(\mathbb{E}[X]-1) && \text{(linearity)}
\end{aligned}$$

Since now $X \sim \text{Geometric}(\frac{1}{2})$, the expected number of rolls until we roll a number greater than 3 is $\mathbb{E}[X] = 2$:

$$= \frac{1}{3}(2-1) = \frac{1}{3}$$

Alternatively, we can use iterated expectation, along with the fact that $\mathbb{E}[Y \mid X] = \frac{1}{3}(X-1)$, to give

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\
&= \mathbb{E}\left[\frac{1}{3}(X-1)\right] \\
&= \frac{1}{3}(\mathbb{E}[X]-1) \\
&= \frac{1}{3}(2-1) = \frac{1}{3}
\end{aligned}$$