1 How Much Counting? A Lot...

a. How many polynomials of degree at most d are there in $GF(p)$ for a prime $p > d$? [**2 points**]

Solution: p^{d+1}. There are $d+1$ coefficients, which can range from 0 to $p-1$.

b. How many polynomials of degree at most d in $GF(p)$ (for a prime $p > d > 5$) go through the points $(2,3)$ and $(4,5)$? [**3 points**]

Solution: $d+1$ points define a polynomial of degree at most d, two of them are already set. Since we're working in $GF(p)$ this means there are p options for each of the remaining $d-1$ points, so the answer is p^{d-1}.

c. The Berkeley library has five distinct books on discrete math and eight distinct books on probability. How many ways are there to choose three books from the library such that at least 2 of the books are on discrete math? Assume the library only has one copy of each book. [**3 points**]

Solution: $\binom{5}{3} + 8\binom{5}{2}$ (which is 90). There are $\binom{5}{3}$ ways to check out 3 discrete math books. If we want a probability book, there are 8 choices, and then $\binom{5}{2}$ ways to pick remaining 2 discrete math books.

d. Nate is buying burritos. When ordering a burrito, he has two choices of rice, three choices of beans, and eight choices of protein. Note that he must pick **exactly** one option from each category. In order to feed the TAs, Nate chooses 13 not-necessarily-distinct burrito combinations. How many possible ways can he choose these 13 burrito combinations? The order of the burritos does not matter. [**3 points**]

Solution: $\binom{60}{47}$. There are $2 \cdot 3 \cdot 8 = 48$ different ways to make a single burrito. Therefore, if we label these 48 different combinations as $x_1, x_2, \ldots x_{48}$, we can now say that

$$x_1 + x_2 + \ldots x_{48} = 13.$$

This problem is now equivalent to distributing 13 balls (TAs) across the 48 bins (burrito varieties). Using stars and bars, we can see that the number of solutions to this equation is $\binom{13+48-1}{48-1} = \binom{60}{47}$.

e. Nate falls drastically ill from his poor burrito choice, and must set up a secret to hide quiz solutions from the wrong hands. Given 2 instructors, 3 TAs, and 10 readers, Nate wants to ensure the secret polynomial can only be solved with the approval of either one member of each group, or all members of any one group. (Ex: 10 readers could unlock the secret; alternatively, one instructor, one TA, and one reader could unlock the secret.)

He sets up the following scheme. Create a degree 27 polynomial in $GF(p)$ for large prime p. Give each instructor 15 points, give each TA 10 points, and give each reader 3 points. Assume all of the points given are distinct. Does Nate's scheme work? Why or why not? [**4 points**]

Solution: Nate's scheme does not work – one counterexample is 9 readers and one TA, who are not supposed to be able to get the secret, but who would have $9 \cdot 3 + 10 = 37$ points, which is more than the 28 necessary to solve a degree 27 polynomial.

2 It's Been a Long Time Coming ...

Nikki is going to the Eras Tour on July 28th!

a. There are 30 lines that enter the arena, numbered $\{1, 2, ..., 30\}$. If Nikki randomly selects a line to stand in, what is the probability its corresponding number is divisible by 3? **[2 points]**

Solution: Uniform probability space. Size of the sample space is 30, there are 10 seat numbers within the set divisible by 3. Answer is $\frac{10}{30} = \frac{1}{3}$.

b. At the concert, each person (including Nikki) has a 40% chance of having a floor ticket, and a 60% chance of having a seat ticket. 100% of the people with floor tickets buy merch, but only 50% of the people with seat tickets buy merch.

(i) What is the probability that Nikki buys merch? **[3 points]**

Solution: Denote M as the event of buying merch, S as the event of having a seat ticket, and F as the event of having a floor ticket. Since $\overline{S} = F$, by total probability

$$\mathbb{P}[M] = \mathbb{P}[M|S]\,\mathbb{P}[S] + \mathbb{P}[M|F]\,\mathbb{P}[F] = 0.5 \cdot 0.6 + 1 \cdot 0.4 = 0.7.$$

(ii) After the concert, you find out that Nikki did indeed buy merch. What is the probability that she had a floor ticket? **[3 points]**

Solution: Denote F as the event of getting floor ticket and M as the event of buying merch.

$$\mathbb{P}[F|M] = \frac{\mathbb{P}[F \cap M]}{\mathbb{P}[M]} = \frac{0.4}{0.7} = \frac{4}{7}.$$

c. At every concert, there is a 60% chance that a proposal happens. There are 44 total concerts.

(i) What is the probability that there is **exactly** 1 concert **without** a proposal? **[3 points]**

Solution: There are $\binom{44}{1} = 44$ ways to choose the concert that doesn't have a proposal. The probability the other 43 concerts has a proposal is $(0.6)^{43}$ and the probability the chosen concert doesn't have a proposal is 0.4. Hence, the probability is $44(0.4)(0.6)^{43}$.

(ii) Call S the number of concerts **without** proposals. What is the distribution of S? Either explicitly name the distribution and parameters, or give $\mathbb{P}[S = s]$ for all possible s. **[3 points]**

Solution: $S \sim \text{Binom}(44, 0.4), P[S = s] = \binom{44}{s}(0.4)^s(0.6)^{44-s}$

d. Gavin owns 8 distinct Taylor Swift vinyls, and wants to arrange them on his shelf. Let us enumerate them as A, B, C, D, E, F, G, and H. If he picks a uniformly random permutation of them, what is the probability that the permutation satisfies **at least one** of the following conditions: **[5 points]**

- E and F are placed next to each other
- B is placed at any point to the left of C

Solution: Denote X as the event representing the first bullet point and Y as the event representing the second.

To compute $\mathbb{P}[X]$ we merge E and F as a single unit: we now have 7 distinct units which we know has a total of 7! arrangements. Since the unit of E and F could can look like "EF" or "FE", we have 2 ways to arrange the unit leading to a total of $2 \cdot 7!$ arrangements. This means that $\mathbb{P}[X] = \frac{2 \cdot 7!}{8!} = \frac{1}{4}$.

We can see that $\mathbb{P}[Y] = \frac{1}{2}$ by symmetry: every arrangement that has B to the left of C has a corresponding arrangement with C to the left of B.

We now compute $\mathbb{P}[X \cap Y]$. We saw in the computation of $\mathbb{P}[X]$ that there were $2 \cdot 7!$ arrangements with E and F being adjacent. By the same argument for $\mathbb{P}[Y]$, half of these $2 \cdot 7!$ arrangements have B to the left of C so $\mathbb{P}[X \cap Y] = \frac{1}{2} \cdot \frac{2 \cdot 7!}{8!} = \frac{1}{8}$.

By Inclusion-Exclusion,

$$\mathbb{P}[X \cup Y] = \mathbb{P}[X] + \mathbb{P}[Y] - \mathbb{P}[X \cap Y] = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}.$$

3 That's So Random!

a. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ and $Z \sim \text{Poisson}(\lambda_3)$ be three independent random variables. What is the distribution of $X + Y + Z$? **[2 points]**

Solution: $X + Y + Z \sim \text{Poisson}(\lambda_1 + \lambda_2 + \lambda_3)$ since you're summing three independent Poisson random variables. Proven in Note 19.

b. Aaron flips a coin until he gets five total heads (not necessarily consecutively). What's the probability that the fifth head he gets happens **on** his twelveth coin flip, given that his second head was on his second coin flip? Assume the coin has probability p of landing heads. **[5 points]**

Solution: Using the fact that it's memoryless, you want to find the probability that in 10 coin flips, Aaron gets three heads AND that his last of the three heads is on the 10th flip. This means that in 9 flips, he got two heads, and then ended with a final head. The probability of this is

$$\binom{9}{2} p^2 (1-p)^7 p = \binom{9}{2} p^3 (1-p)^7.$$

c. Provide a combinatorial proof that

$$\binom{n}{4} = \sum_{k=2}^{n-2} \binom{k-1}{1} \binom{n-k}{2}$$

Answers which use **any** algebraic manipulation will receive no credit. **[4 points]**

Solution: We show the two sides are equivalent.
LHS: We select 4 people from a total of n people for a team.

RHS: Suppose we number the people from 1 to n. Suppose person k is the second person chosen for the team when ordered by their number. That means the first person was chosen from the previous $k-1$ people and the remaining two spots must be filled from the remaining $n-k$ people. This is what $\binom{k-1}{1}\binom{n-k}{2}$ represents. Since k can be arbitrary from 2 to $n-2$ (to allow equal space for a first and a third/fourth person), we iterate with a summation.

d. Let $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(p)$ be two independent random variables. For any $n \geq 2$, what is $P(X+Y=n)$?
Provide justification, noting that to receive full credit, your final answer should **not** include a summation. **[5 points]**

Solution: There's two ways to approach this. We can do the math explicitly, which gets us

$$\begin{aligned}
\mathbb{P}[X+Y=n] &= \sum_{k=1}^{n-1} \mathbb{P}[X=k \cap Y=n-k] \\
&= \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) \\
&= \sum_{k=1}^{n-1} (1-p)^{k-1}p(1-p)^{n-k-1}p \\
&= (1-p)^{n-2}p^2 \sum_{k=1}^{n-1} 1 \\
&= (n-1)(1-p)^{n-2}p^2.
\end{aligned}$$

Alternate solution: We can also think about it conceptually, where the summation is the number of trials until we get two successes. So in the first $n-1$ trials, we must get exactly one success – the number of ways to do this is $\binom{n-1}{1}p(1-p)^{n-2}$ and then we must also end with a success – multiplying by p. So the solution is $\binom{n-1}{1}p(1-p)^{n-2}p = (n-1)(1-p)^{n-2}p^2$