

1 Strings

Note 10 What is the number of strings you can construct given:

- (a) n ones, and m zeroes?
- (b) n_1 A's, n_2 B's and n_3 C's?
- (c) n_1, n_2, \dots, n_k respectively of k different letters?

Solution:

- (a) $\binom{n+m}{n}$
- (b) $(n_1 + n_2 + n_3)! / (n_1! \cdot n_2! \cdot n_3!)$
- (c) $(n_1 + n_2 + \dots + n_k)! / (n_1! \cdot n_2! \cdot \dots \cdot n_k!)$.

2 You'll Never Count Alone

Note 10 (a) An anagram of LIVERPOOL is any re-ordering of the letters of LIVERPOOL, i.e., any string made up of the letters L, I, V, E, R, P, O, O, L in any order. For example, IVLERPOOL and POLIVOLRE are anagrams of LIVERPOOL but PIVEOLR and CHELSEA are not. The anagram does not have to be an English word.

How many different anagrams of LIVERPOOL are there?

- (b) How many solutions does $y_0 + y_1 + \dots + y_k = n$ have, if each y must be a non-negative integer?
- (c) How many solutions does $y_0 + y_1 + \dots + y_k = n$ have, if each y must be a positive integer?

Solution:

- (a) In this 9 letter word, the letters L and O are each repeated 2 times while the other letters appear once. Hence, the number $9!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 L's among themselves and another factor of $2!$ for the number of ways of permuting the 2 O's among themselves). Hence, there are $9! / (2!)^2$ different anagrams.

- (b) $\binom{n+k}{k}$. We can imagine this as a sequence of n ones and k plus signs: y_0 is the number of ones before the first plus, y_1 is the number of ones between the first and second plus, etc. We can now count the number of sequences using the “balls and bins” method (also known as “stars and bars”).
- (c) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

Alternatively, we can derive a method similar to stars and bars/balls and bins; here, the restriction to positive integers means that we cannot have any empty groups. In particular, instead of arranging all of the objects (i.e. all the stars and all the bars), we can instead choose where to place the bars.

Looking at the “gaps” between the stars (i.e. the 1’s), we have a total of $n-1$ places to put the bars in between the n stars. Selecting k of these positions (we can’t have two bars occupy the same gap, otherwise we’d have an empty group), we have a total of $\binom{n-1}{k}$ ways to group the 1’s.

3 The Count

- (a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?
- (b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?
- (c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0’s. How many possible passwords can he make?

Solution:

- (a) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.
- (b) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

- (c) One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the i^{th} digit, then the $i - 1^{th}$ digit must be a 1, and the other $(10 - 5 - 1 = 4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of valid passwords is $2^5 + 5 \cdot 2^4 = 112$.