1 Joint Practice

Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} Ax^2y^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a positive constant.

(a) What is the value of A?

(b) What is the marginal density of X?

(c) What is $\text{cov}(X,Y)$?

Solution:

(a) Since $f_{X,Y}$ is a joint density, we know that it must integrate to 1. Since $f_{X,Y}$ is only nonzero on the unit square, we can set up and solve the following integral:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 Ax^2y^2\,dx\,dy = \frac{1}{9}\cdot A,$$

hence, we can see that $\boxed{A = 9}$.

(b) Since the joint density can only be nonzero when X is between 0 and 1, we know that outside of this interval, the marginal density of X must also be zero. Inside this interval, we can find the marginal density of X by integrating the joint density with respect to Y :

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 9x^2y^2\,dy = 3x^2.$$

Thus, we can see that the marginal density of X is given by

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Verifying our answer, one can see that this function is both nonnegative and integrates to 1.

(c) There are two ways of approaching this. The first way is to use the fact that

$$\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

In order to apply this formula, we need to first find these values. We first find that

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 9x^3y^3\,dx\,dy = \frac{9}{16}.$$

Moreover, we can compute that

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}.$$

Finally, in order to compute $\mathbb{E}[Y]$, we first find the marginal density of Y. We can do this in a similar fashion to the previous part by integrating the joint density with respect to X. Since the joint density is zero when Y is not between 0 and 1, we know that the marginal density of Y must also be zero outside of this interval. When Y is inside this interval, we have that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 9x^2y^2\,dx = 3y^2,$$

hence the full marginal density is

$$f_Y(y) = \begin{cases} 3y^2 & \text{if } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

This then allows to compute

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 3y^3\,dy = \frac{3}{4},$$

hence

$$\text{cov}(X,Y) = \frac{9}{16} - \frac{3}{4}\cdot\frac{3}{4} = \boxed{0}.$$

The second way of approaching this problem is to first compute the marginal density of Y. Upon doing so, one can check that for any pair of values x, y, we have that

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y),$$

meaning that X and Y are independent random variables. Thus, since the covariance between two independent random variables is always zero, we can conclude the desired result and finish.

2 Max of Uniforms

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

Solution:

(a) $Pr[X \leq x] = x^n$ since in order for $\max(X_1, ...X_n) < x$, we must have $X_i < x$ for all i. Since they are independent, we can multiply together the probabilities of each of them being less than x, which is x itself, as their distributions are uniform.

(b) Taking the derivative of the cdf, we have $f_X(x) = nx^{n-1}$

(c)

$$\begin{aligned}\mathbb{E}[X] &= \int_0^1 x f_X(x) \\ &= \int_0^1 nx^n dx \\ &= \frac{n}{n+1}\end{aligned}$$

(d)

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x) = \int_0^1 nx^{n+1} dx = \frac{n}{n+2} \\ Var[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2}\end{aligned}$$

3 Exponential Expectation

(a) Let $X \sim \text{Exp}(\lambda)$. Use induction to show that $\mathbb{E}[X^k] = k!/\lambda^k$ for every $k \in \mathbb{N}$.

(b) For any $|t| < \lambda$, compute $\mathbb{E}[e^{tX}]$ directly from the definition of expectation.

(c) Using part (a), compute $\sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} t^k$.

(d) Let $M(t) = \mathbb{E}[e^{tX}]$ be a function defined for all t such that $|t| < \lambda$. What is $\left.\frac{dM(t)}{dt}\right|_{t=0}$? What is $\left.\frac{d^2M(t)}{dt^2}\right|_{t=0}$? How does each of these relate to the mean and variance of an $\text{Exp}(\lambda)$ distribution?

Solution:

(a) The base case is $\mathbb{E}[X] = 1/\lambda$, which we already know. Using integration by parts,

$$\begin{aligned}
\mathbb{E}[X^{k+1}] &= \int_0^\infty x^{k+1} \cdot \lambda e^{-\lambda x} \, \mathrm{d}x \\
&= -x^{k+1} e^{-\lambda x} \Big|_0^\infty + (k+1) \int_0^\infty x^k e^{-\lambda x} \, \mathrm{d}x \\
&= \frac{k+1}{\lambda} \int_0^\infty x^k \cdot \lambda e^{-\lambda x} \, \mathrm{d}x \\
&= \frac{k+1}{\lambda} \mathbb{E}[X^k] \\
&= \frac{(k+1)!}{\lambda^{k+1}}
\end{aligned}$$

which proves the inductive step.

(b) For any $|t| < \lambda$.

$$\begin{aligned}
\mathbb{E}[\exp(tX)] &= \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, \mathrm{d}x \\
&= \frac{\lambda}{\lambda - t} \int_0^\infty (\lambda - t) e^{-(\lambda - t)x} \, \mathrm{d}x \\
&= \frac{\lambda}{\lambda - t} \\
&= \frac{1}{1 - t/\lambda}
\end{aligned}$$

(c) We have,

$$\sum_{k=0}^\infty \frac{\mathbb{E}[X^k]}{k!} t^k = \sum_{k=0}^\infty \frac{t^k}{\lambda^k} = \frac{1}{1 - t/\lambda}$$

for any $|t| < \lambda$ (if $|t| \geq \lambda$ then this series does not converge). This is the same as what we found in part (b)! Recall the power series expansion

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!}.$$

If X is any random variable, and we plug in tX for x in this identity and take expectations (remembering linearity of course!), we get

$$\mathbb{E}[\exp\{(tX)\}] = \mathbb{E}\left[\sum_{k=0}^\infty \frac{t^k X^k}{k!}\right] = \sum_{k=0}^\infty \frac{t^k \, \mathbb{E}[X^k]}{k!}$$

for whichever t the series on the right side converges.

(d)

$$\left.\frac{dM(t)}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{1}{\lambda} = \mu_1$$

$$\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \left.\frac{d}{dt}\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda-t)^3}\right|_{t=0} = \frac{2}{\lambda^2} = \mu_2$$

μ_1 is the mean of an $\text{Exp}(\lambda)$ distribution, and $\mu_2 - \mu_1^2$ is the variance of that distribution.

μ_2 is called the second moment of the distribution. In general, $\left.\frac{d^nM(t)}{dt^n}\right|_{t=0} = \mu_n = \mathbb{E}[X^n]$ is called the n^{th} moment of the random variable X, and $M(t) = \mathbb{E}[e^{tX}]$ is called the moment-generating function (mgf) of X. Just like the pdf and cdf, the mgf of a distribution also uniquely characterizes the probability distribution.