Nth-Order Dynamics Target Observability From Angle Measurements

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Necessary and sufficient conditions are presented for the observability of the target's Nth-order dynamics, given direction measurements. The derivations are extremely simple and do not require the examination of an observability matrix and nonlinear differential equations. Previously published observability requirements for the first-order dynamics are shown to be necessary, but not sufficient.

Manuscript received September 18, 1987; revised November 21, 1987.

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0018-9251/88/0500-0305 $1.00 © 1988 IEEE

I. INTRODUCTION

The estimation of a target trajectory from passive angle measurements is a widely investigated problem. In the general three-dimensional case, two angle measurements, such as azimuth and elevation, may be used for target state estimation. A basic requirement for the target motion analysis (TMA) is the system observability, i.e., the existence of a unique tracking solution.

Previous publications on the TMA observability requirements consider only the constant velocity trajectory (first-order dynamics) case, in two [1] and three dimensions [2]. The basic equations relating the target dynamics to the angle measurements are nonlinear. However the observability problem may be formulated as linear relations in the unknown dynamics, and nonlinear in the measurements. Such a formulation is presented in [1, 2], but the analysis there yields criteria derived via complex nonlinear differential equations. Some tedious mathematics enable the analysis of the first-order motion only, giving what is claimed to be both necessary and sufficient observability conditions.

We show that the three-dimensional observability requirements, which are equivalent to the uniqueness criterion for the solution of functional equations, can be stated directly. Thus we avoid analyzing an observability matrix which yields a set of nonlinear differential equations [1, 2]. The presented formulation enables the direct derivation of the necessary and sufficient observability conditions for the general Nth-order dynamics. Previously published requirements are shown to be necessary, but not sufficient for the TMA observability.

II. PROBLEM STATEMENT

Let the target and own-craft trajectories in Cartesian coordinates be represented by the three-dimensional vectorial functions $\mathbf{s}(t)$ and $\mathbf{w}(t)$, respectively (Fig. 1). The function $\mathbf{w}(t)$ is assumed known and $\mathbf{s}(t)$ is to be estimated using target to own-craft direction measurement

$$\mathbf{u}(t) = \mathbf{r}(t)/\|\mathbf{r}(t)\| \tag{1}$$

where $\mathbf{r}(t) = \mathbf{s}(t) - \mathbf{w}(t)$ and $\|\mathbf{r}(t)\|$ denotes the norm of the vector $\mathbf{r}(t)$. Vector $\mathbf{r}(t)$ is not identically zero, namely this meaningless degenerate case is precluded.

In practice, $\mathbf{u}(t)$ is obtained from two angle measurements. For example, if the two angles consist of azimuth $\beta(t)$ and elevation $\phi(t)$, $\mathbf{u}(t)$ can be expressed as

$$\mathbf{u}(t) = [\sin \beta(t) \cos \phi(t), \cos \beta(t) \cos \phi(t), \sin \phi(t)]^{\mathrm{T}} \tag{2}$$

where the superscript T denotes transposition.

It is assumed that the target motion can be modelled over the observation time $[0, t_f]$ by Nth-order dynamics, i.e.,

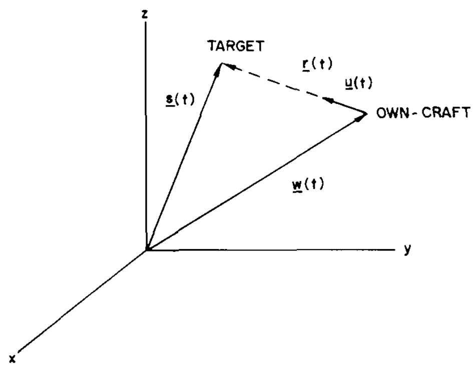

Fig. 1. Target, own-craft geometry.

$$\mathbf{s}(t) = \mathcal{S}\,\mathbf{t} \tag{3}$$

where $\mathbf{t} = [1\ t\ t^2 \dots t^N]^{\mathrm{T}}$ and $\mathcal{S}$ is a $3 \times (N+1)$ matrix of unknown coefficients to be estimated. Equation (3) represents, for $N = 0$, a stationary target to be located, and for $N = 1$, a constant velocity target to be tracked [1, 2]; for $N = 2$ and $N = 3$ the trajectory of an accelerating and jerking target, respectively, are to be estimated etc.

The problem addressed in this paper is the target observability associated with the above scenario, i.e., the required conditions on own-craft trajectory in order for the tracking solution to be unique.

III. OBSERVABILITY CONDITIONS

Let the known own-craft trajectory be defined by

$$\mathbf{w}(t) = \mathcal{W}\,\mathbf{t} - \mathbf{g}(t0 \tag{4}$$

where the $3 \times (N+1)$ matrix $\mathcal{W}$ describes own-craft Nth-order dynamics during the observation time and $\mathbf{g}(t) = [g_1(t), g_2(t), g_3(t)]\mathrm{T}$ represents own-craft "maneuver." The series expansion of $g_i(t)$ around $t = 0$ does not contain power of t less than $N + 1$, i.e., the kth derivatives at $t = 0$ satisfy

$$g_i^{(k)}(0) = 0, \qquad \text{for } i = 1, 2, 3, \text{ and } k = 0, 1, \dots, N. \tag{5}$$

Note that maneuvering of own-craft means developing velocity for $N = 0$, acceleration for $N = 1$, jerk for $N = 2$ etc.

Equations (3) and (4) yield

$$\mathbf{r}(t) = \mathcal{R}\,\mathbf{t} + \mathbf{g}(t) \tag{6}$$

where $\mathcal{R} = \mathcal{S} - \mathcal{W}$. Estimating the target trajectory is equivalent to finding the matrix $\mathcal{R}$, which corresponds to the Nth-order dynamics of the relative motion between the target and own-craft.

It follows from (1) that $\mathbf{r}(t)$ and $\mathbf{u}(t)$ satisfy

$$\mathbf{r}(t) = \lambda(t)\,\mathbf{u}(t) \tag{7}$$

where $\lambda(t)$ is a scalar function. From (6) and (7), we have

$$\mathcal{R}\,\mathbf{t} + \mathbf{g}(t0 = \lambda(t)\,\mathbf{u}(t). \tag{8}$$

Note that the above equation, although nonlinear in the measurements, is linear in the unknown $\mathcal{R}$. In order to derive the uniqueness conditions required for the solution of $\mathcal{R}$, assume there exist two distinct solutions, $\mathcal{R}_1$ and $\mathcal{R}_2$, associated with scalar functions $\lambda_1(t)$ and $\lambda_2(t)$, respectively. Substituting both solutions in (8) and subtracting, we obtain

$$[\lambda_1(t) - \lambda_2(t)]\,\mathbf{u}(t) = (\mathcal{R}_1 - \mathcal{R}_2)\,\mathbf{t}. \tag{9}$$

Thus the solution of (8) is unique if and only if there exist no two distinct matrices $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying (9).

Finally, from (7) and (9) we obtain the necessary and sufficient conditions for three-dimensional, Nth-order dynamics observability

$$\mathbf{r}(t) \neq \alpha(t)\,\mathcal{A}\,\mathbf{t} = \alpha(t) \begin{bmatrix} \sum_{i=1}^{N+1} & a_{1i} & t^{i-1} \\ \sum_{i=1}^{N+1} & a_{2i} & t^{i-1} \\ \sum_{i=1}^{N+1} & a_{3i} & t^{i-1} \end{bmatrix} \tag{10}$$

where $\alpha(t)$ is an arbitrary scalar function and $\mathcal{A}$ is a $3 \times (N+1)$ matrix of coefficients.

For $\alpha(t) \equiv 1$, (10) implies the well-known fact that, in the absence of a maneuver, i.e., where $\mathbf{g}(t) \equiv 0$, the target is not observable. As already stated, maneuvering in the general case means the existence of nonzero derivatives of $w(t)$ higher than the order of the model dynamics to be estimated. However only those maneuvers satisfying (10) are acceptable, namely, there exist maneuvers which still leave the target unobservable. For example, own-craft is required to move in order to locate a stationary target (nonzero velocity for the $N = 0$ case), and its motion should not be in the line of sight direction.

Note that in order to define an unobservable maneuver using (10), we cannot select $\alpha(t)$ and $\mathcal{A}$ independently. The choice of the pair $\{\alpha(t), \mathcal{A}\}$ is obviously restricted by relation (5).

The observability of a planar trajectory from azimuth angle measurements is obtained directly considering two-dimensional vectors and the corresponding $2 \times (N+1)$ matrix $\mathcal{A}$ in (10).

Finally, we emphasize that the requirement of (10) reduces to that stated in [1, 2] for the first-order dynamics ($N = 1$) if the matrix $\mathcal{A}$ is replaced by $\mathcal{R}$. However, substituting $\mathcal{A} = \mathcal{R}$ in (10) is a particular case and thus it is only a necessary observability condition, but not a sufficient one, as claimed in [1, 2]. To demonstrate this point consider the following example.

Example: Assume that the trajectory of a constant velocity target ($N = 1$) is to be estimated. Let the relative motion between the target and own-craft be defined by

$$\mathcal{R} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & r \end{bmatrix}; \qquad \mathbf{g}(t) = [t^2, t^2, t^2]^{\mathrm{T}} \tag{11}$$

where the matrix $\mathcal{R}$ defines the first-order relative dynamics and $\mathbf{g}(t)$ represents own-craft maneuver as before.

The observability condition of [2] demands

$$\mathbf{g}(t) \neq \alpha(t)\, \mathcal{R}\, \mathbf{t} \tag{12a}$$

or equivalently

$$\mathbf{r}(t) \neq [1 + \alpha(t)]\, \mathcal{R}\, \mathbf{t} \tag{12b}$$

where $\alpha(t)$ is a scalar function. Substituting (11) into (12a) yields

$$[t^2, t^2, t^2]^{\mathrm{T}} \neq \alpha(t)\, [1 + 2t,\ 2 + 3t,\ 3 + 4t]^{\mathrm{T}} \tag{13}$$

which is obviously true. Thus, according to [2], (11) represents an observable trajectory.

However, since

$$\mathbf{r}(t) = \begin{bmatrix} 1 + 2t + t^2 \\ 2 + 3t + t^2 \\ 3 + 4t + t^2 \end{bmatrix} = (t + 1) \begin{bmatrix} 1 + t \\ 2 + t \\ 3 + t \end{bmatrix} \tag{14}$$

(10) is not satisfied for this trajectory, rendering it unobservable!

Indeed, the direction vector measurement given by

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{r}(t) / \|\mathbf{r}(t)\| \\ &= [1 + t,\ 2 + t,\ 3 + t]^{\mathrm{T}} / \\ &\quad \|[1 + t,\ 2 + t,\ 3 + t]\| \end{aligned} \tag{15}$$

and $\mathbf{g}(t)$ of (11) are satisfied by any relative target trajectory of the form

$$\mathbf{r}(t) = (c + t)\, [1 + t,\ 2 + t,\ 3 + t]^{\mathrm{T}}. \tag{16}$$

Thus

$$\mathcal{R} = \begin{bmatrix} c & c + 1 \\ 2c & c + 2 \\ 3c & c + 3 \end{bmatrix} \tag{17}$$

is a possible solution for any constant c, namely, the system is unobservable.

IV. CONCLUSIONS

The necessary and sufficient observability conditions for the general Nth-order dynamics target, given direction measurements, have been found. Although the general case is examined, the presented derivations are extremely simple and do not require the solution of complex nonlinear differential equations [1, 2].

Previously published observability criteria for the first-order dynamics case have been shown to be necessary, but not sufficient.

In practice, angle derivatives as well as angle measurements are often available. Such information is of significant importance in the design of an accurate state estimator, but would not alter the observability status of target dynamics.

The actual design of a target state estimator from angle measurements, e.g., an extended Kalman filter, is an obvious extension of the present results. In this design problem, the observability notion discussed in this paper should be generalized to include a metric indicating degree of target observability. Such a metric would provide the means for the evaluation of the allowed sensor noise level and the target trajectory deviation from the assumed linear dynamics, in conjunction with (10). A possible metric is a measure of the problem singularity, as reflected in (10). The presentation of such an analysis is beyond the scope of this paper and will be discussed in future publications.

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