0.1 Single User Channel Capacities

Definition 0.1

The capacity of a continuous-time AWGN channel is

$$C = log(1 + \frac{P}{N_0 W}) \tag{1}$$

$$C_{awgn} = log(1 + SNR) \quad bits/s/Hz \tag{2}$$

☞ **Remark**

Unit: bits per complex dimension, proved by sphere packing or mutual information.

Consider a SIMO channel with one transmit antenna and L receive antennas:

$$y_l[m] = h_l x[m] + w_l[m] \quad l = 1, \ldots, L \tag{3}$$

Definition 0.2

The capacity of a SIMO channel is

$$C = log(1 + \frac{P\|\mathbf{h}\|^2}{N_0}) \quad bits/s/Hz \tag{4}$$

☞ **Remark**

We arrive at the equation via receiver beam-forming, where the idea is to post-process with the conjugate of channel as:

$$\tilde{y}[m] = \mathbf{h}^* y[m] = \|\mathbf{h}\|^2 x[m] + h^* w[m] \tag{5}$$

Note that $h = [h_1, \ldots, h_L]^t$

Definition 0.3

The capacity of a MISO channel is

$$C = log(1 + \frac{P\|\mathbf{h}\|^2}{N_0}) \quad bits/s/Hz \tag{6}$$

☞ **Remark**

It is the same because we transmit

$$x[m] = \frac{\mathbf{h}^*}{\|\mathbf{h}\|} \tilde{x}[m] \tag{7}$$

Consider the frequency selective channel:

$$y[m] = \sum_{l=0}^{L-1} h_l x[m-l] + w[m] \tag{8}$$

which can be converted into a parallel channel with N_c independent sub-carriers by adding a cyclic prefix of L-1. Whereh_l is tap of channel.

☞ **Remark**
This is of course, because that frequency selectivity is determined by multipath fading, which is related to the delay spread. A continuous time time-invariant multi-path fading channel is modelled as:

$$y(t) = \sum_i a_i(t)x(t-\tau_i(t)) + w(t) \tag{9}$$

Definition 0.4
The capacity of frequency-selective channel is

$$C_{N_c} = \underset{P_0,\ldots,P_{N_c-1}}{argmax} \sum_{n=1}^{N_c-1} log(1 + \frac{P_n|\tilde{h}_n|^2}{N_0}) \tag{10}$$

Which is solved by typical water-filling convex optimization.

Now consider a complex base-band representation of a flat fading channel:

$$y[m] = h[m]x[m] + w[m] \tag{11}$$

where $h[m]$ is the fading process, and $E[|h[m]|^2]$. We first consider slow fading channel, where the channel gain is random but remains constant for all time, $h[m] = h$ for all m (quasi-static).

Definition 0.5
The capacity of slow-fading flat-fading channel is in fact

$$C = log(1 + |h|^2 SNR) \tag{12}$$

Definition 0.6
The capacity of slow-fading flat-fading channel with L transmit and receive diversity is

$$C = log(1 + ||\mathbf{h}||^2 SNR) \tag{13}$$

Both expression follows AWGN capacity except for channel gain. We know consider fast fading flat fading channel in equation 11. The channel gain $h[m] = h_l$ remains constant over the lth coherence period and is identically independently distributed across different coherence periods.

Definition 0.7

The capacity of a fast-fading flat-fading channel is

$$C = E[log(1 + |h|^2 SNR)] \quad bits/s/Hz \tag{14}$$

If, however, the transmitter has information on channel statistics, one can readily employ water-filling algorithm in the format of

$$p_l^* = (\frac{1}{\lambda} - \frac{N_0}{|h_l|^2})^+ \tag{15}$$

It follows that:

Definition 0.8

The capacity of a fast-fading flat-fading channel with transmitter side information is:

$$C = E[log(1 + \frac{P*(h)|h|^2}{N_0})] \quad bits/s/Hz \tag{16}$$

0.2 Multi-User Capacity

For multiple user, instead of capacity, we define a capacity region and a total capacity bound of all users. Due to SIC methods, User 2 can achieve non-zero capacity while User 1 achieves full capacity.

Definition 0.9

The general K-user uplink capacity bound is

$$\sum_{k \in S} R_k < log(1 + \frac{\sum_{k \in S} P_k}{N_0}) \quad for \quad all \quad S \subset 1, ..., k \tag{17}$$

With sum capacity as

$$C_{sum} = log(1 + \frac{\sum_{k=1}^{K} P_k}{N_0}) \quad bits/s/Hz \tag{18}$$

Definition 0.10

The general K-user downlink capacity bound is

$$\sum_{k \in S} R_k < log(1 + \frac{P|h|^2}{N_0}) \quad for \quad all \quad S \subset 1, ..., K \tag{19}$$

☞ **Remark**

Notice first that uplink does not have h term, This is of course because in uplink, each link has a different power dictated by the individual user power, and we factor in h channel gain in SNR as well. Uplink total capacity scales with total number of users but downlink total capacity is bounded by single user capacity because in downlink all links have same power dictated by single power constraint P. Notice, however, that if uplink users are subjected to total power constraint $\sum_{k=1}^{K} P_k = P$, and that $E[|h|] = 1$, then two capacities are identical.

We now consider the uplink fading channel with K users:

$$y[m] = \sum_{k=1}^{K} h_k[m]x_k[m] + w[m] \tag{20}$$

where $h_k[m]_m$ is the fading process of user k. Assume fading processes are independent and that $E[|h_k[m]|^2] = 1$. We first consider slow-fading channel where $h_k[m] = h_k$ for all m. Continue...

0.3 MIMO Capacity via SVD

MIMO time-invariant channel is described by

$$y = Hx + w \tag{21}$$

where channel matrix $H \in n_r \times n_t$ is deterministic and assumed to be constant at all times and known to both the transmitter and the receiver. The channel matrix can be expressed as singular value decomposition as:

$$H = U \Lambda V^* \tag{22}$$

$$H = \sum_{i=1}^{n_{min}} \lambda_i u_i v_i^* \tag{23}$$

Convince yourself that with some pre-processing and post-processing the equivalent MIMO channel capacity is

$$C = \sum_{i=1}^{n_{min}} log(1 + \frac{P_i^* \lambda_i^2}{N_0}) \quad bits/s/Hz \tag{24}$$

0.4 MIMO Capacity

Assume the time-invariant channel model

$$y[m] = Hx[m] + w[m], \quad m = 1, 2, \ldots \tag{25}$$

Definition 0.11

The capacity of a MIMO time-invariant channel with n_t transmit antennas and n_r receive antennas is

$$R < logdet(I_{n_r} + \frac{1}{N_0} HK_x H^*) \quad bits/s/Hz \tag{26}$$

where

$$K_x = Qdiag(P_1, \ldots, P_{n_t})Q^* \tag{27}$$

and Q is the V-BLAST unitary matrix where if $Q = V$ in SVD, then the powers are given by the waterfilliing allocations, then we have the capacity-achieving architecture and where if $Q = I_{n_r}$, then independent data streams are sent on the different transmit antennas. The expression gives the upper bound on total rate.

We now consider fast-fading MIMO channel

$$y[m] = H[m]x[m] + w[m] \tag{28}$$

where $H[m]$ is a random fading process. Assume we know CSI at transmitter, in which case the capacity is equivalent to solving the convex problem

$$C = \max_{K_x : Tr[K_x] \leqslant P} E[logdet(I_{n_r} + \frac{1}{N_0} HK_xH^*)] \tag{29}$$

The optimum covariance is

$$K_x = (\frac{P}{n_t})I_{n_r} \tag{30}$$

Hence,

Definition 0.12

With equal powers, the MIMO capacity with receive CSI is

$$C = E[logdet(I_{n_r} + \frac{SNR}{n_t} HH^*)] \tag{31}$$

where $SNR = \frac{P}{N_0}$ is the common SNR at teach receive antenna.

We now consider the case with full CSI, where we can now water-fill the transmission. Notice that in fast-fading situation, we now water-fill over both time and space.

Definition 0.13

The capacity with full CSI in fast-fading is given by

$$C = \sum_{i=1}^{n_{min}} E[log(1 + \frac{P^*(\lambda_i)\lambda_i^2}{N_0})] \tag{32}$$

Definition 0.14

If MMSE-SIC receiver is used for demodulating the streams and the SINR and rate for stream k are $SINR_k$ and $log(1 + SINR_k$

$$logdet(I_{n_r} + HK_xH^*) \tag{33}$$

We now turn our attention to slow fading MIMO channel

$$y[m] = Hx[m] + w[m] \tag{34}$$

☞ Remark
H is fixed over time but random. Assuming a total power of P and suppose we want to communicate at a target rate R bits/s/Hz. And further assume that the transmitter were aware of the channel realization, then we use V-BLAST architecture.

Definition 0.15
The Capacity of a MIMO slow-fading channel is

$$C = logdet(I_{n_r} + \frac{1}{N_0}HK_xH^*) \tag{35}$$

0.5 Universal ST Codes

A key measure of the performance capability of a slow fading channel is the maximum diversity gain that can be extracted from it. A Rayleigh faded MIMO channel with n_t transmit and n_r receive antennas has a maximum diversity gain of $n_t * n_r$. A key performance benefit of a fast fading MIMO channel is the spatial multiplexing capability it provides through the additional degrees of freedom. The capacity of a Rayleigh fading channel scales with $n_{min}logSNR$, where $n_{min} = min(n_t, n_r)$ is the number of spatial degrees of freedom in the channel.

☞ Remark
Here we attempt to draw a clear distinction between two sources of diversity. In diversity techiniques, the same information is sent across independent fading channels to combat fading. This increases the chance of properly receiving the transmitted data and reliability of the entire system. This technique is referred to as inducing a spatial diversity in the communication system. For example, a 2 by 2 MIMO system achieves a spatial diversity of 4 if antennas are properly spaced.
Multiplexing gain, or spatial multiplexing, or degrees of freedom on the other hand, each spatial channel carries independent information and thereby increasing the data rate of the system. This type of gain governs the total capacity of the system is up bounded by $min(N_t, N_r)$. For example, a 3 by 3 MIMO system achieves a spatial multiplexing gain of 3.

Here we investigate the Diversity-Multiplexing tradeoff of various schemes and properties of a universal Space-Time Code.

Definition 0.16
For a slow fading channel, A diversity gain $d^*(r)$ is achieved at multiplexing gain r if

$$R = rlog(SNR) \tag{36}$$

and

$$p_{out}(R) \approx SNR^{-d^*(r)} \tag{37}$$

Definition 0.17

A Space-Time coding scheme is a family of codes, indexed by the signal-to-noise SNR. It attains a multiplexing gain r and a diversity gain d if the data rate scales as

$$R = r log(SNR) \tag{38}$$

and the error probability scales as

$$p_e \approx SNR^{-d} \tag{39}$$

We use QAM as an example to illustrate this tradeoff. Assuming a data rate of R, there are now $2^{R/2}$ constellation points in each of the real and imaginary dimensions.

☞ **Remark**

With data rate 1, meaning one bit of information is carried in one symbol, there are a total of 2^1 constellation points, with data rate R, there is then a total of 2^R constellations. Hence there are $2^{R/2}$ constellation in each of the real or imaginary dimension.

The minimum distance of the constellation is approximately:

$$D_{min} \approx \frac{\sqrt{SNR}}{2^{R/2}} \tag{40}$$

$$p_e \approx \frac{2^R}{SNR} \tag{41}$$

which yields a diversity-multiplexing tradeoff of

$$d_{qam}(r) = 1 - r, \quad r \in [0, 1] \tag{42}$$

☞ **Remark**

Two ends of the tradeoff curve can be interpreted as increasing the reliability for a fixed data rate, or increasing data rate for a fixed reliability.

Definition 0.18

The fundamental tradeoff of a slow fading Rayleigh channel is

$$d^*(r) = 1 - r \quad r \in [0, 1] \tag{43}$$

And hence QAM scheme trades off diversity and multiplexing gains optimally

Consider the slow fading parallel channel with independently identically distributed Rayleigh fading on each sub-channel:

$$y_l[m] = h_l x_l[m] + w_l[m], \quad l = 1, \ldots, L \tag{44}$$

Definition 0.19

The optimal diversity-multiplexing tradeoff for the parallel channel with L diversity branches is

$$d^*(r) = L(1-r), \quad r \in [0,1] \tag{45}$$

Now consider a MISO channel with, again, independently identically distributed Rayleigh fading with n_t transmit antenna:

$$y[m] = \mathbf{h}^* \mathbf{x}[m] + w[m] \tag{46}$$

Definition 0.20

The optimal diversity-multiplexing tradeoff for the i.i.d Rayleigh fading MISO channel is

$$d^*(r) = n_t(1-r), \quad r \in [0,1] \tag{47}$$