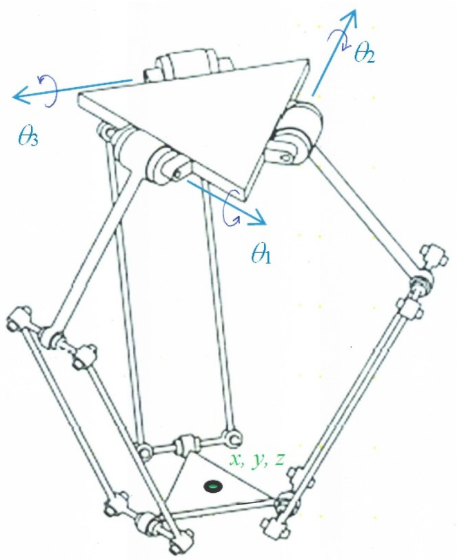
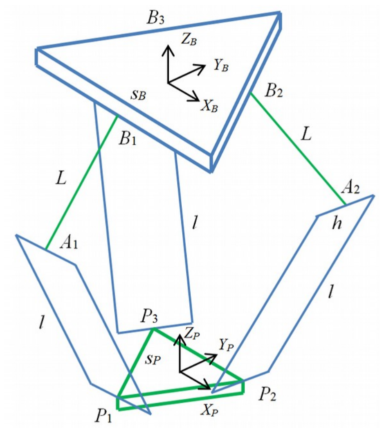
ABB FlexPicker IRB 360-1/1600

RUU legs (Revolute + 2 Universal joints) 4 DOF

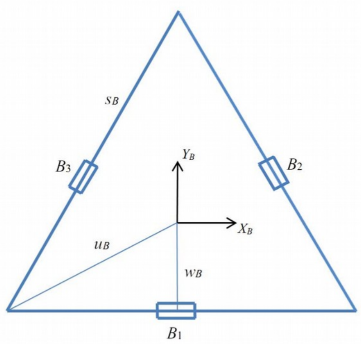


Delta Parallel Robot Diagram

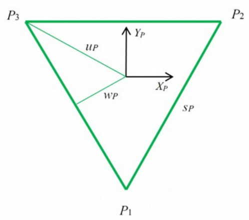


The fixed base Cartesian reference frame is $\{B\}$, whose origin is located in the center of the base equilateral triangle. The moving platform Cartesian reference frame is $\{P\}$, whose origin is located in the center of the platform equilateral triangle. The orientation of $\{P\}$ is always identical to the orientation of $\{B\}$ so rotation matrix ${}^{B}_{P}\mathbf{R}=\mathbf{I}_3$ is constant. The joint variables are $\mathbf{\Theta}=\{\theta_1 \quad \theta_2 \quad \theta_3\}^T$, and the Cartesian variables are ${}^{B}\mathbf{P}_P=\{x \quad y \quad z\}^T$. The design shown has high symmetry, with three upper leg lengths L and three lower lengths l (the parallelogram four-bar mechanisms major lengths).

The Delta Robot fixed base and platform geometric details are shown on the next page.



Delta Robot Fixed Base Details



Delta Robot Moving Platform Details

$$ {}^{B}\mathbf{B}_1=\begin{Bmatrix}0\\-w_B\\0\end{Bmatrix} \qquad {}^{B}\mathbf{B}_2=\begin{Bmatrix}\frac{\sqrt{3}}{2}w_B\\\frac{1}{2}w_B\\0\end{Bmatrix} \qquad {}^{B}\mathbf{B}_3=\begin{Bmatrix}-\frac{\sqrt{3}}{2}w_B\\\frac{1}{2}w_B\\0\end{Bmatrix} $$

$$ {}^{P}\mathbf{P}_1=\begin{Bmatrix}0\\-u_P\\0\end{Bmatrix} \qquad {}^{P}\mathbf{P}_2=\begin{Bmatrix}\frac{s_P}{2}\\w_P\\0\end{Bmatrix} \qquad {}^{P}\mathbf{P}_3=\begin{Bmatrix}-\frac{s_P}{2}\\w_P\\0\end{Bmatrix} $$

The vertices of the fixed-based equilateral triangle are:

$$ {}^{B}\mathbf{b}_1=\begin{Bmatrix}\frac{s_B}{2}\\-w_B\\0\end{Bmatrix} \qquad {}^{B}\mathbf{b}_2=\begin{Bmatrix}0\\u_B\\0\end{Bmatrix} \qquad {}^{B}\mathbf{b}_3=\begin{Bmatrix}-\frac{s_B}{2}\\-w_B\\0\end{Bmatrix} $$

where:

$$ w_B=\frac{\sqrt{3}}{6}s_B \qquad u_B=\frac{\sqrt{3}}{3}s_B \qquad w_P=\frac{\sqrt{3}}{6}s_P \qquad u_P=\frac{\sqrt{3}}{3}s_P $$

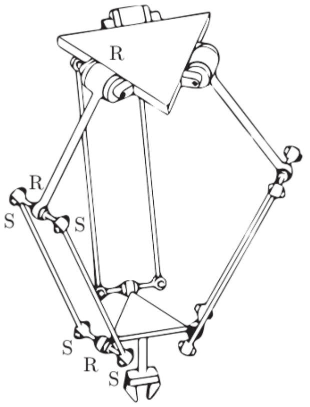


Figure 2.8: The Delta robot.

Grübler's formula for the number of degrees of freedom of the robot is

$$\begin{aligned}\text{dof} &= \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}} \\ &= m(N-1) - \sum_{i=1}^{J}(m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i. \end{aligned} \tag{2.4}$$

Example 2.7 (Delta robot)**.** The Delta robot of Figure 2.8 consists of two platforms – the lower one mobile, the upper one stationary – connected by three legs. Each leg contains a parallelogram closed chain and consists of three revolute joints, four spherical joints, and five links. Adding the two platforms, there are $N = 17$ links and $J = 21$ joints (nine revolute and 12 spherical). By Grübler's formula,

$$\text{dof} = 6(17-1-21) + 9(1) + 12(3) = 15.$$

Of these 15 degrees of freedom, however, only three are visible at the end-effector on the moving platform. In fact, the parallelogram leg design ensures that the moving platform always remains parallel to the fixed platform, so that the Delta robot acts as an x–y–z Cartesian positioning device. The other 12 internal degrees of freedom are accounted for by torsion of the 12 links in the parallelograms (each of the three legs has four links in its parallelogram) about their long axes.

From the kinematic diagram above, the following three vector-loop closure equations are written for the Delta Robot:

$$\{{}^{B}\mathbf{B}_i\} + \{{}^{B}\mathbf{L}_i\} + \{{}^{B}\mathbf{l}_i\} = \{{}^{B}\mathbf{P}_P\} + [{}^{B}_{P}\mathbf{R}]\{{}^{P}\mathbf{P}_i\} = \{{}^{B}\mathbf{P}_P\} + \{{}^{P}\mathbf{P}_i\} \qquad i = 1,2,3$$

where $[{}^{B}_{P}\mathbf{R}] = [\mathbf{I}_3]$ since no rotations are allowed by the Delta Robot.

The three applicable constraints state that the lower leg lengths must have the correct, constant length l (the virtual length through the center of each parallelogram):

$$l_i = \left\| \{{}^{B}\mathbf{l}_i\} \right\| = \left\| \{{}^{B}\mathbf{P}_P\} + \{{}^{P}\mathbf{P}_i\} - \{{}^{B}\mathbf{B}_i\} - \{{}^{B}\mathbf{L}_i\} \right\| \qquad i = 1,2,3$$

It will be more convenient to square both sides of the constraint equations above to avoid the square-root in the Euclidean norms:

$$l_i^2 = \left\| \{{}^{B}\mathbf{l}_i\} \right\|^2 = l_{ix}^2 + l_{iy}^2 + l_{iz}^2 \qquad i = 1,2,3$$

Again, the Cartesian variables are ${}^{B}\mathbf{P}_P = \{x \quad y \quad z\}^T$. The constant vector values for points P_i and B_i were given previously. The vectors $\{{}^{B}\mathbf{L}_i\}$ are dependent on the joint variables $\mathbf{\Theta} = \{\theta_1 \quad \theta_2 \quad \theta_3\}^T$:

$${}^{B}\mathbf{L}_1 = \begin{Bmatrix} 0 \\ -L\cos\theta_1 \\ -L\sin\theta_1 \end{Bmatrix} \qquad {}^{B}\mathbf{L}_2 = \begin{Bmatrix} \frac{\sqrt{3}}{2}L\cos\theta_2 \\ \frac{1}{2}L\cos\theta_2 \\ -L\sin\theta_2 \end{Bmatrix} \qquad {}^{B}\mathbf{L}_3 = \begin{Bmatrix} -\frac{\sqrt{3}}{2}L\cos\theta_3 \\ \frac{1}{2}L\cos\theta_3 \\ -L\sin\theta_3 \end{Bmatrix}$$

Substituting all above values into the vector-loop closure equations yields:

$$\{{}^{B}\mathbf{l}_1\} = \begin{Bmatrix} x \\ y + L\cos\theta_1 + a \\ z + L\sin\theta_1 \end{Bmatrix} \qquad \{{}^{B}\mathbf{l}_2\} = \begin{Bmatrix} x - \frac{\sqrt{3}}{2}L\cos\theta_2 + b \\ y - \frac{1}{2}L\cos\theta_2 + c \\ z + L\sin\theta_2 \end{Bmatrix} \qquad \{{}^{B}\mathbf{l}_3\} = \begin{Bmatrix} x + \frac{\sqrt{3}}{2}L\cos\theta_3 - b \\ y - \frac{1}{2}L\cos\theta_3 + c \\ z + L\sin\theta_3 \end{Bmatrix}$$

where: $$a = w_B - u_P$$ $$b = \frac{s_P}{2} - \frac{\sqrt{3}}{2} w_B$$ $$c = w_P - \frac{1}{2} w_B$$

And the three constraint equations yield the kinematics equations for the Delta Robot:

$$2L(y+a)\cos\theta_1 + 2zL\sin\theta_1 + x^2 + y^2 + z^2 + a^2 + L^2 + 2ya - l^2 = 0$$
$$-L(\sqrt{3}(x+b) + y + c)\cos\theta_2 + 2zL\sin\theta_2 + x^2 + y^2 + z^2 + b^2 + c^2 + L^2 + 2xb + 2yc - l^2 = 0$$
$$L(\sqrt{3}(x-b) - y - c)\cos\theta_3 + 2zL\sin\theta_3 + x^2 + y^2 + z^2 + b^2 + c^2 + L^2 - 2xb + 2yc - l^2 = 0$$

The three absolute vector knee points are found using ${}^B\mathbf{A}_i = {}^B\mathbf{B}_i + {}^B\mathbf{L}_i$, $i = 1,2,3$:

$${}^B\mathbf{A}_1 = \begin{Bmatrix} 0 \\ -w_B - L\cos\theta_1 \\ -L\sin\theta_1 \end{Bmatrix} \qquad {}^B\mathbf{A}_2 = \begin{Bmatrix} \frac{\sqrt{3}}{2}(w_B + L\cos\theta_2) \\ \frac{1}{2}(w_B + L\cos\theta_2) \\ -L\sin\theta_2 \end{Bmatrix} \qquad {}^B\mathbf{A}_3 = \begin{Bmatrix} -\frac{\sqrt{3}}{2}(w_B + L\cos\theta_3) \\ \frac{1}{2}(w_B + L\cos\theta_3) \\ -L\sin\theta_3 \end{Bmatrix}$$

Inverse Position Kinematics (IPK) Solution

The 3-dof Delta Robot inverse position kinematics (IPK) problem is stated: Given the Cartesian position of the moving platform control point (the origin of $\{P\}$), ${}^B\mathbf{P}_P = \{x \quad y \quad z\}^T$, calculate the three required actuated revolute joint angles $\mathbf{\Theta} = \{\theta_1 \quad \theta_2 \quad \theta_3\}^T$. The IPK solution for parallel robots is often straightforward; the IPK solution for the Delta Robot is not trivial, but can be found analytically. Referring to the Delta Robot kinematic diagram above, the IPK problem can be solved independently for each of the three **R**UU legs. Geometrically, each leg IPK solution is the intersection between a known circle (radius L, centered on the base triangle **R** joint point ${}^B\mathbf{B}_i$) and a known sphere (radius l, centered on the moving platform vertex ${}^P\mathbf{P}_i$).

This solution may be done geometrically/trigonometrically. However, we will now accomplish this IPK solution analytically, using the three constraint equations applied to the vector loop-closure equations (derived previously). The three independent scalar IPK equations are of the form:

$$E_i\cos\theta_i + F_i\sin\theta_i + G_i = 0 \qquad i = 1,2,3$$

where:

The IPK solution is more useful for Delta Robot control, to specify where the tool should be in XYZ .

$$E_1 = 2L(y+a)$$
$$F_1 = 2zL$$
$$G_1 = x^2 + y^2 + z^2 + a^2 + L^2 + 2ya - l^2$$

$$E_2 = -L(\sqrt{3}(x+b) + y + c)$$
$$F_2 = 2zL$$
$$G_2 = x^2 + y^2 + z^2 + b^2 + c^2 + L^2 + 2(xb + yc) - l^2$$

$$E_3 = L(\sqrt{3}(x-b) - y - c)$$
$$F_3 = 2zL$$
$$G_3 = x^2 + y^2 + z^2 + b^2 + c^2 + L^2 + 2(-xb + yc) - l^2$$

The equation $E_i \cos\theta_i + F_i \sin\theta_i + G_i = 0$ appears a lot in robot and mechanism kinematics and is readily solved using the **Tangent Half-Angle Substitution**.

If we define $t_i = \tan\dfrac{\theta_i}{2}$ then $\cos\theta_i = \dfrac{1-t_i^2}{1+t_i^2}$ and $\sin\theta_i = \dfrac{2t_i}{1+t_i^2}$

Substitute the **Tangent Half-Angle Substitution** into the *EFG* equation:

$$E_i\left(\frac{1-t_i^2}{1+t_i^2}\right) + F_i\left(\frac{2t_i}{1+t_i^2}\right) + G_i = 0 \qquad E_i(1-t_i^2) + F_i(2t_i) + G_i(1+t_i^2) = 0$$

$$(G_i - E_i)t_i^2 + (2F_i)t_i + (G_i + E_i) = 0$$

quadratic formula: $t_{i_{1,2}} = \dfrac{-F_i \pm \sqrt{E_i^2 + F_i^2 - G_i^2}}{G_i - E_i}$

Solve for θ_i by inverting the original Tangent Half-Angle Substitution definition:

$$\theta_i = 2\tan^{-1}(t_i)$$

Two θ_i solutions result from the $\pm$ in the quadratic formula. Both are correct since there are two valid solutions – knee left and knee right. This yields two IPK branch solutions for each leg of the Delta Robot, for a total of 8 possible valid solutions. Generally the one solution with all knees kinked out instead of in will be chosen.

Forward Position Kinematics (FPK) Solution

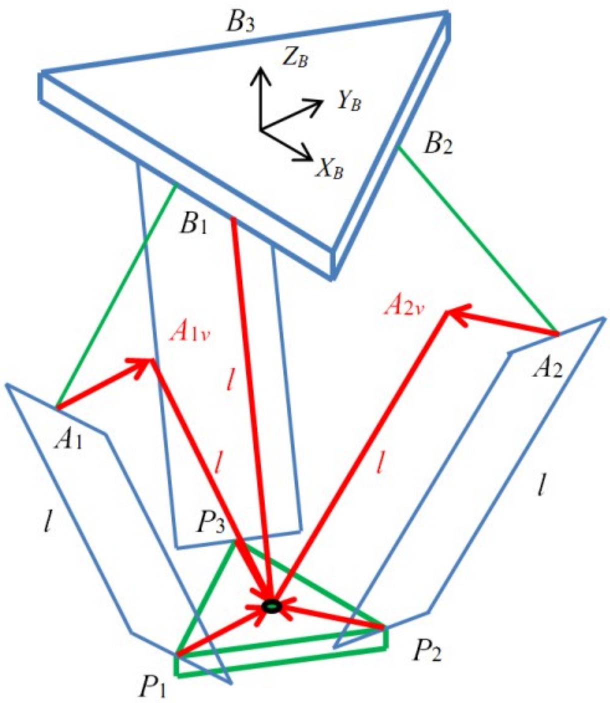
The 3-dof Delta Robot forward position kinematics (FPK) problem is stated: Given the three actuated joint angles $\mathbf{\Theta}=\{\theta_1 \quad \theta_2 \quad \theta_3\}^T$, calculate the resulting Cartesian position of the moving platform control point (the origin of $\{P\}$), ${}^B\mathbf{P}_P=\{x \quad y \quad z\}^T$. The FPK solution for parallel robots is generally very difficult. It requires the solution of multiple coupled nonlinear algebraic equations, from the three constraint equations applied to the vector loop-closure equations (derived previously). Multiple valid solutions generally result.

Thanks to the translation-only motion of the 3-dof Delta Robot, there is a straightforward analytical solution for which the correct solution set is easily chosen. Since $\mathbf{\Theta}=\{\theta_1 \quad \theta_2 \quad \theta_3\}^T$ are given, we calculate the three absolute vector knee points using ${}^B\mathbf{A}_i={}^B\mathbf{B}_i+{}^B\mathbf{L}_i$, $i=1,2,3$. Referring to the Delta Robot FPK diagram below, since we know that the moving platform orientation is constant, always horizontal with $\left[{}^B_P\mathbf{R}\right]=\left[\mathbf{I}_3\right]$, we define three virtual sphere centers ${}^B\mathbf{A}_{iv}={}^B\mathbf{A}_i-{}^P\mathbf{P}_i$, $i=1,2,3$:

$$
{}^B\mathbf{A}_{1v}=\begin{Bmatrix}0\\ -w_B-L\cos\theta_1+u_P\\ -L\sin\theta_1\end{Bmatrix}\quad
{}^B\mathbf{A}_{2v}=\begin{Bmatrix}\dfrac{\sqrt{3}}{2}(w_B+L\cos\theta_2)-\dfrac{s_P}{2}\\ \dfrac{1}{2}(w_B+L\cos\theta_2)-w_P\\ -L\sin\theta_2\end{Bmatrix}\quad
{}^B\mathbf{A}_{3v}=\begin{Bmatrix}-\dfrac{\sqrt{3}}{2}(w_B+L\cos\theta_3)+\dfrac{s_P}{2}\\ \dfrac{1}{2}(w_B+L\cos\theta_3)-w_P\\ -L\sin\theta_3\end{Bmatrix}
$$

and then the Delta Robot FPK solution is the intersection point of three known spheres. Let a sphere be referred as a vector center point $\{\mathbf{c}\}$ and scalar radius r, $(\{\mathbf{c}\}, r)$. Therefore, the FPK unknown point $\left\{{}^B\mathbf{P}_P\right\}$ is the intersection of the three known spheres:

$$
(\left\{{}^B\mathbf{A}_{1v}\right\}, l) \qquad (\left\{{}^B\mathbf{A}_{2v}\right\}, l) \qquad (\left\{{}^B\mathbf{A}_{3v}\right\}, l).
$$



Delta Robot FPK Diagram

Appendix A presents an analytical solution for the intersection point of the three given spheres, from Williams et al.[3] This solution also requires the solving of coupled transcendental equations. The appendix presents the equations and analytical solution methods, and then discusses imaginary solutions, singularities, and multiple solutions that can plague the algorithm, but all turn out to be no problem in this design.

In particular, with this existing three-spheres-intersection algorithm, if all three given sphere centers $\left\{{}^{B}\mathbf{A}_{iv}\right\}$ have the same Z height (a common case for the Delta Robot), there will be an algorithmic singularity preventing a successful solution (dividing by zero). One way to fix this problem is to simple rotate coordinates so all $\left\{{}^{B}\mathbf{A}_{iv}\right\}$ Z values are no longer the same, taking care to reverse this coordinate transformation after the solution is accomplished. However, we present another solution (Appendix B) for the intersection of three spheres assuming that all three sphere Z heights are identical, to be used in place of the primary solution when necessary.

Another applicable problem to be addressed is that the intersection of three spheres yields two solutions in general (only one solution if the spheres meet tangentially, and zero solutions if the center distance is too great for the given sphere radii l – in this latter case the solution is imaginary and the input data is not consistent with Delta Robot assembly). The spheres-intersection algorithm calculates both solution sets and it is possible to automatically make the computer choose the correct solution by ensuring it is below the base triangle rather than above it.

This three-spheres-intersection approach to the FPK for the Delta Robot yields results identical to solving the three kinematics equations for ${}^{B}\mathbf{P}_{P}=\{x \quad y \quad z\}^{T}$ given $\mathbf{\Theta}=\{\theta_1 \quad \theta_2 \quad \theta_3\}^{T}$.

<

Revolute-Input Delta Robot Velocity Kinematics Equations

The revolute-input Delta Robot velocity kinematics equations come from the first time derivative of the three position constraint equations presented earlier:

$$2L\dot{y}\cos\theta_1 - 2L(y+a)\dot{\theta}_1\sin\theta_1 + 2L\dot{z}\sin\theta_1 + 2Lz\dot{\theta}_1\cos\theta_1 + 2x\dot{x} + 2(y+a)\dot{y} + 2z\dot{z} = 0$$
$$-L(\sqrt{3}\dot{x}+\dot{y})\cos\theta_2 + L(\sqrt{3}(x+b)+y+c)\dot{\theta}_2\sin\theta_2 + 2L\dot{z}\sin\theta_2 + 2Lz\dot{\theta}_2\cos\theta_2 + 2(x+b)\dot{x} + 2(y+c)\dot{y} + 2z\dot{z} = 0$$
$$L(\sqrt{3}\dot{x}-\dot{y})\cos\theta_3 - L(\sqrt{3}(x-b)-y-c)\dot{\theta}_3\sin\theta_3 + 2L\dot{z}\sin\theta_3 + 2Lz\dot{\theta}_3\cos\theta_3 + 2(x-b)\dot{x} + 2(y+c)\dot{y} + 2z\dot{z} = 0$$

Re-written:

$$x\dot{x} + (y+a)\dot{y} + L\dot{y}\cos\theta_1 + z\dot{z} + L\dot{z}\sin\theta_1 = L(y+a)\dot{\theta}_1\sin\theta_1 - Lz\dot{\theta}_1\cos\theta_1$$
$$2(x+b)\dot{x} + 2(y+c)\dot{y} - L(\sqrt{3}\dot{x}+\dot{y})\cos\theta_2 + 2z\dot{z} + 2L\dot{z}\sin\theta_2 = -L(\sqrt{3}(x+b)+y+c)\dot{\theta}_2\sin\theta_2 - 2Lz\dot{\theta}_2\cos\theta_2$$
$$2(x-b)\dot{x} + 2(y+c)\dot{y} + L(\sqrt{3}\dot{x}-\dot{y})\cos\theta_3 + 2z\dot{z} + 2L\dot{z}\sin\theta_3 = L(\sqrt{3}(x-b)-y-c)\dot{\theta}_3\sin\theta_3 - 2Lz\dot{\theta}_3\cos\theta_3$$

Written in matrix-vector form:

$$[\mathbf{A}]\{\dot{\mathbf{X}}\} = [\mathbf{B}]\{\dot{\mathbf{\Theta}}\}$$

$$\begin{bmatrix} x & y+a+L\cos\theta_1 & z+L\sin\theta_1 \\ 2(x+b)-\sqrt{3}L\cos\theta_2 & 2(y+c)-L\cos\theta_2 & 2(z+L\sin\theta_2) \\ 2(x-b)+\sqrt{3}L\cos\theta_3 & 2(y+c)-L\cos\theta_3 & 2(z+L\sin\theta_3) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}$$

where:

$$b_{11} = L\left[(y+a)\sin\theta_1 - z\cos\theta_1\right]$$
$$b_{22} = -L\left[(\sqrt{3}(x+b)+y+c)\sin\theta_2 + 2z\cos\theta_2\right]$$
$$b_{33} = L\left[(\sqrt{3}(x-b)-y-c)\sin\theta_3 - 2z\cos\theta_3\right]$$

Appendix A. Three-Spheres Intersection Algorithm

We now derive the equations and solution for the intersection point of three given spheres. This solution is required in the forward pose kinematics solution for many cable-suspended robots and other parallel robots. Let us assume that the three given spheres are ($\mathbf{c}_1$,r_1), ($\mathbf{c}_2$,r_2), and ($\mathbf{c}_3$,r_3). That is, center vectors $\mathbf{c}_1=\{x_1 \quad y_1 \quad z_1\}^T$, $\mathbf{c}_2=\{x_2 \quad y_2 \quad z_2\}^T$, $\mathbf{c}_3=\{x_3 \quad y_3 \quad z_3\}^T$, and radii r_1, r_2, and r_3 are known (The three sphere center vectors must be expressed in the same frame, {0} in this appendix; the answer will be in the same coordinate frame). The equations of the three spheres are:

$$\begin{aligned}(x-x_1)^2+(y-y_1)^2+(z-z_1)^2&=r_1^2\\(x-x_2)^2+(y-y_2)^2+(z-z_2)^2&=r_2^2\\(x-x_3)^2+(y-y_3)^2+(z-z_3)^2&=r_3^2\end{aligned} \tag{A.1}$$

Equations (A.1) are three coupled nonlinear equations in the three unknowns x, y, and z. The solution will yield the intersection point $\mathbf{P}=\{x \quad y \quad z\}^T$. The solution approach is to expand equations (A.1) and combine them in ways so that we obtain $x=f(y)$ and $z=f(y)$; we then substitute these functions into one of the original sphere equations and obtain one quadratic equation in y only. This can be readily solved, yielding two y solutions. Then we again use $x=f(y)$ and $z=f(y)$ to determine the remaining unknowns x and z, one for each y solution. Let us now derive this solution.

First, expand equations (A.1) by squaring all left side terms. Then subtract the third from the first and the third from the second equations, yielding (notice this eliminates the squares of the unknowns):

$$a_{11}x+a_{12}y+a_{13}z=b_1 \tag{A.2}$$

$$a_{21}x+a_{22}y+a_{23}z=b_2 \tag{A.3}$$

where:

$$\begin{aligned}a_{11}&=2(x_3-x_1) & a_{21}&=2(x_3-x_2) & b_1&=r_1^2-r_3^2-x_1^2-y_1^2-z_1^2+x_3^2+y_3^2+z_3^2\\a_{12}&=2(y_3-y_1) & a_{22}&=2(y_3-y_2) & b_2&=r_2^2-r_3^2-x_2^2-y_2^2-z_2^2+x_3^2+y_3^2+z_3^2\\a_{13}&=2(z_3-z_1) & a_{23}&=2(z_3-z_2)\end{aligned}$$

Solve for z in (A.2) and (A.3):

$$z=\frac{b_1}{a_{13}}-\frac{a_{11}}{a_{13}}x-\frac{a_{12}}{a_{13}}y \tag{A.4}$$

$$z=\frac{b_2}{a_{23}}-\frac{a_{21}}{a_{23}}x-\frac{a_{22}}{a_{23}}y \tag{A.5}$$

Subtract (A.4) from (A.5) to eliminate z and obtain $x=f(y)$:

$$x=f(y)=a_4y+a_5 \tag{A.6}$$

where:

$$a_4=-\frac{a_2}{a_1} \qquad a_5=-\frac{a_3}{a_1} \qquad a_1=\frac{a_{11}}{a_{13}}-\frac{a_{21}}{a_{23}} \qquad a_2=\frac{a_{12}}{a_{13}}-\frac{a_{22}}{a_{23}} \qquad a_3=\frac{b_2}{a_{23}}-\frac{b_1}{a_{13}}$$

Substitute (A.6) into (A.5) to eliminate x and obtain $z=f(y)$:

$$z=f(y)=a_6y+a_7 \tag{A.7}$$

where:

$$a_6=\frac{-a_{21}a_4-a_{22}}{a_{23}} \qquad a_7=\frac{b_2-a_{21}a_5}{a_{23}}$$

Now substitute (A.6) and (A.7) into the first equation in (A.1) to eliminate x and z and obtain a single quadratic in y only:

$$ay^2+by+c=0 \tag{A.8}$$

where:

$$\begin{aligned}a&=a_4^2+1+a_6^2\\b&=2a_4(a_5-x_1)-2y_1+2a_6(a_7-z_1)\\c&=a_5(a_5-2x_1)+a_7(a_7-2z_1)+x_1^2+y_1^2+z_1^2-r_1^2\end{aligned}$$

There are two solutions for y:

$$y_\pm=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{A.9}$$

To complete the intersection of three spheres solution, substitute both y values y_+ and y_- from (A.9) into (A.6) and (A.7):

$$x_\pm=a_4y_\pm+a_5 \tag{A.10}$$

$$z_\pm=a_6y_\pm+a_7 \tag{A.11}$$

In general there are two solutions, one corresponding to the positive and the second to the negative in (A.9). Obviously, the + and – solutions cannot be switched:

$$\{x_+ \quad y_+ \quad z_+\}^T \qquad\qquad \{x_- \quad y_- \quad z_-\}^T \tag{A.12}$$

Example

Let us now present an example to demonstrate the solutions in the intersection of three spheres algorithm. Given three spheres $(\mathbf{c},r)$:

$$\left(\{0 \quad 0 \quad 0\}^T, \sqrt{2}\right) \qquad \left(\{3 \quad 0 \quad 0\}^T, \sqrt{5}\right) \qquad \left(\{1 \quad -3 \quad 1\}^T, 3\right) \tag{A.13}$$

The intersection of three spheres algorithm yields the following two valid solutions:

$$\{x_+ \quad y_+ \quad z_+\}^T = \{1 \quad 0 \quad 1\}^T \qquad \{x_- \quad y_- \quad z_-\}^T = \{1 \quad -0.6 \quad -0.8\}^T \tag{A.14}$$

These two solutions may be verified by a 3D sketch. This completes the intersection of three spheres algorithm. In the next subsections we present several important topics related to this three-spheres intersection algorithm: imaginary solutions, singularities, and multiple solutions.

Imaginary Solutions

The three spheres intersection algorithm can yield imaginary solutions. This occurs when the radicand $b^2 - 4ac$ in (A.9) is less than zero; this yields imaginary solutions for $y_\pm$, which physically means not all three spheres intersect. If this occurs in the hardware, there is either a joint angle sensing error or a modeling error, since the hardware should assemble properly.

A special case occurs when the radicand $b^2 - 4ac$ in (A.9) is equal to zero. In this case, both solutions have degenerated to a single solution, i.e. two spheres meet tangentially in a single point, and the third sphere also passes through this point.

Singularities

The three spheres intersection algorithm and hence the overall forward pose kinematics solution is subject to singularities. These are all algorithmic singularities, i.e. there is division by zero in the mathematics, but no problem exists in the hardware (no loss or gain in degrees of freedom). This subsection derives and analyzes the algorithmic singularities for the three spheres intersection algorithm presented above. Different possible three spheres intersection algorithms exist, by combining different equations starting with (A.1) and eliminating and solving for different variables first. Each has a different set of algorithmic singularities. We only analyze the algorithm presented above.

Inspecting the algorithm, represented in equations (A.1) – (A.12), we see there are four singularity conditions, all involving division by zero.

Singularity Conditions

$$\begin{aligned} a_{13} &= 0 \\ a_{23} &= 0 \\ a_1 &= 0 \\ a &= 0 \end{aligned} \tag{A.15}$$

The first two singularity conditions:

$$a_{13} = 2(z_3 - z_1) = 0 \tag{A.16}$$

$$a_{23} = 2(z_3 - z_2) = 0 \tag{A.17}$$

are satisfied when the centers of spheres 1 and 3 or spheres 2 and 3 have the same z coordinate, i.e. $z_1 = z_3$ or $z_2 = z_3$. Therefore, in the nominal case where all four virtual sphere centers have the same z height, this three-spheres intersection algorithm is always singular. An alternate solution is presented in Appendix B to overcome this problem.

The third singularity condition,

$$a_1 = \frac{a_{11}}{a_{13}} - \frac{a_{21}}{a_{23}} = 0 \tag{A.18}$$

Simplifies to:

$$\frac{x_3 - x_1}{z_3 - z_1} = \frac{x_3 - x_2}{z_3 - z_2} \tag{A.19}$$

For this condition to be satisfied, the centers of spheres 1, 2, and 3 must be collinear in the XZ plane. In general, singularity condition 3 lies along the edge of the useful workspace and thus presents no problem in hardware implementation if the system is properly designed regarding workspace limitations.

The fourth singularity condition,

$$a = a_4^2 + 1 + a_6^2 = 0 \tag{A.20}$$

Is satisfied when:

$$a_4^2 + a_6^2 = -1 \tag{A.21}$$

It is impossible to satisfy this condition as long as a_4 and a_6 from (A.6) and (A.7) are real numbers, as is the case in hardware implementations. Thus, the fourth singularity condition is never a problem.

Multiple Solutions

In general the three spheres intersection algorithm yields two distinct, correct solutions ($\pm$ in (A.9 – A.11)). Generally only one of these is the correct valid solution, determined by the admissible Delta Robot assembly configurations.

Appendix B. Simplified Three-Spheres Intersection Algorithm

We now derive the equations and solution for the intersection point of three given spheres, assuming all three spheres have identical vertical center heights. Assume that the three given spheres are $(\mathbf{c}_1, r_1)$, $(\mathbf{c}_2, r_2)$, and $(\mathbf{c}_3, r_3)$. That is, center vectors $\mathbf{c}_1 = \{x_1 \quad y_1 \quad z_1\}^T$, $\mathbf{c}_2 = \{x_2 \quad y_2 \quad z_2\}^T$, $\mathbf{c}_3 = \{x_3 \quad y_3 \quad z_3\}^T$, and radii r_1, r_2, and r_3 are known. The three sphere center vectors must be expressed in the same frame, $\{B\}$ here, and the answer will be in the same coordinate frame. The equations of the three spheres to intersect are (choosing the first three spheres):

$$(x-x_1)^2 + (y-y_1)^2 + (z-z_n)^2 = r_1^2 \tag{B.1}$$
$$(x-x_2)^2 + (y-y_2)^2 + (z-z_n)^2 = r_2^2 \tag{B.2}$$
$$(x-x_3)^2 + (y-y_3)^2 + (z-z_n)^2 = r_3^2 \tag{B.3}$$

Since all Z sphere-center heights are the same, we have $z_1 = z_2 = z_3 = z_n$. The unknown three-spheres intersection point is $\mathbf{P} = \{x \quad y \quad z\}^T$. Expanding (1-3) yields:

$$x^2 - 2x_1x + x_1^2 + y^2 - 2y_1y + y_1^2 + z^2 - 2z_nz + z_n^2 = r_1^2 \tag{B.4}$$
$$x^2 - 2x_2x + x_2^2 + y^2 - 2y_2y + y_2^2 + z^2 - 2z_nz + z_n^2 = r_2^2 \tag{B.5}$$
$$x^2 - 2x_3x + x_3^2 + y^2 - 2y_3y + y_3^2 + z^2 - 2z_nz + z_n^2 = r_3^2 \tag{B.6}$$

Subtracting (6) from (4) and (6) from (5) yields:

$$2(x_3 - x_1)x + 2(y_3 - y_1)y + x_1^2 + y_1^2 - x_3^2 - y_3^2 = r_1^2 - r_3^2 \tag{B.7}$$
$$2(x_3 - x_2)x + 2(y_3 - y_2)y + x_2^2 + y_2^2 - x_3^2 - y_3^2 = r_2^2 - r_3^2 \tag{B.8}$$

All non-linear terms of the unknowns x, y cancelled out in the subtractions above. Also, all z-related terms cancelled out in the above subtractions since all sphere-center z heights are identical. Equations (7-8) are two linear equations in the two unknowns x, y, of the following form.

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c \\ f \end{Bmatrix} \tag{B.9}$$

where:

$$\begin{aligned}
a &= 2(x_3 - x_1) \\
b &= 2(y_3 - y_1) \\
c &= r_1^2 - r_3^2 - x_1^2 - y_1^2 + x_3^2 + y_3^2 \\
d &= 2(x_3 - x_2) \\
e &= 2(y_3 - y_2) \\
f &= r_2^2 - r_3^2 - x_2^2 - y_2^2 + x_3^2 + y_3^2
\end{aligned}$$

The unique solution for two of the unknowns x, y is:

$$x = \frac{ce - bf}{ae - bd}$$

$$y = \frac{af - cd}{ae - bd} \tag{B.10}$$

Returning to (1) to solve for the remaining unknown z:

$$Az^2 + Bz + C = 0 \tag{B.11}$$

where:

$$A = 1$$
$$B = -2z_n$$
$$C = z_n^2 - r_1^2 + (x - x_1)^2 + (y - y_1)^2$$

Knowing the unique values x, y, the two possible solutions for the unknown z are found from the quadratic formula:

$$z_{p,m} = \frac{-B \pm \sqrt{B^2 - 4C}}{2} \tag{B.12}$$

For the Delta Robot, ALWAYS choose the z height solution that is below the base triangle, i.e. negative z, since that is the only physically-admissible solution.

This simplified three-spheres intersection algorithm solution for x, y, z fails in two cases:

i) When the determinant of the coefficient matrix in the x, y, linear solution (B.10) is zero.

$$ae - bd = 2(x_3 - x_1)2(y_3 - y_2) - 2(y_3 - y_1)2(x_3 - x_2) = 0 \tag{B.13}$$

This is an algorithmic singularity whose condition can be simplified as follows. (B.13) becomes:

$$(x_3 - x_1)(y_3 - y_2) = (y_3 - y_1)(x_3 - x_2) \tag{B.14}$$

If (B.14) is satisfied there will be an algorithmic singularity. Note that the algorithmic singularity condition (B.14) is only a function of constant terms. Therefore, this singularity can be avoided by design, i.e. proper placement of the robot base locations in the XY plane. For a symmetric Delta Robot, this particular algorithmic singularity is avoided by design.

ii) When the radicand in (B.12) is negative, the solution for z will be imaginary. The condition $B^2 - 4C < 0$ yields:

$$(x - x_1)^2 + (y - y_1)^2 > r_1^2 \tag{B.15}$$

When this inequality is satisfied, the solution for z will be imaginary, which means that the robot will not assemble for that configuration. Note that (B.15) is an inequality for a circle. This singularity will NEVER occur if valid inputs are given for the FPK problem, i.e. the Delta Robot assembles.

Conformal Geometric Algebra (CGA) is a specific 5D representation of 3D space that embeds geometric primitives and conformal transformations as elements of the same algebra

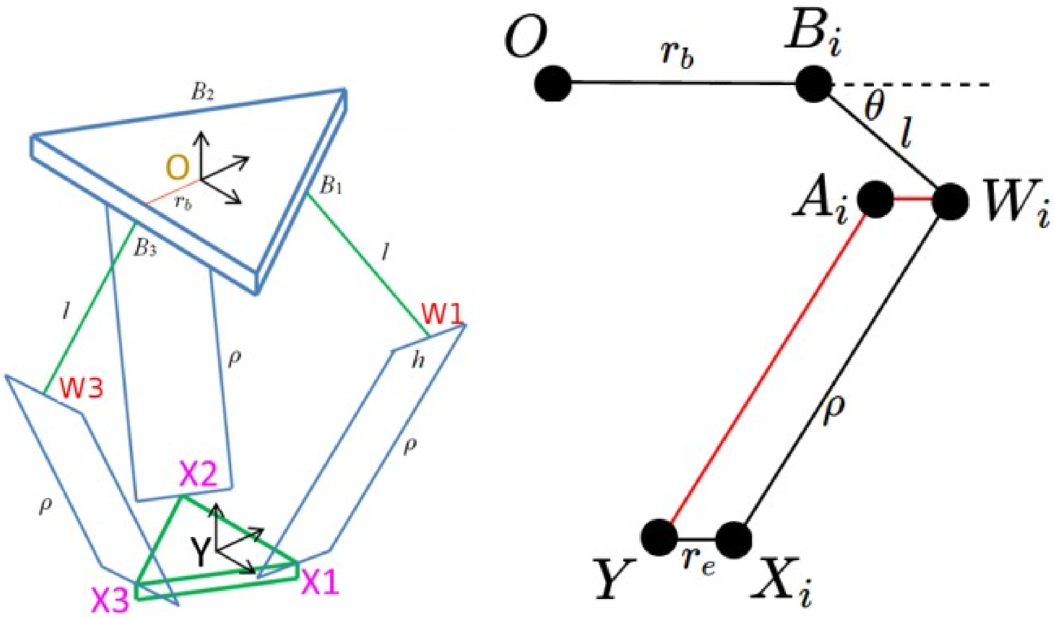


Fig. 1. Left: the 3D geometry of the delta robot. Right: The geometry of a single arm in plane.

The static part of the robot is a base plate to which three motors are rigidly attached, we will assume a space in which the origin is at the centre of this plate. Each motor shaft is rigidly attached to an ‘upper arm’ of length l;

we will number each upper arm i ∈ [1, 2, 3]. The connection point of the motor and upper arm will be labelled B i . The arm can only rotate in plane about the motor axis as the motor shaft and upper arm are rigidly connected.

We will refer to the other end of this upper arm as the ‘elbow point’ and will label it W i . At the elbow point each arm is rigidly attached to a central point of a horizontal rod we will refer to as the ‘elbow rod’.

At each end of the elbow rod a ball joint connects to a ‘forearm’ piece. The two forearm pieces for each arm are the same
length and, at the other end from the elbow rod, are connected to a rigid plate that we will refer to as the end-effector plate. The point half-way between where the two forearm rods connect to the end-effector plate is labelled X i .

We will label the point at the centre of the end-effector plate Y . Assuming the robot is infinitely stiff, the end plate is constrained, due to this specific arrangement of the forearms, to always remain parallel to the base plate and to have its in-plane orientation fixed as well. The Delta robot is therefore a purely translational mechanism.

The labels have assigned here will also serve as the notation for points in conformal space; for example, the centre of the end-effector plate has 3D position, y with Y = F (y).

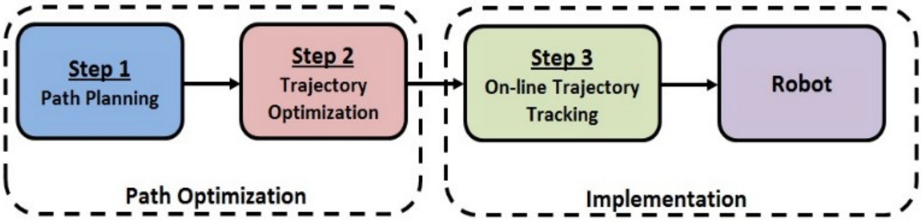


Figure 2.1: Three steps for path optimization for robotic manipulators

2.1 Path Planning

Path planning is the first step of path optimization for robotic manipulators. Most of the path planning techniques consider the manipulator kinematics and do not incorporate the manipulator's dynamics information. While planning motion, the pick and place positions and the intermediate path must be well defined. The main objective achieved in path planning is the collision avoidance from obstacles.

The workspace of robotic manipulators can be divided into two types:

- Workplace without obstacles
- Workplace with obstacles

Path planning in the absence of obstacles is just connecting the starting point to ending point with a straight line. The main task in this scenario is to set the velocity profile along the straight line.

An algorithm is called complete if it always finds a solution or determines that none exists.