The free spectral range (FSR) of the imbalanced interferometer

Given the imbalanced interferometer transfer function,

$$\frac{I_o}{I_i} = \frac{1}{2}\left[1 + \cos\left(\beta \Delta L\right)\right], \tag{1}$$

we wish to find the spacing between adjacent peaks, known as the free spectral range,

$$\mathrm{FSR} = \Delta\lambda = \lambda_{m+1} - \lambda_m. \tag{2}$$

We define the phase shift term in the interferometer transfer function as

$$\delta = \beta \Delta L, \text{ where } \beta = \frac{2\pi n}{\lambda}. \tag{3}$$

The phase difference between adjacent peaks is

$$\Delta\delta = \delta_m - \delta_{m+1} = 2\pi. \tag{4}$$

Typically there are many peaks in our spectrum, hence we are working at a high order of the interferometer, namely, $\delta \gg 2\pi$.

The propagation constant at each peak is β_m and β_{m+1}, hence

$$2\pi = \delta_m - \delta_{m+1} = \beta_m \Delta L - \beta_{m+1} \Delta L \tag{5}$$

and so

$$\Delta\beta = \beta_m - \beta_{m+1} = \frac{2\pi}{\Delta L} \tag{6}$$

Next, we approximate that β varies linearly with λ:

$$\Delta\beta = \beta_m - \beta_{m+1} \approx -\frac{d\beta}{d\lambda}\Delta\lambda \tag{7}$$

Combining Eqs. 6 and 7, this allows us to find an expression for $\Delta\lambda$,

$$\Delta\lambda \approx \; -\Delta\beta\left(\frac{d\beta}{d\lambda}\right)^{-1} \; = \; -\frac{2\pi}{\Delta L}\left(\frac{d\beta}{d\lambda}\right)^{-1} \tag{8}$$

Next, we find an expression for $\frac{d\beta}{d\lambda}$. Since $\beta = \frac{2\pi n}{\lambda}$,

$$\frac{d\beta}{d\lambda} = \; \frac{2\pi}{\lambda}\frac{dn}{d\lambda} + 2\pi n(-\lambda^{-2}) = \; -\frac{2\pi}{\lambda^2}\left(n - \frac{dn}{d\lambda}\lambda\right) \tag{9}$$

We can now solve for $\Delta\lambda$, the free spectral range, by inserting Eq. 9 into Eq. 8,

$$\mathrm{FSR} = \Delta\lambda = \frac{\lambda^2}{\Delta L\left(n - \lambda\frac{dn}{d\lambda}\right)} = \frac{\lambda^2}{\Delta L n_g} \tag{10}$$

where we have defined the group index, n_g, as

$$n_g = n - \lambda\frac{dn}{d\lambda}. \tag{11}$$

We can convert $\Delta\lambda$ [nm] to $\Delta\nu$ [GHz] by the following,

$$\Delta\nu \approx -\frac{c\Delta\lambda}{\lambda^2} = \frac{c}{\Delta L n_g} \tag{12}$$

This expression is similar to the free spectral range of a Fabry-Perot cavity.

Alternative derivation

Note that we arrive at the same result if we start with β defined in terms of frequency, rather than wavelength.

The free spectral range (FSR) defined in terms of the radial frequency is $\Delta\omega = \omega_2 - \omega_1$, and is the spacing between adjacent peaks of the interferometer transfer function, δ_2, δ_1:

$$2\pi = \delta_2 - \delta_1 = \beta_2 \Delta L - \beta_1 \Delta L, \quad \beta = n\frac{\omega}{c} \tag{13}$$

Assume n varies linearly with ω, hence $n = n_0 + \dfrac{\Delta\omega}{2}\dfrac{dn}{d\omega}$, where $\Delta\omega = \omega_2 - \omega_1$, $n_0 = n(\omega_0)$, $\omega_0 = \dfrac{\omega_2 + \omega_1}{2}$

$$\begin{aligned} \frac{2\pi c}{\Delta L} &= n_2\omega_2 - n_1\omega_1 = \ \omega_2\left(n_0 + \frac{\Delta\omega}{2}\frac{dn}{d\omega}\right) - \omega_1\left(n_0 - \frac{\Delta\omega}{2}\frac{dn}{d\omega}\right) \\ &= \Delta\omega\left(n_0 + \omega_0\frac{dn}{d\omega}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} \Delta\omega &= \frac{2\pi c}{\Delta L\left(n + \omega\frac{dn}{d\omega}\right)} = \frac{2\pi c}{\Delta L n_g} \\ \text{FSR} = \Delta\nu &= \frac{c}{\Delta L\left(n + \nu\frac{dn}{d\nu}\right)} = \frac{c}{\Delta L n_g} \ \text{[Hz]} \end{aligned} \tag{15}$$

We can convert this to a wavelength FSR,

$$\text{FSR} = \Delta\lambda \approx -\frac{\lambda^2\Delta\nu}{c} = \frac{\lambda^2}{\Delta L n_g} \ \text{[m]} \tag{16}$$