Greedy Algorithms: Grouping Children

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Higher School of Economics

Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

Outline

MinGroups(C)

```
m ← len(C)
for each partition into groups
C = G_1 ∪ G_2 ∪ ··· ∪ G_k:
    good ← true
    for i from 1 to k:
        if max(G_i) − min(G_i) > 1:
            good ← false
    if good:
        m ← min(m, k)
return m
```

Running time

Lemma

The number of operations in `MinGroups`(C) is at least 2^n, where n is the number of children in C.

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- Size of C is $\boldsymbol{n}$
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- There are 2^n different G_1
- Thus, at least 2^n operations $\square$

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operations!
- We will improve this significantly

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Covering points by segments

Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

Output: The minimum number of segments of unit length needed to cover all the points.

Example

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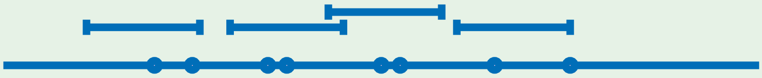

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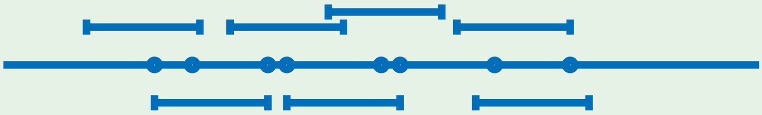
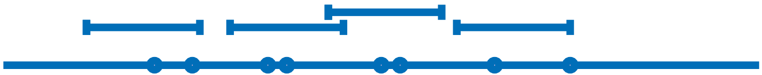


Safe move: cover the leftmost point with a unit segment which starts in this point.



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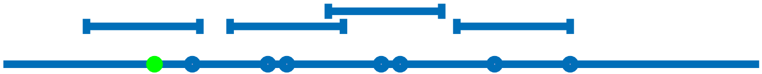

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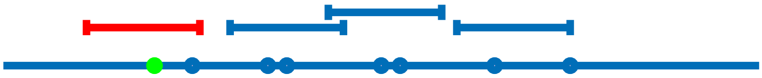

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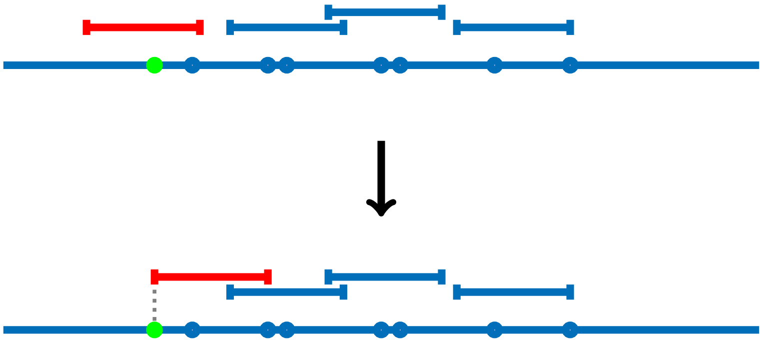

Assume $x_1 \leq x_2 \leq \ldots \leq x_n$

```
PointsCoverSorted(x_1, ..., x_n)

R ← {}, i ← 1
while i ≤ n:
   [ℓ, r] ← [x_i, x_i + 1]
   R ← R ⋃ {[ℓ, r]}
   i ← i + 1
   while i ≤ n and x_i ≤ r:
      i ← i + 1
return R
```

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- Overall, running time is $O(n)$ $\square$

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- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
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- Sort + `PointsCoverSorted` is $O(n \log n)$

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- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe move is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$