Divide-and-Conquer: Master Theorem

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$\downarrow$$

$$T(n) = O(\log n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$\downarrow$$

$$T(n) = O(n^2)$$

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$\downarrow$$

$$T(n) = O(n^{\log_2 3})$$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$\downarrow$$

$$T(n) = O(n \log n)$$

Master Theorem

Theorem

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$

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If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:

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$$T(n) = \begin{cases} & \\ & \\ & \end{cases}$$

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$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ \\ \end{cases}$$

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If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

Master Theorem Example 1

$$T(n) = 4\,T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

$$b = 2$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n^{1})$$

$$a = 4$$

$$b = 2$$

$$d = 1$$

Master Theorem Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4$$

$$b = 2$$

$$d = 1$$

Since $d < \log_b a$, $T(n) = O(n^{\log_b a}) = O(n^2)$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

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$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3$$

$$b = 2$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n^1)$$

$$a = 3$$
$$b = 2$$
$$d = 1$$

Master Theorem Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$
$$a = 3$$
$$b = 2$$
$$d = 1$$

Since $d < \log_b a$,
$T(n) = O(n^{\log_b a}) = O(n^{\log_2 3})$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2$$

$$b = 2$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^1)$$

$$a = 2$$

$$b = 2$$

$$d = 1$$

Master Theorem Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$
$$a = 2$$
$$b = 2$$
$$d = 1$$

Since $d = \log_b a$,
$T(n) = O(n^d \log n) = O(n \log n)$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

Master Theorem Example 4

$$T(n) = 1\,T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1$$

$$b = 2$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(n^0)$$

$$a = 1$$
$$b = 2$$
$$d = 0$$

Master Theorem Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$
$$a = 1$$
$$b = 2$$
$$d = 0$$

Since $d = \log_b a$, $T(n) = O(n^d \log n) = O(n^0 \log n) = O(\log n)$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

$$d = 2$$

Master Theorem Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2$$

$$b = 2$$

$$d = 2$$

Since $d > \log_b a$, $T(n) = O(n^d) = O(n^2)$

Outline

Master Theorem

If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:
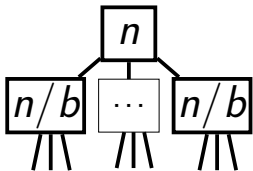
$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

$$\boxed{n}$$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$
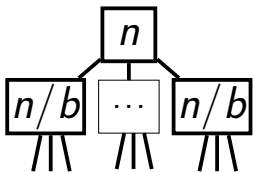
level

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

level

0

1

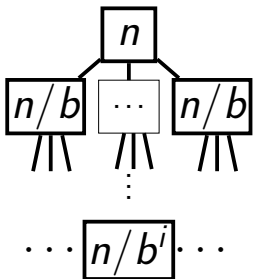$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$
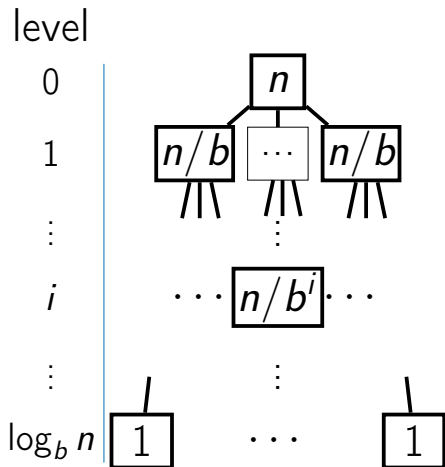
level

0   n

1   n/b $\cdots$ n/b

$\vdots$

i $\cdots$ n/b^i $\cdots$

$$T(n) = aT\left(\left\lceil \tfrac{n}{b} \right\rceil\right) + O(n^d)$$

level

0   n

1   n/b $\cdots$ n/b

$\vdots$ $\vdots$

i $\cdots$ n/b^i $\cdots$
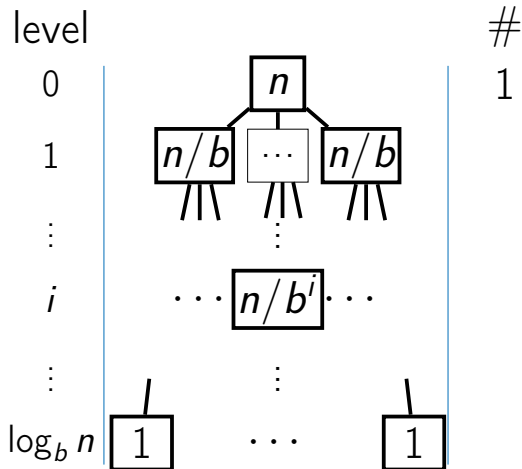
$\vdots$ $\vdots$
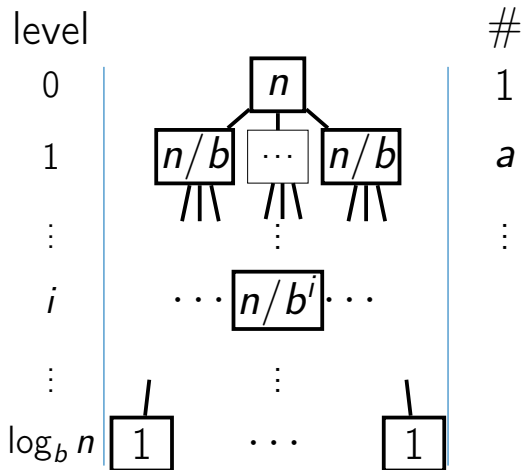
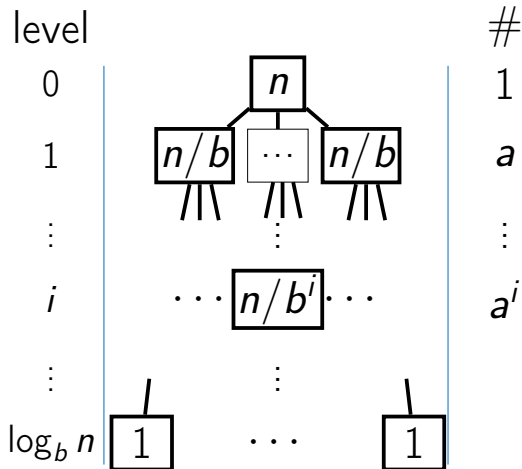$\log_b n$ 1 $\cdots$ 1

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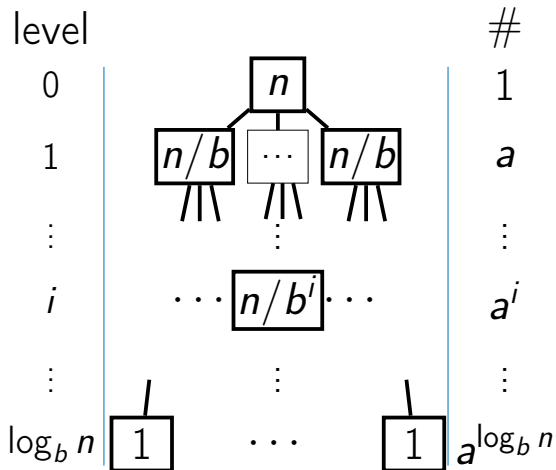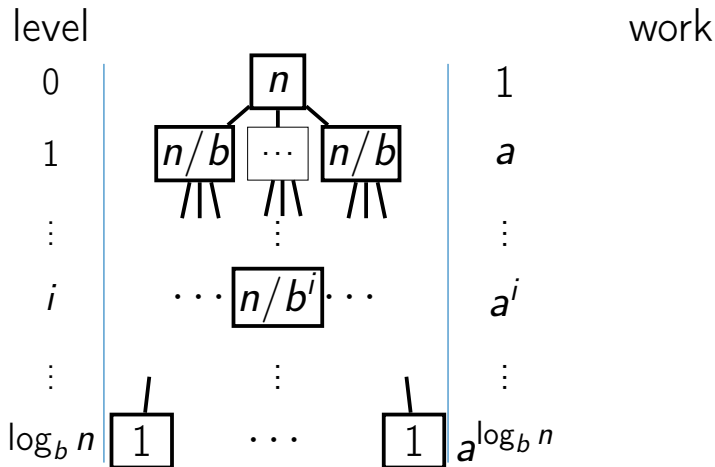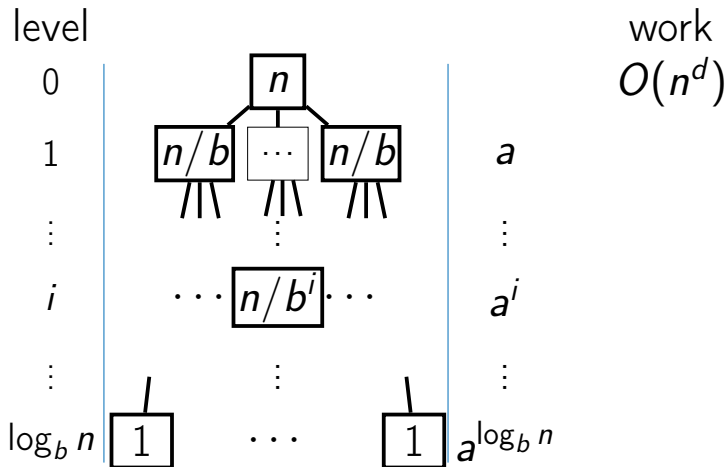| level		#
0	n	1
1	n/b $\cdots$ n/b	a
$\vdots$	$\vdots$	$\vdots$
i	$\cdots$ n/b^i $\cdots$	a^i
$\vdots$	$\vdots$	$\vdots$
$\log_b n$	1 $\cdots$ 1	$a^{\log_b n}$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

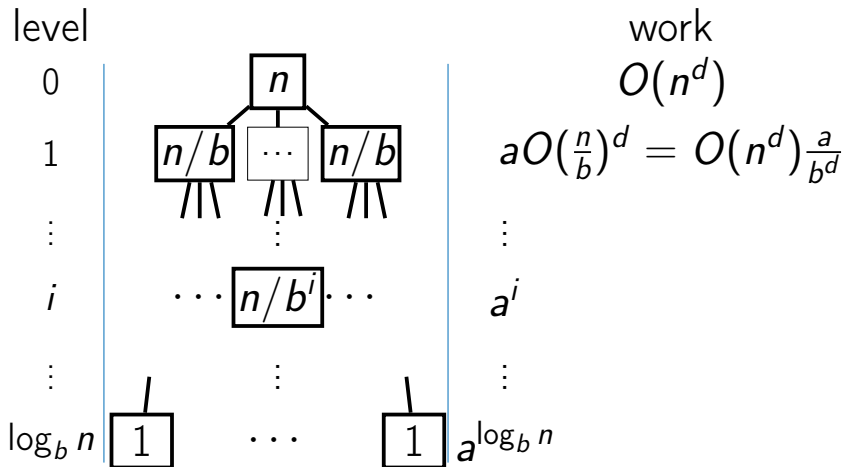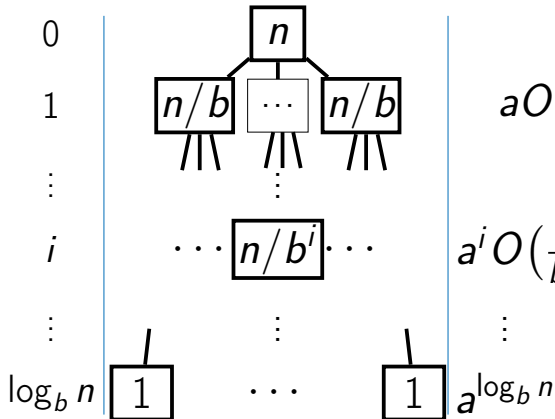$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

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level		work
0	n	$O(n^d)$
1	$n/b \cdots n/b$	$aO(\frac{n}{b})^d = O(n^d)\frac{a}{b^d}$
$\vdots$	$\vdots$	$\vdots$
i	$\cdots n/b^i \cdots$	a^i
$\vdots$	$\vdots$	$\vdots$
$\log_b n$	$1 \cdots 1$	$a^{\log_b n}$

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$$

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0	n	$O(n^d)$
1	$n/b \cdots n/b$	$aO(\frac{n}{b})^d = O(n^d)\frac{a}{b^d}$
$\vdots$	$\vdots$	$\vdots$
i	$\cdots n/b^i \cdots$	$a^i O\left(\frac{n}{b^i}\right)^d = O(n^d)\left(\frac{a}{b^d}\right)^i$
$\vdots$	$\vdots$	$\vdots$
$\log_b n$	$1 \quad \cdots \quad 1$	$a^{\log_b n}$

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level		work
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1	$n/b \;\cdots\; n/b$	$aO(\tfrac{n}{b})^d = O(n^d)\tfrac{a}{b^d}$
$\vdots$	$\vdots$	$\vdots$
i	$\cdots\; n/b^i \;\cdots$	$a^i O\left(\tfrac{n}{b^i}\right)^d = O(n^d)\left(\tfrac{a}{b^d}\right)^i$
$\vdots$	$\vdots$	$\vdots$
$\log_b n$	$1 \quad \cdots \quad 1$	$a^{\log_b n} = O(n^{\log_b a})$

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level		work
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1	$n/b \cdots n/b$	$aO(\frac{n}{b})^d = O(n^d)\frac{a}{b^d}$
$\vdots$	$\vdots$	$\vdots$
i	$\cdots n/b^i \cdots$	$a^i O\left(\frac{n}{b^i}\right)^d = O(n^d)\left(\frac{a}{b^d}\right)^i$
$\vdots$	$\vdots$	$\vdots$
$\log_b n$	$1 \quad \cdots \quad 1$	$a^{\log_b n} = O(n^{\log_b a})$

Total: $\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$

Geometric Series

For $r \neq 1$:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

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$$= \left\{ \vphantom{\begin{cases}1\\2\\3\end{cases}} \right.$$

Geometric Series

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$$= \begin{cases} O(a) & \text{if } r < 1 \end{cases}$$

Geometric Series

For $r \neq 1$:

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$$= a \frac{1 - r^n}{1 - r}$$

$$= \begin{cases} O(a) & \text{if } r < 1 \\ O(ar^{n-1}) & \text{if } r > 1 \end{cases}$$

Case 1: $\frac{a}{b^d} < 1$ $(d > \log_b a)$

$$\sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d} \right)^i$$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$
$$= O(n^d)$$

Case 2: $\frac{a}{b^d} = 1$ ($d = \log_b a$)

$$\sum_{i=0}^{\log_b n} O(n^d) \left(\frac{a}{b^d}\right)^i$$

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$$= \sum_{i=0}^{\log_b n} O(n^d)$$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$

$$= \sum_{i=0}^{\log_b n} O(n^d)$$

$$= (1 + \log_b n) O(n^d)$$

Case 2: $\frac{a}{b^d} = 1$ $(d = \log_b a)$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$

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$$= O(n^d \log n)$$

Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

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$$= O\left(O(n^d)\left(\frac{a}{b^d}\right)^{\log_b n}\right)$$

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$$= O\left(O(n^d)\left(\frac{a}{b^d}\right)^{\log_b n}\right)$$

$$= O\left(O(n^d)\frac{a^{\log_b n}}{b^{d\log_b n}}\right)$$

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$$= O\left(O(n^d)\frac{a^{\log_b n}}{b^{d \log_b n}} \right)$$

$$= O\left(O(n^d)\frac{n^{\log_b a}}{n^d} \right)$$

Case 3: $\frac{a}{b^d} > 1$ ($d < \log_b a$)

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If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ (for constants $a > 0, b > 1, d \geq 0$), then:

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