Divide-and-Conquer: Polynomial Multiplication

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

Multiplying Polynomials

Example

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$

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Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$
$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Multiplying polynomials

Input: Two $n-1$ degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$$

Output:

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Output: The product polynomial:

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Output: $C = (15, 13, 33, 9, 10)$

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MultPoly(A, B, n)

$product \leftarrow Array[2n-1]$
`for` i `from` 0 `to` $2n-2$`:`
  $product[i] \leftarrow 0$

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`for` i `from` 0 `to` $n-1$`:`
  `for` j `from` 0 `to` $n-1$`:`
    $product[i+j] \leftarrow product[i+j] + A[i] \times B[j]$

MultPoly(A, B, n)

```
product ← Array[2n − 1]
for i from 0 to 2n − 2:
  product[i] ← 0
for i from 0 to n − 1:
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return product
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Runtime: $O(n^2)$

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- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$
  $= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

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$$\begin{aligned} AB = &(4x^2 + 11x + 6)x^4 + \\ &(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + \\ &6x^2 + 11x + 4 \\ =&4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4 \end{aligned}$$

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D_0E_1 = Mult2$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

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$R[0..n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = Mult2$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

D_1E_0 = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2}...n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

Function Mult2(A, B, n, a_l, b_l)

$R =$ array$[0..2n-1]$

if $n = 1$:

$R[0] = A[a_l] * B[b_l]$; return R

$R[0..n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = Mult2$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

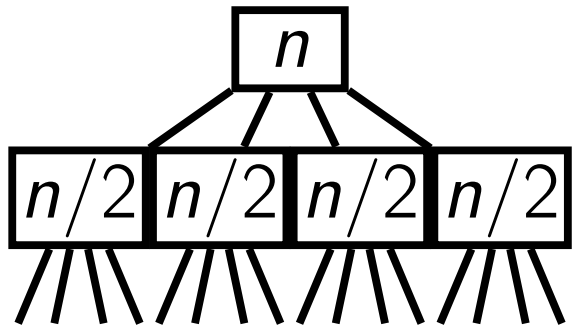
D_1E_0 = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2}...n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

return R

n

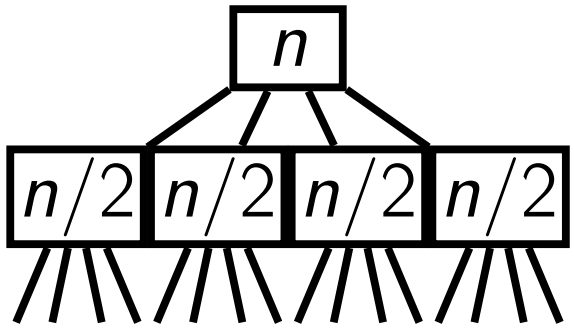
level

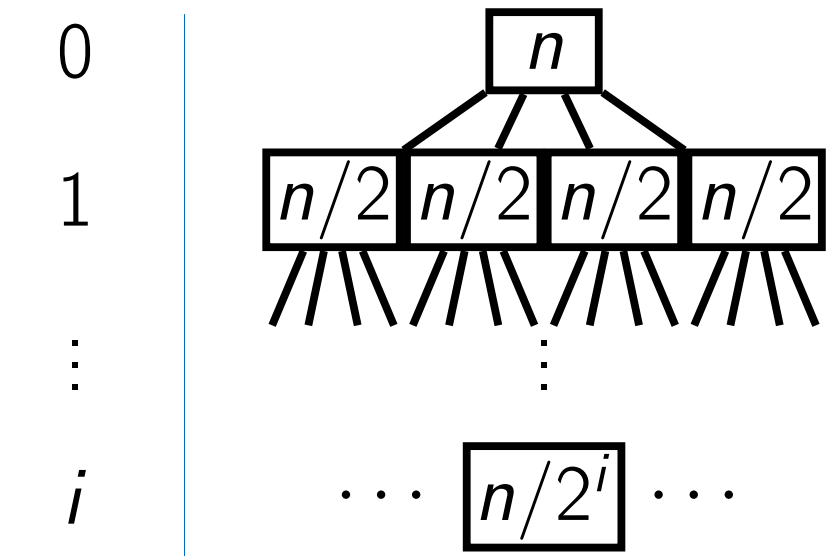


level

0

1




level
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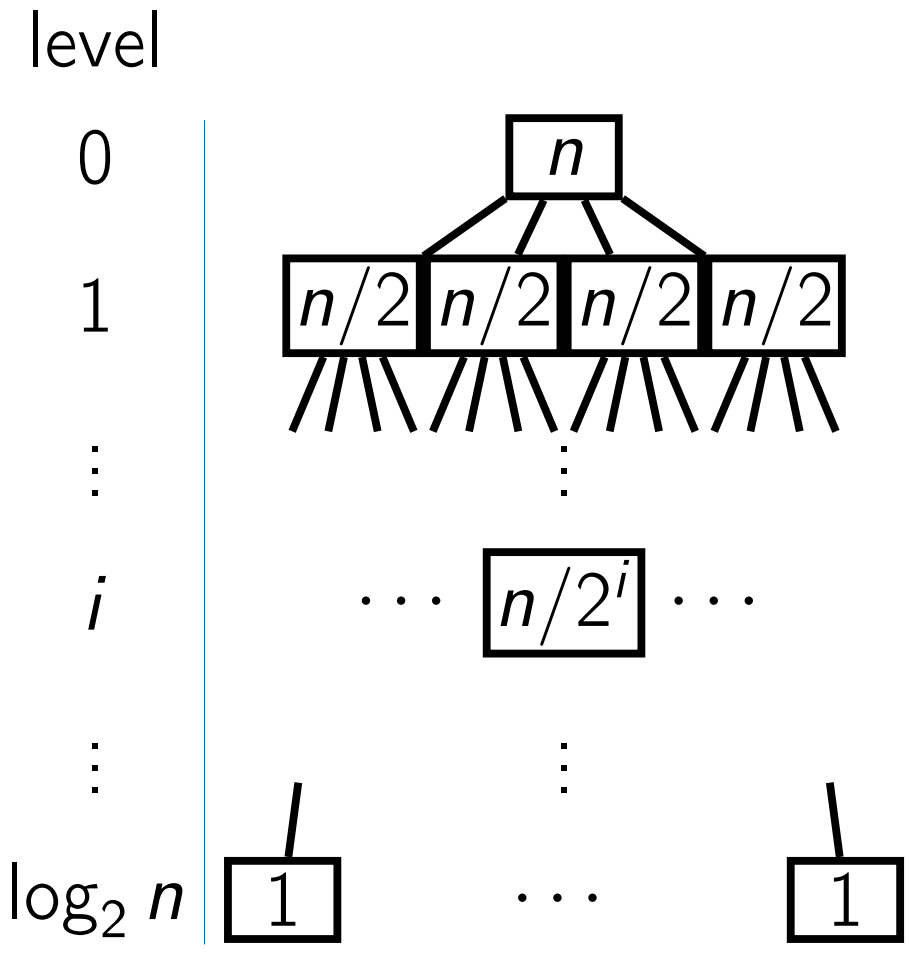
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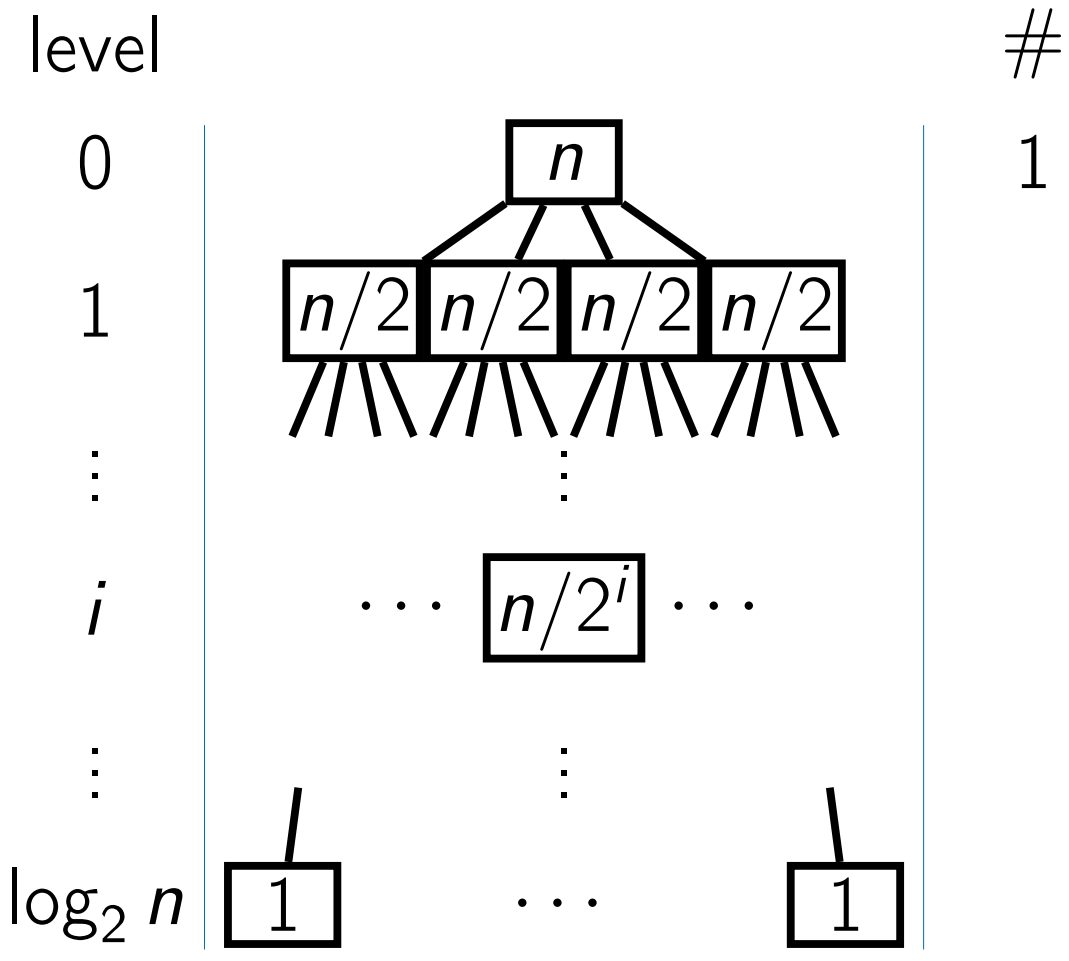
level
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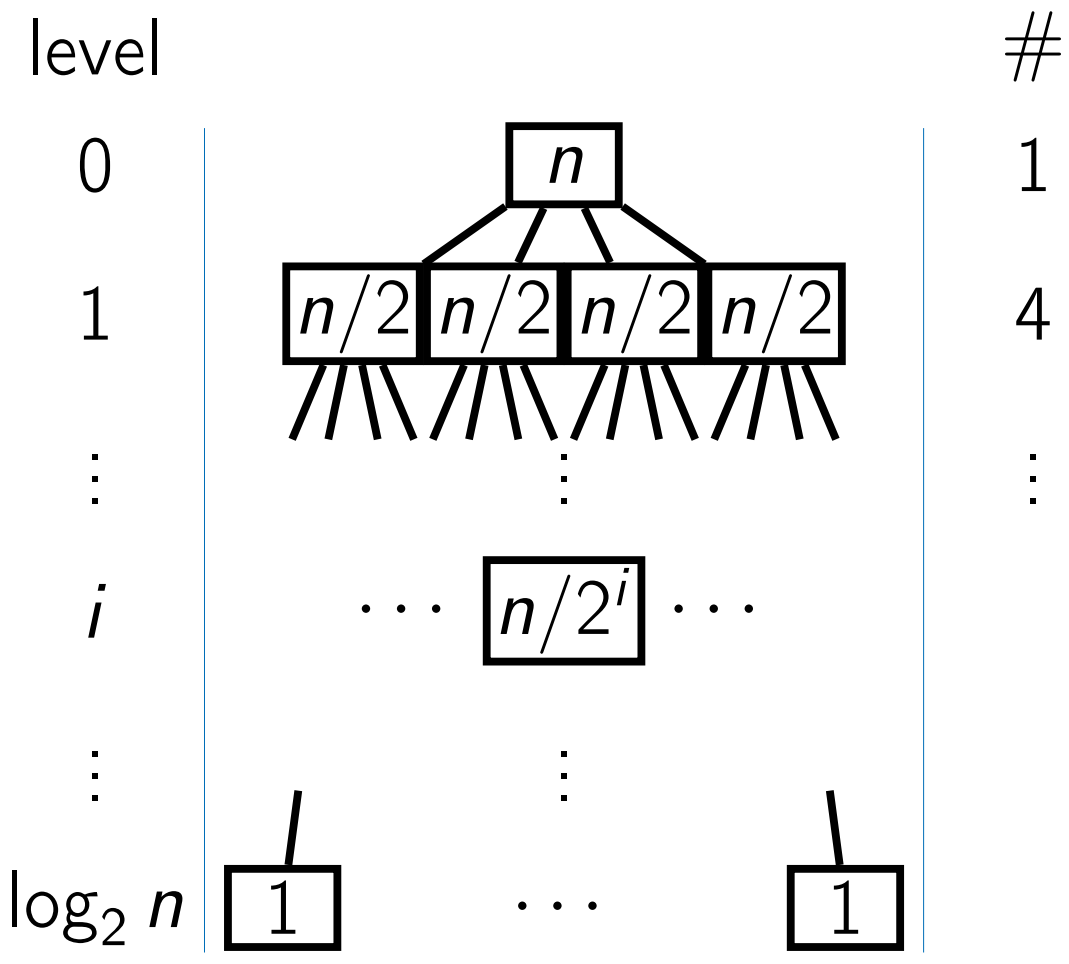
level
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4
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n/2^i
log2 n
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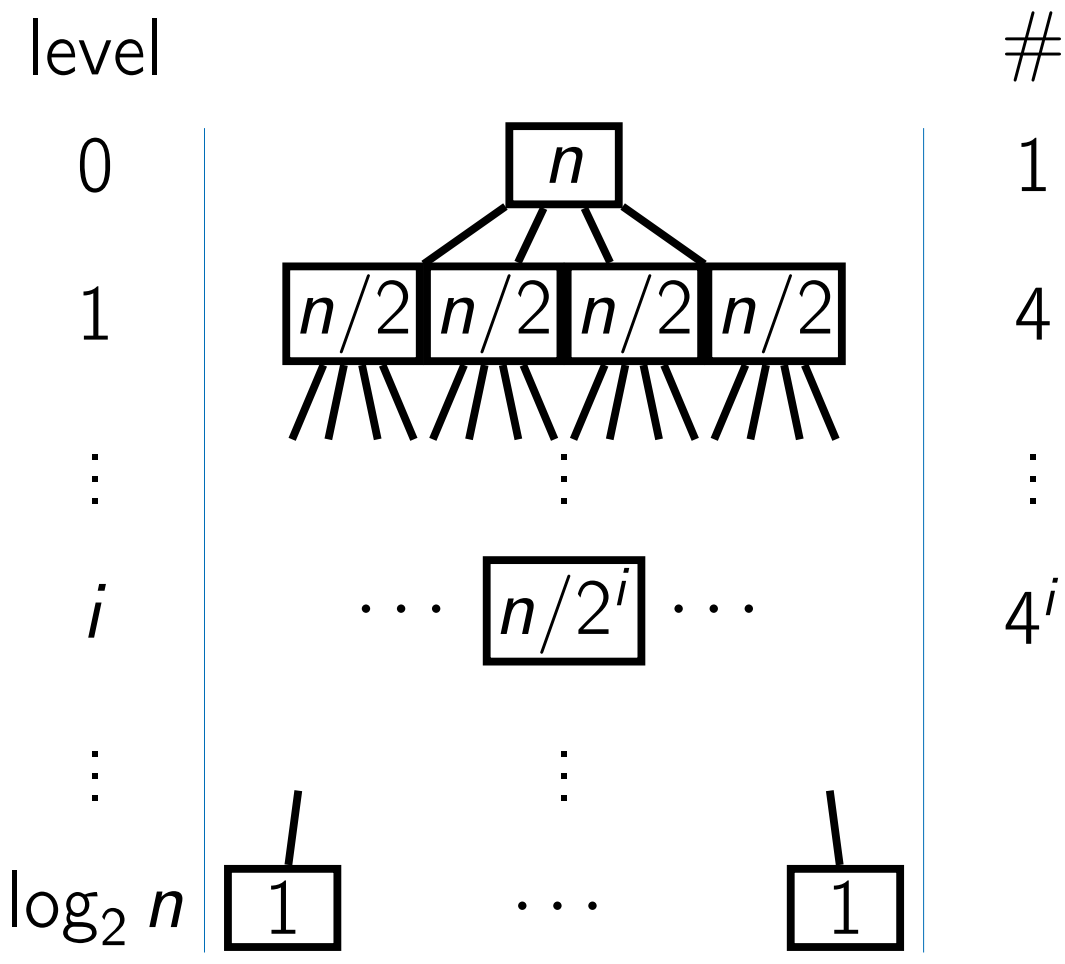
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log_2 n
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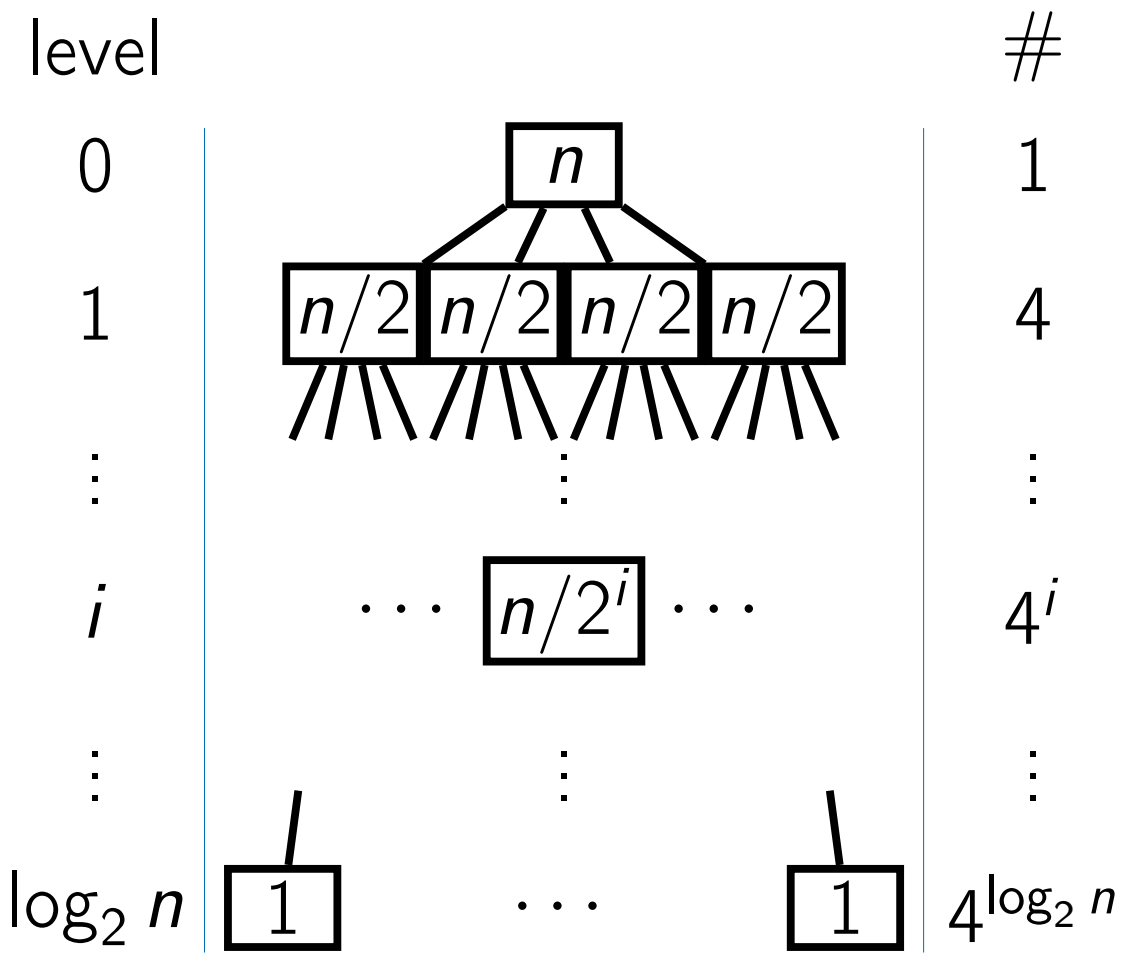
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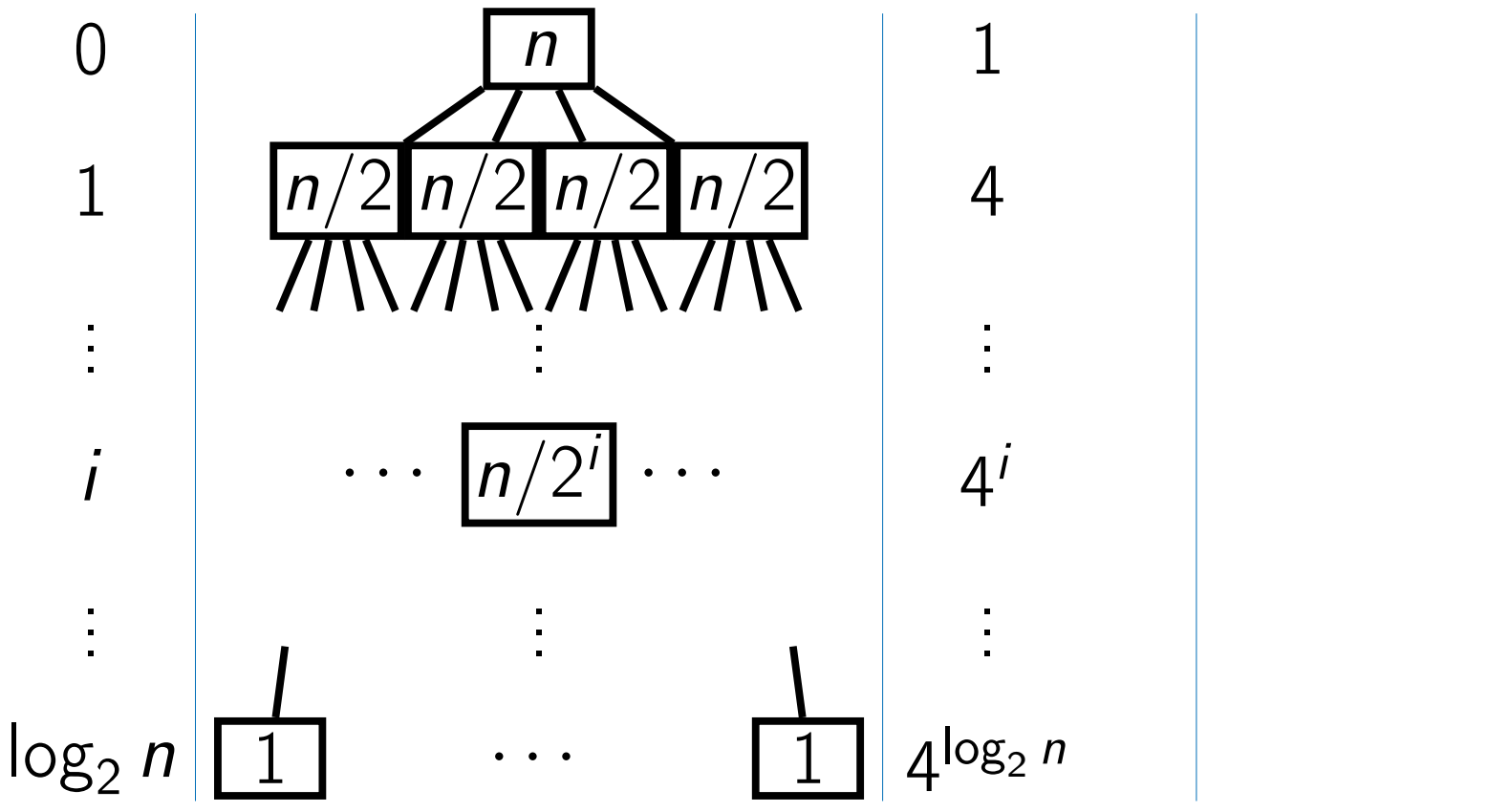
level
#
work
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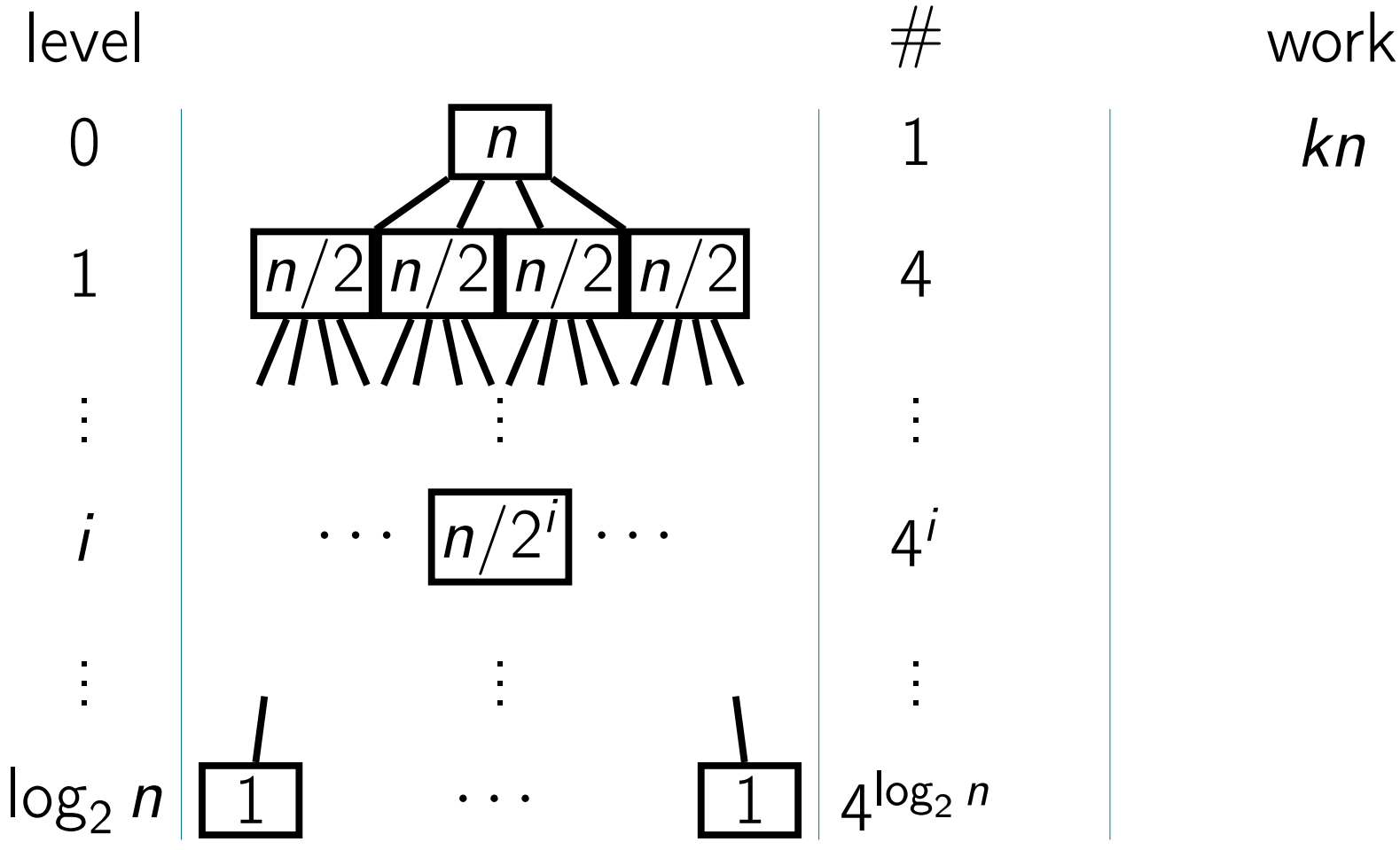
$\cdots$
1
$\cdots$
1
1
4
4^i
$4^{\log_2 n}$


level
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$\cdots$ $n/2^i$ $\cdots$
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4^i
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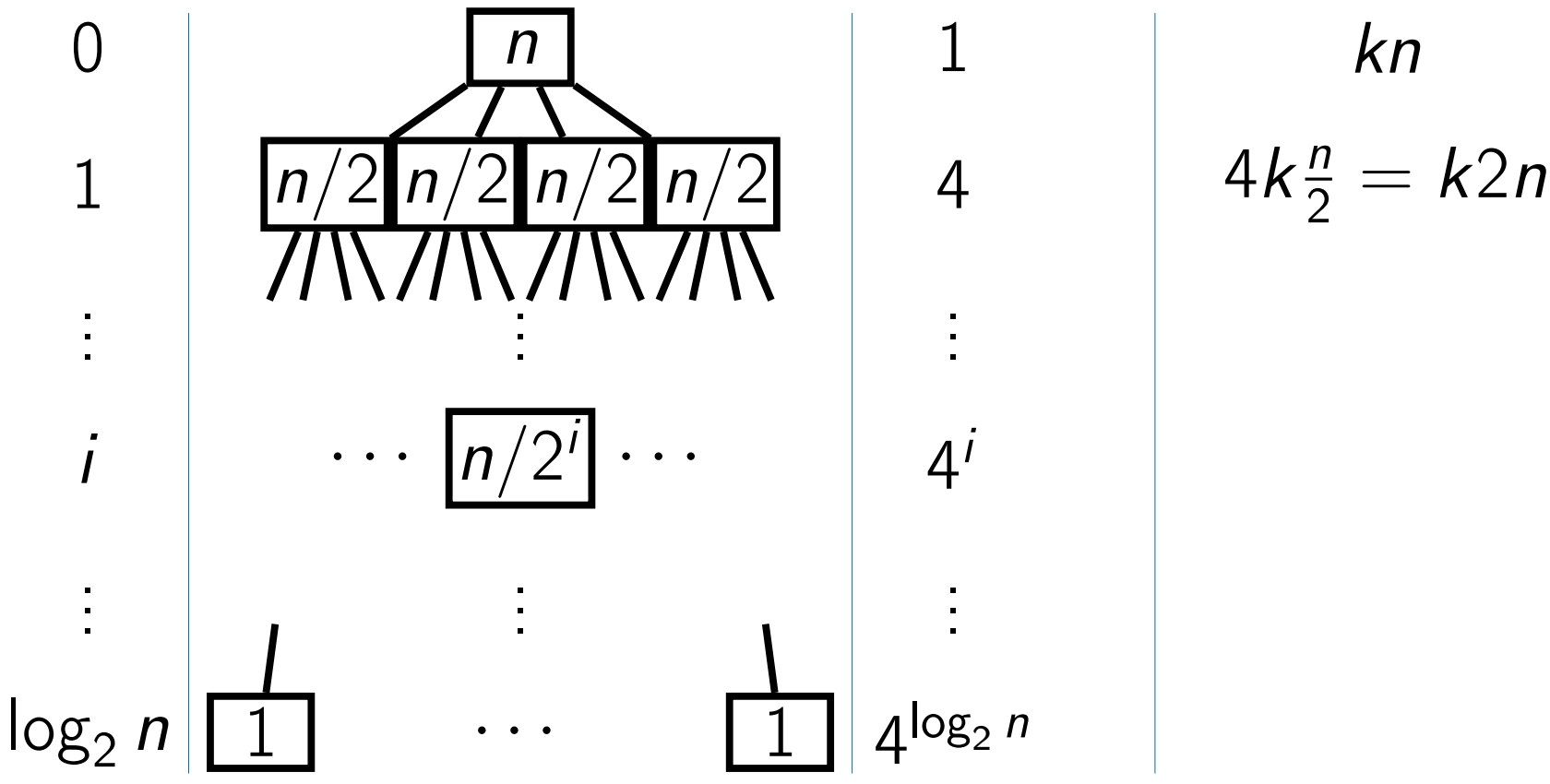
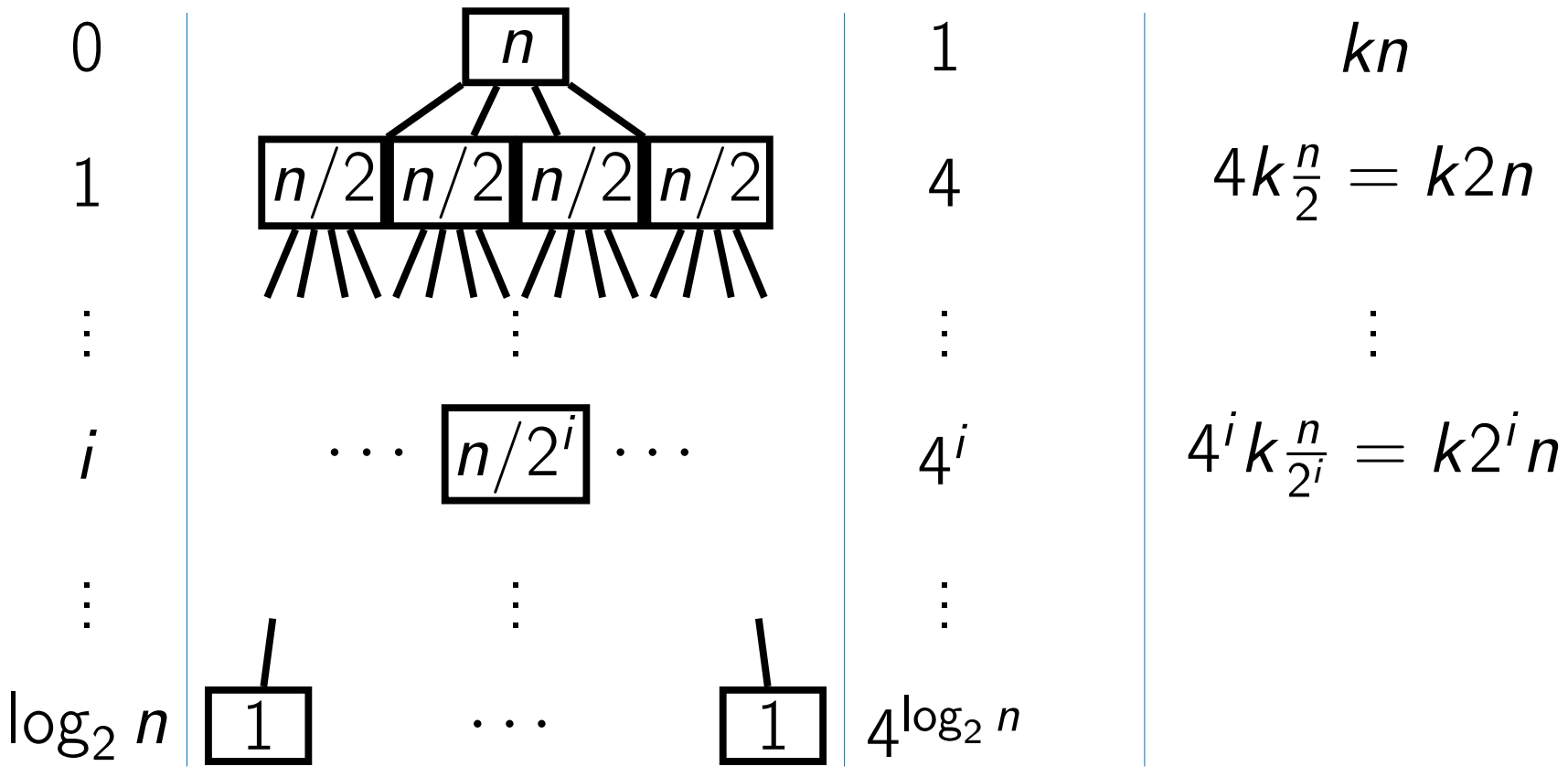
kn


level
#
work
0
1
kn
1
4
$4k\frac{n}{2} = k2n$
i
4^i
$\log_2 n$
1
1
$4^{\log_2 n}$


level
#
work
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$4k\frac{n}{2} = k2n$
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4^i
$4^i k\frac{n}{2^i} = k2^i n$
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$4^{\log_2 n}$
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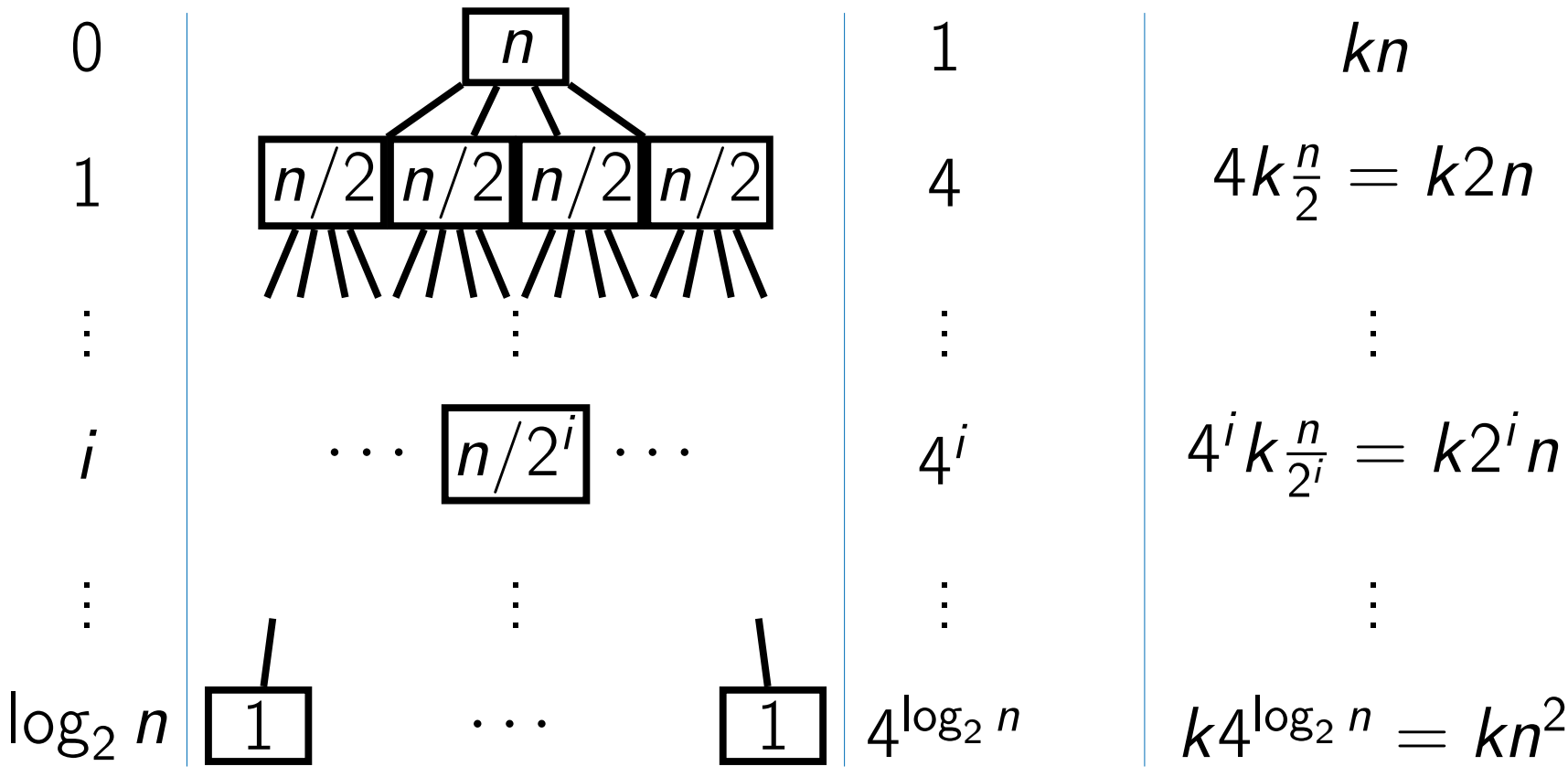
level
#
work
0
n
1
kn
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n/2
n/2
4
$4k\frac{n}{2} = k2n$
i
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4^i
$4^i k\frac{n}{2^i} = k2^i n$
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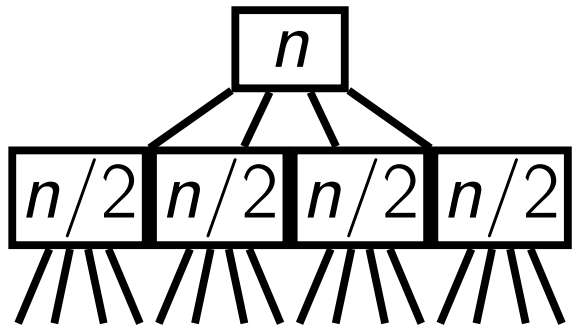
1
$\cdots$
1
$4^{\log_2 n}$
$k4^{\log_2 n} = kn^2$



level		#	work
0		1	kn
1		4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \quad n/2^i \quad \cdots$	4^i	$4^i k\frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$1 \quad \cdots \quad 1$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

Total: $\sum_{i=0}^{\log_2 n} 4^i k\frac{n}{2^i} = \Theta(n^2)$

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Karatsuba approach

Karatsuba approach

$$A(x) = a_1 x + a_0$$

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$$B(x) = b_1 x + b_0$$

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$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Karatsuba approach

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Needs 4 multiplications

Rewrite as:

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0) x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

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Needs 4 multiplications

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Needs 3 multiplications

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Needs 4 multiplications

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Needs 3 multiplications

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$D_1E_1 = 4x^2 + 11x + 6$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) =$$

Karatsuba Example

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)$$

Karatsuba Example

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$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

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$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= 24x^2 + 52x + 24$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

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$$= 24x^2 + 52x + 24$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

$$(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$$

$$)x^2 +$$

Karatsuba Example

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$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= 24x^2 + 52x + 24$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

$$(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$$

$$- (6x^2 + 11x + 4))x^2 +$$

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$$(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$$

$$- (6x^2 + 11x + 4))x^2 +$$

$$6x^2 + 11x + 4$$

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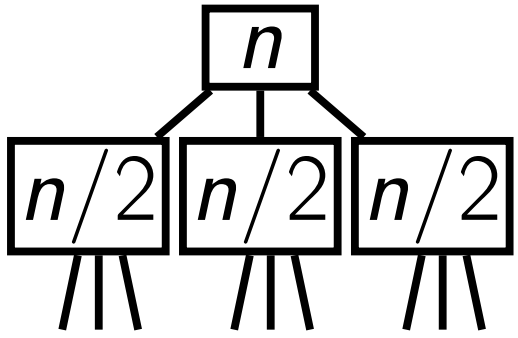
$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = &(4x^2 + 11x + 6)x^4 + \\ &(24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ &\qquad - (6x^2 + 11x + 4))x^2 + \\ &6x^2 + 11x + 4 \\ =&4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4\end{aligned}$$

n

level



level

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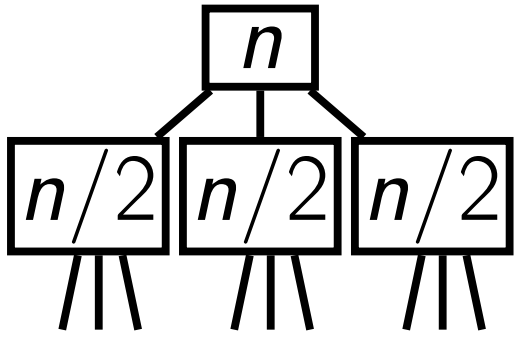

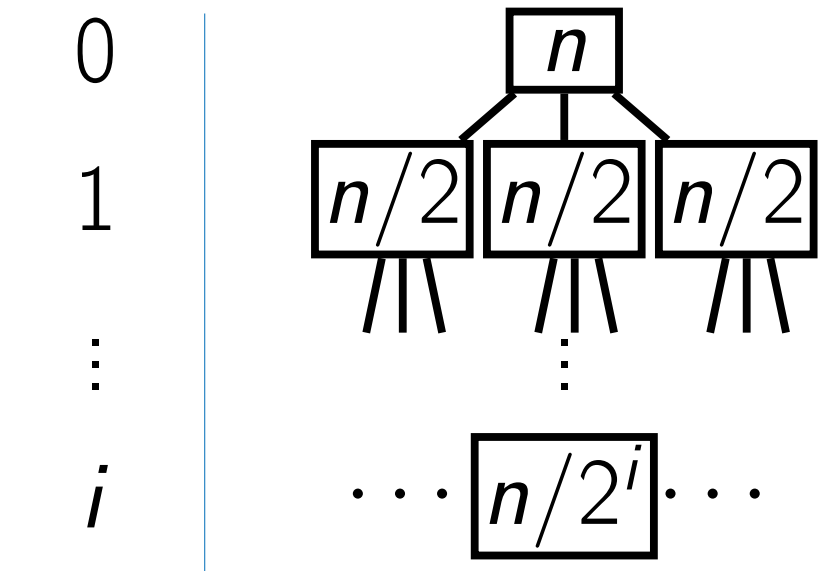
level
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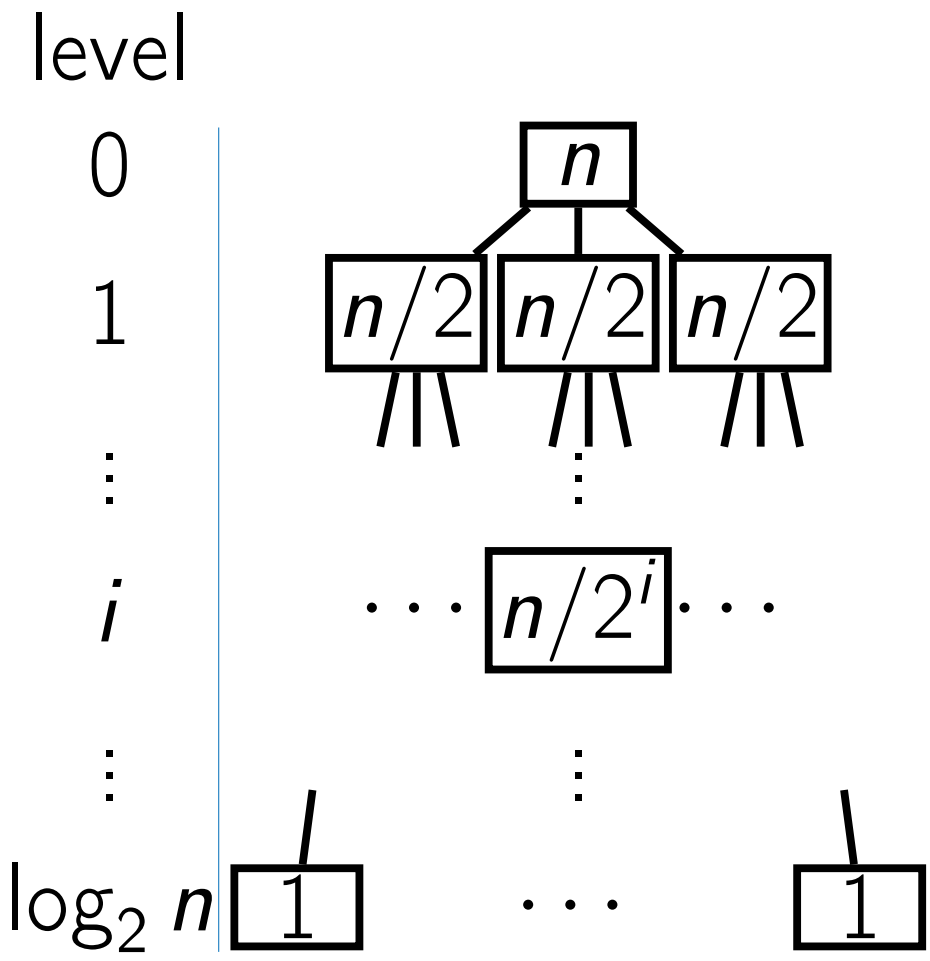
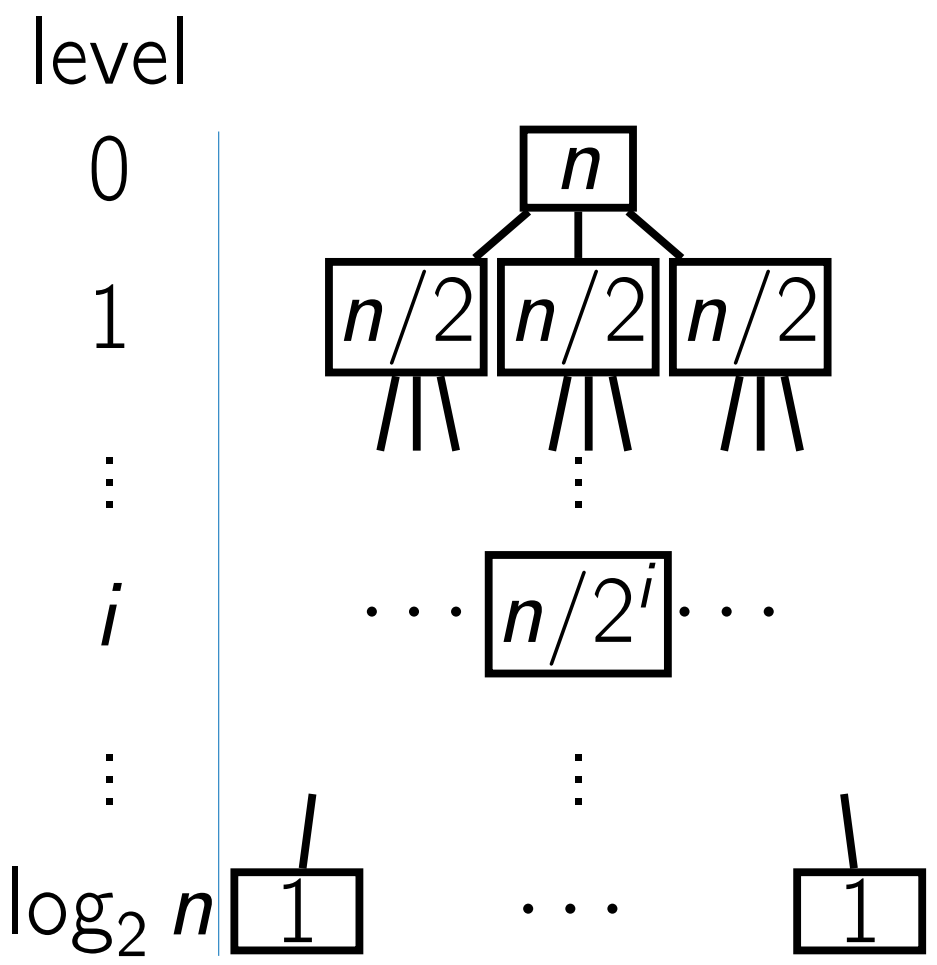
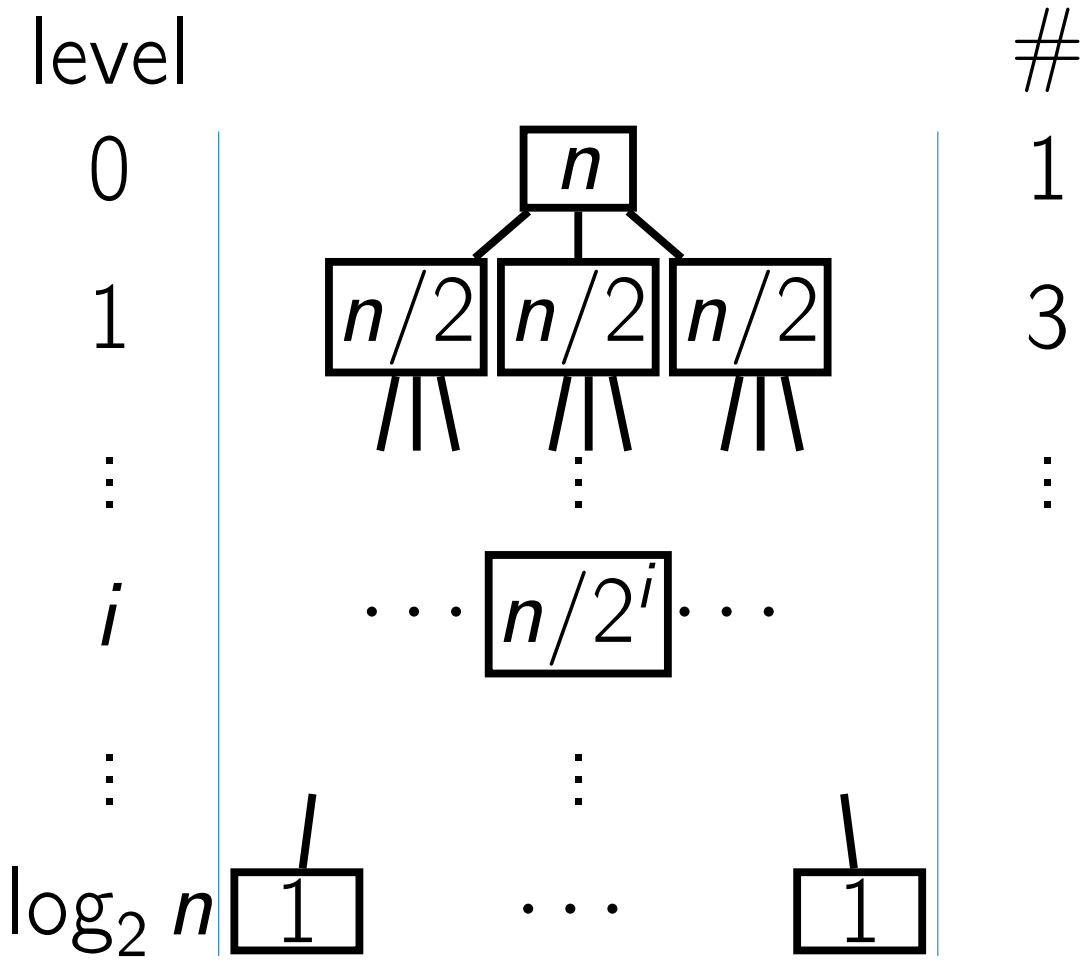
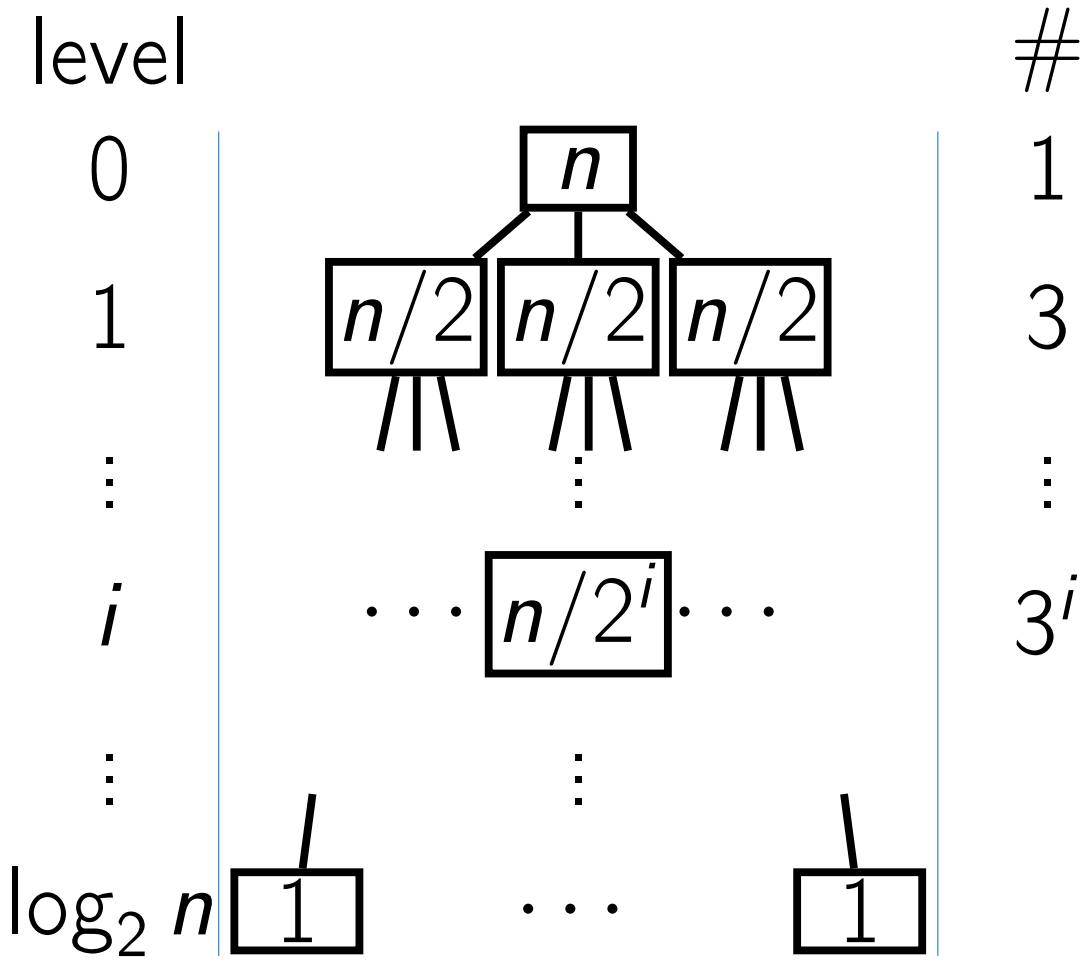
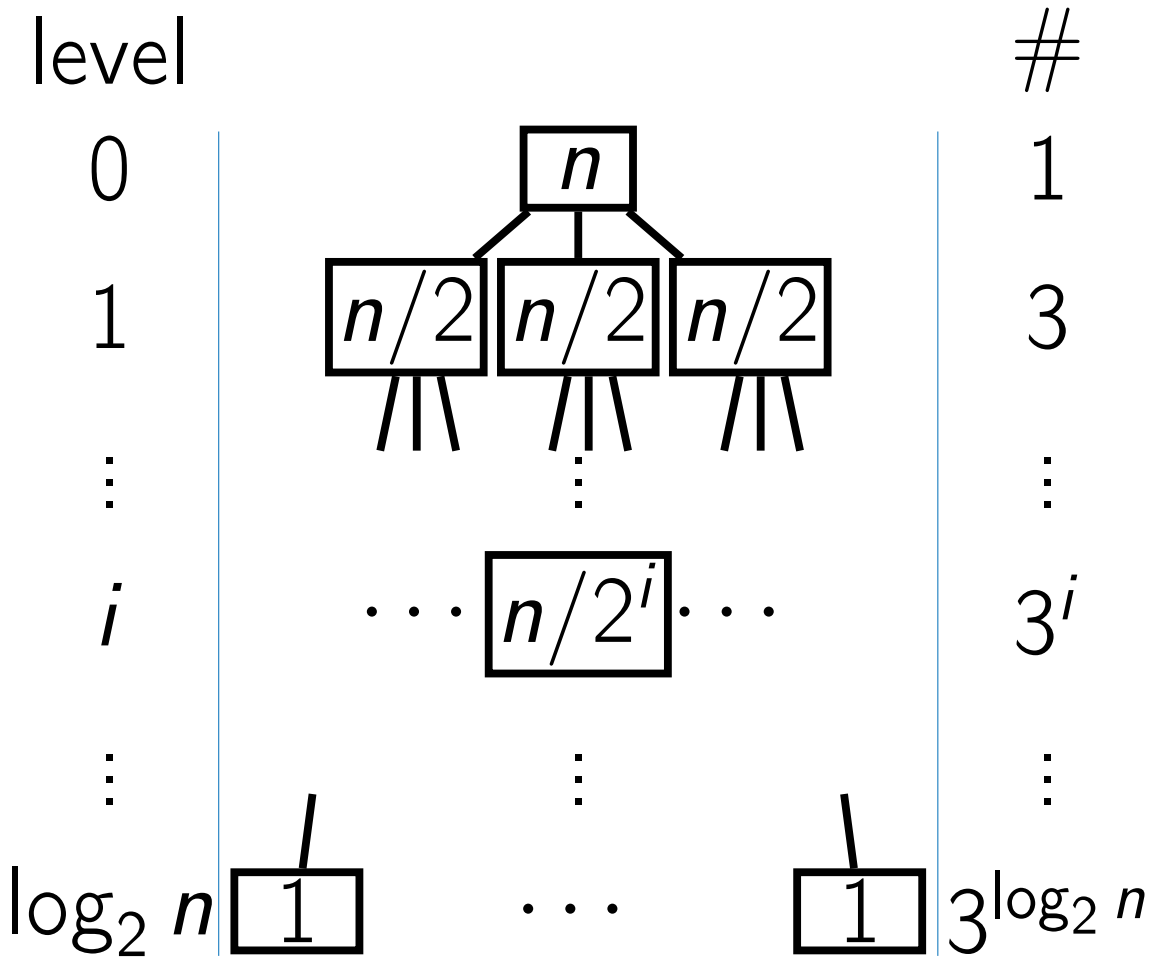
i
n
n/2
n/2
n/2
· · ·
n/2^i
· · ·

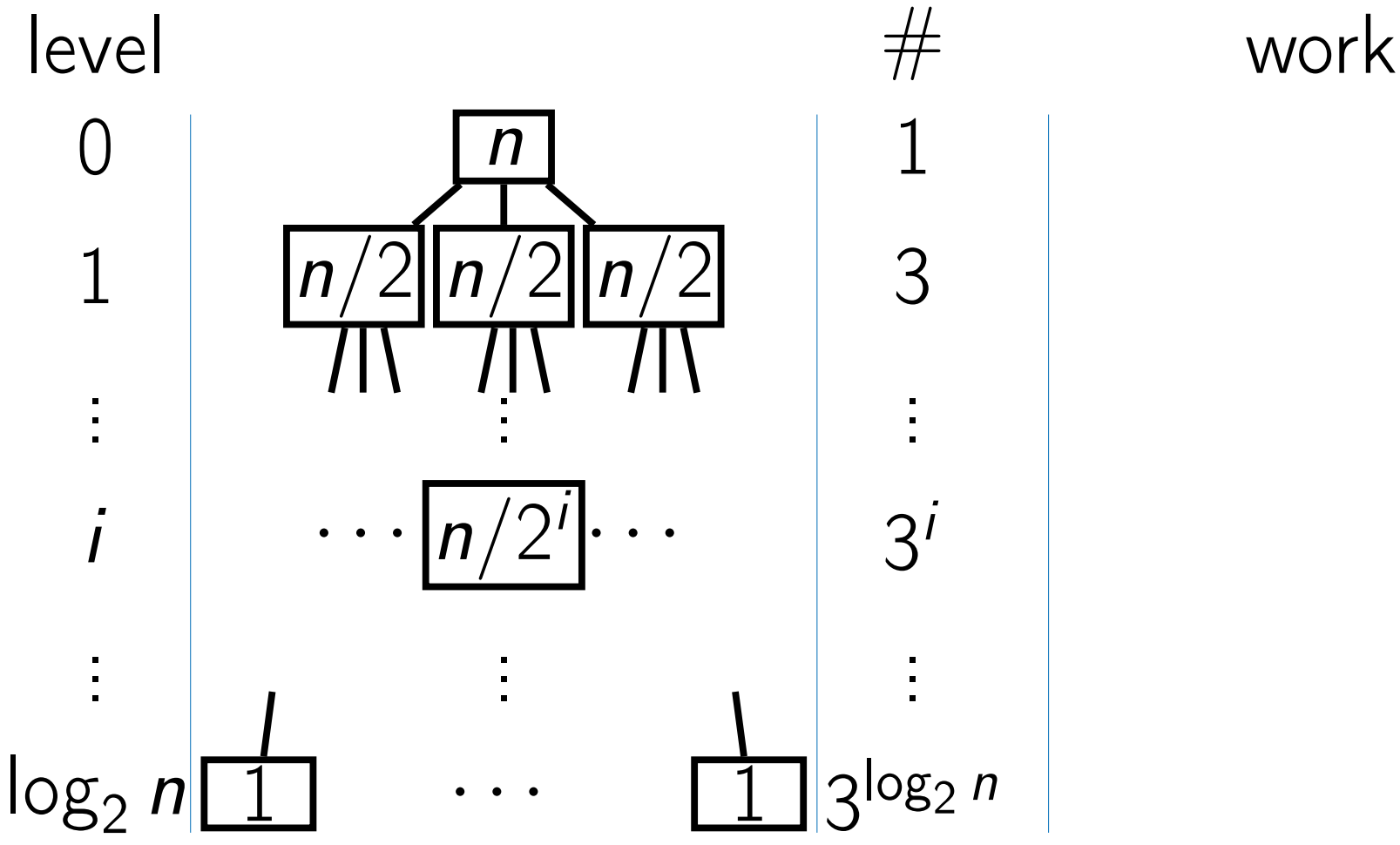

level
0
1
i
log2 n
n
n/2
n/2
n/2
n/2^i
1
1

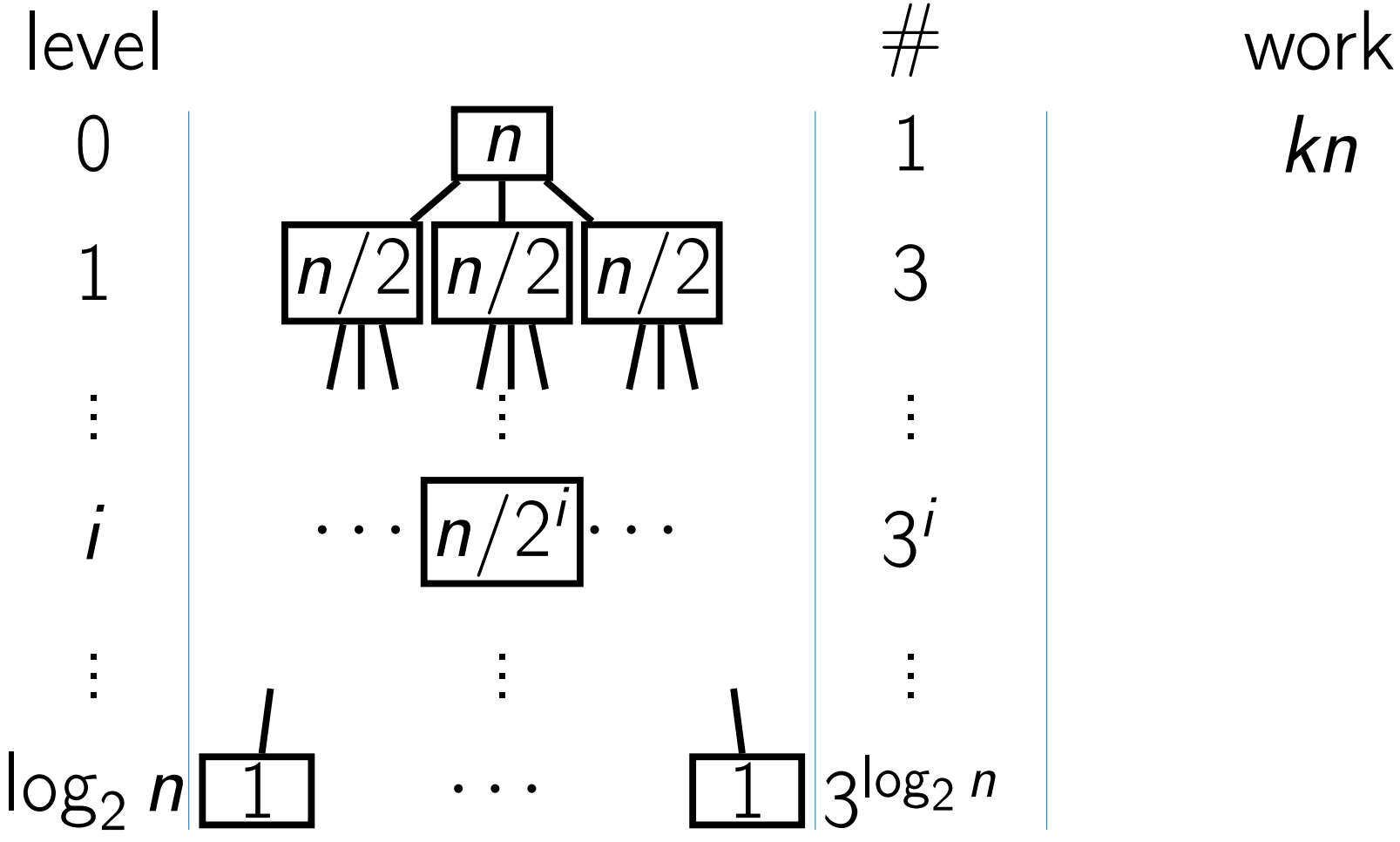

level
#
0
1
i
log2 n
n
n/2
n/2
n/2
n/2^i
1
1

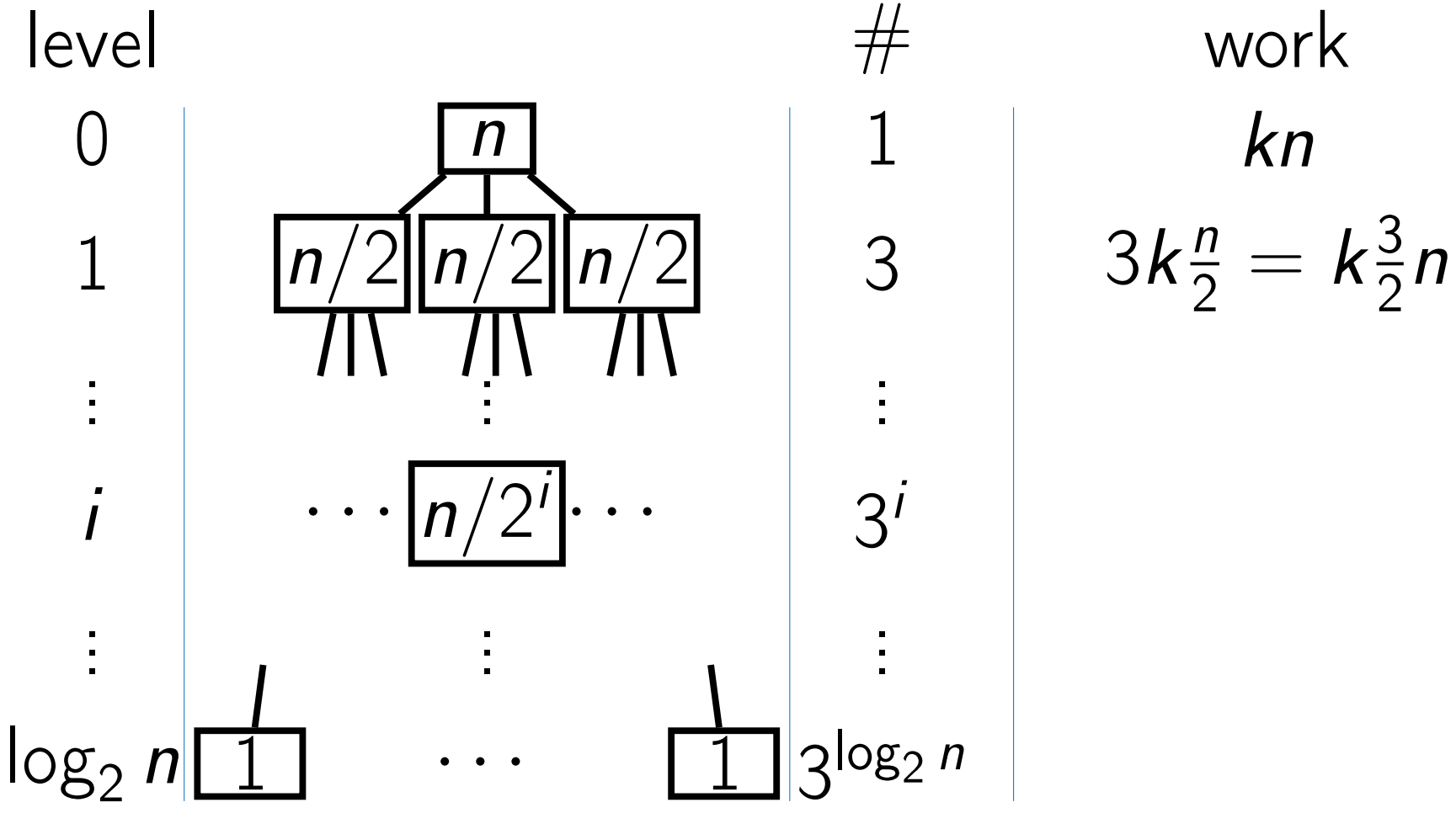

level
#
0
1
i
log2 n
n
n/2
n/2
n/2
n/2^i
1
1
1
3

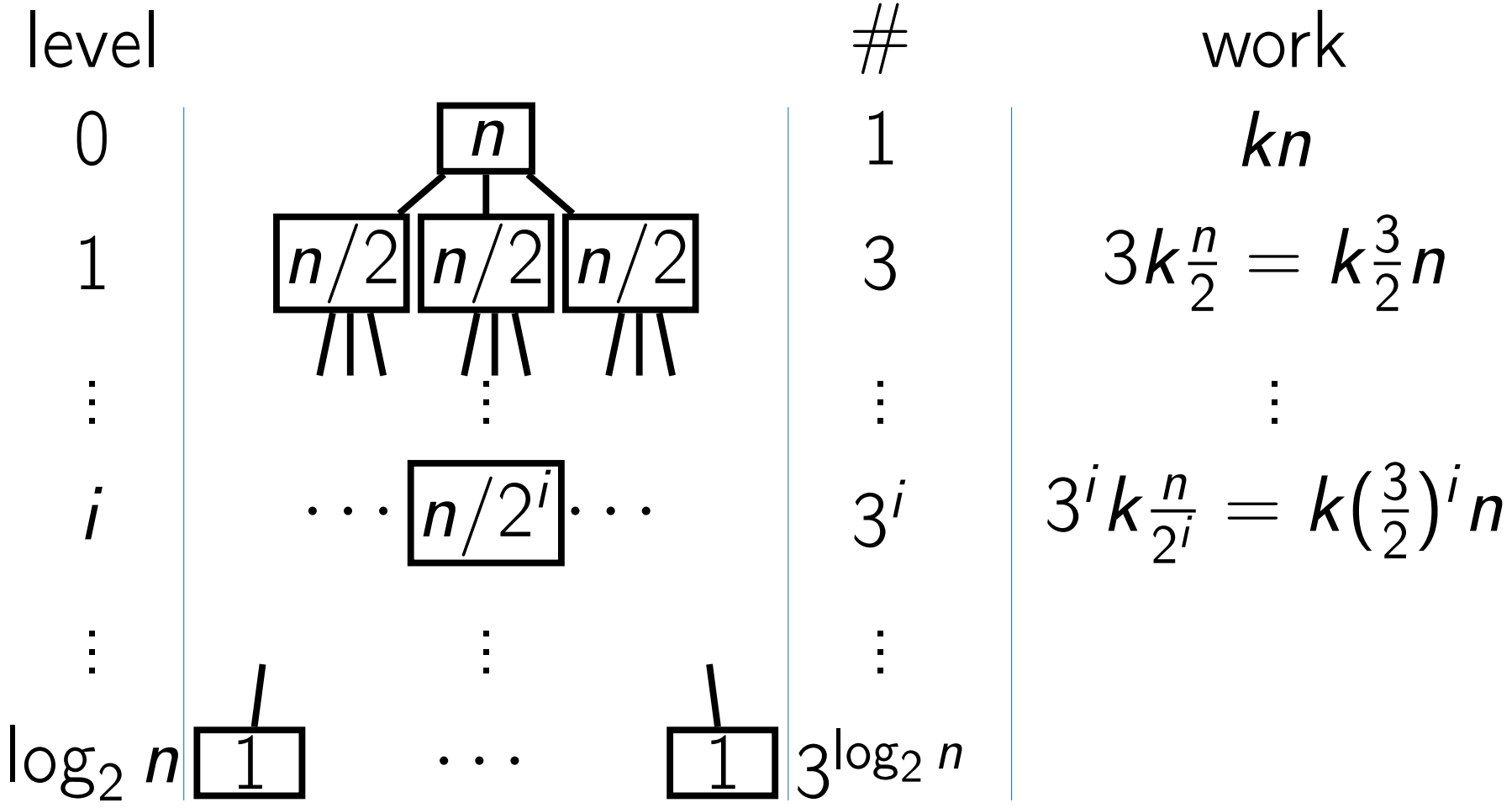

level
#
0
1
i
$\log_2 n$
n
$n/2$
$n/2$
$n/2$
$n/2^i$
1
1
1
3
3^i

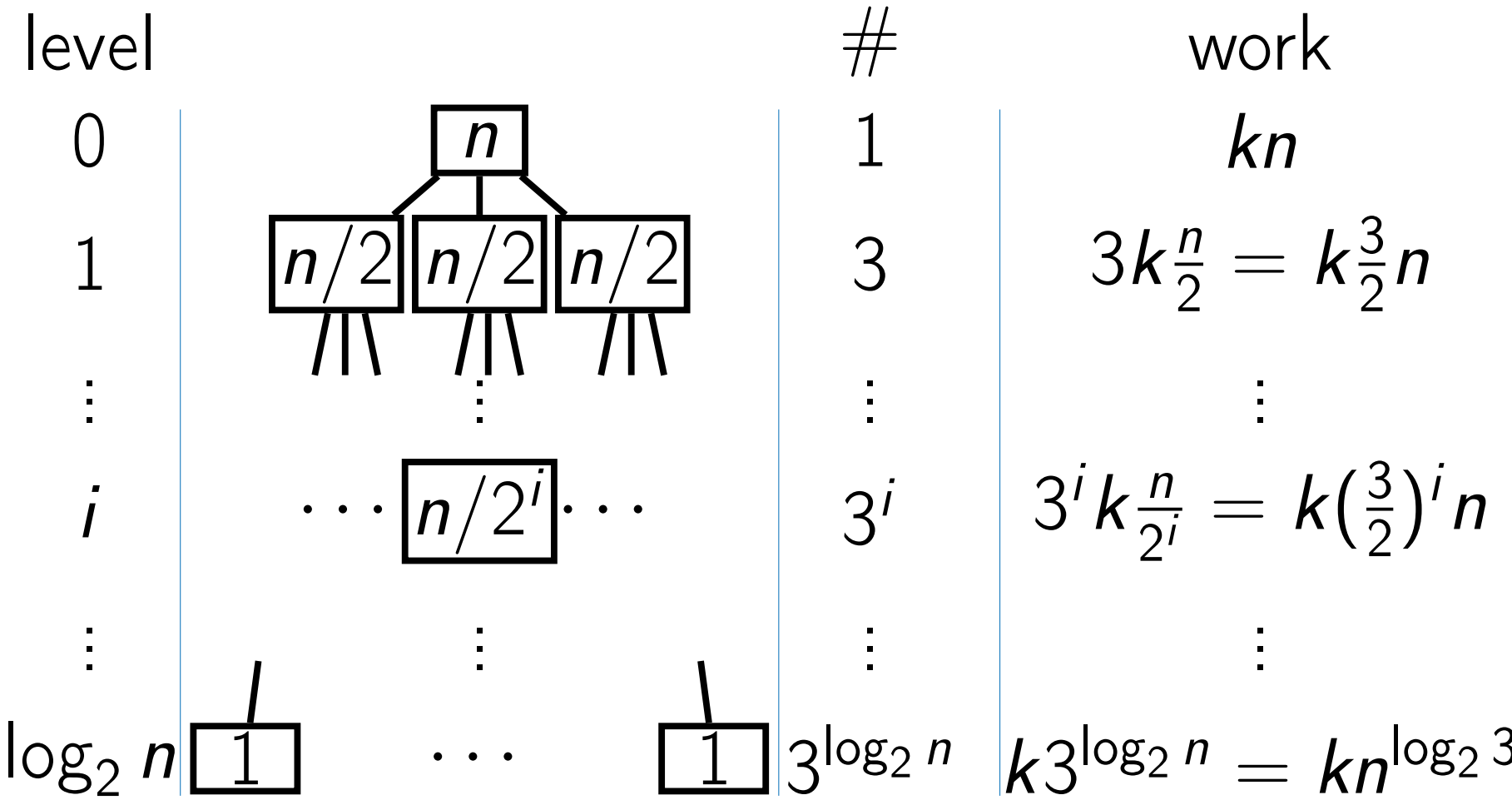

level
#
0
1
i
$\log_2 n$
n
$n/2$
$n/2$
$n/2$
$n/2^i$
1
1
1
3
3^i
$3^{\log_2 n}$

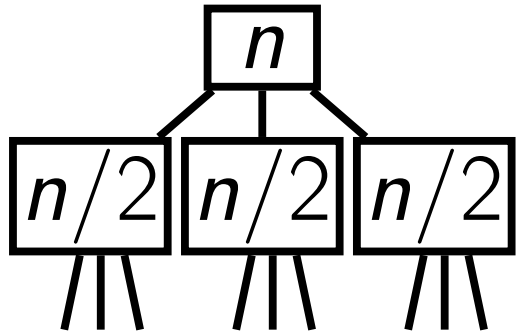

level
#
work
0
1
i
log2 n
n
n/2
n/2
n/2
n/2^i
1
1
1
3
3^i
3^log2 n


level
#
work
0
1
i
$\log_2 n$
n
$n/2$
$n/2$
$n/2$
$n/2^i$
1
1
1
3
3^i
$3^{\log_2 n}$
kn


level
#
work
0
1
kn
1
3
$3k\frac{n}{2} = k\frac{3}{2}n$
n
$n/2$
$n/2$
$n/2$
i
3^i
$n/2^i$
$\log_2 n$
1
1
$3^{\log_2 n}$


level
#
work
0
1
kn
n
1
3
$3k\frac{n}{2} = k\frac{3}{2}n$
$n/2$
$n/2$
$n/2$
i
3^i
$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
$\cdots$ $n/2^i$ $\cdots$
$\log_2 n$
1
$\cdots$
1
$3^{\log_2 n}$


level
#
work
0
1
kn
n
1
3
$3k\frac{n}{2} = k\frac{3}{2}n$
n/2
n/2
n/2
i
3^i
$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
$\cdots$ n/2^i $\cdots$
$\log_2 n$
$3^{\log_2 n}$
$k3^{\log_2 n} = kn^{\log_2 3}$
1
1
$\cdots$

level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots$ $n/2^i$ $\cdots$	3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	1 $\cdots$ 1	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

Total: $\sum_{i=0}^{\log_2 n} 3^i k\frac{n}{2^i} = \Theta(n^{\log_2 3})$

$= \Theta(n^{1.58})$