Divide-and-Conquer: Searching in an Array

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

a problem to be solved

Divide: Break into non-overlapping subproblems of the same type

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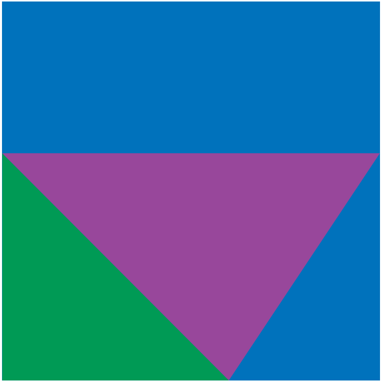
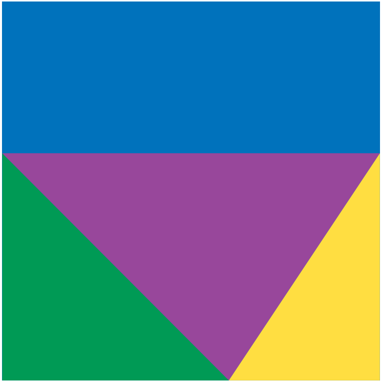
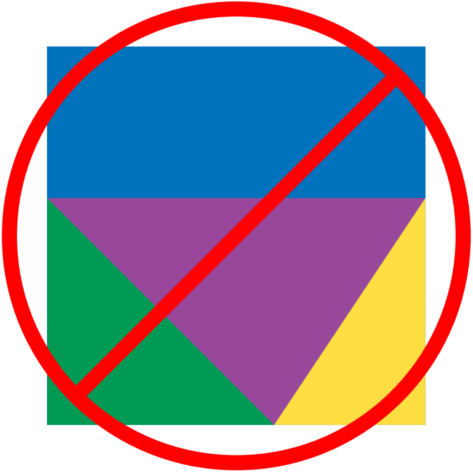
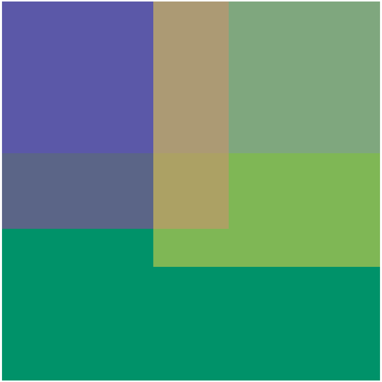
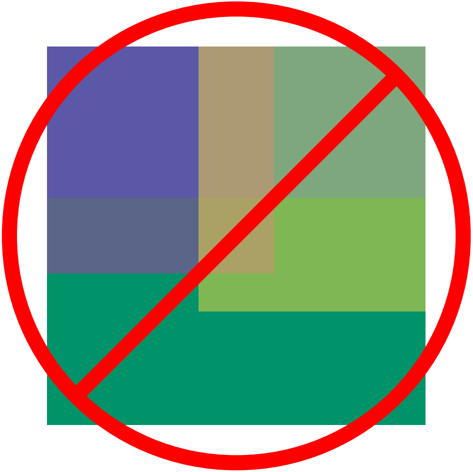






not the
same type





overlapping

Divide: break apart

Divide: break apart

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: combine

✓

1. Break into non-overlapping subproblems of the same type
2. Solve subproblems
3. Combine results

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

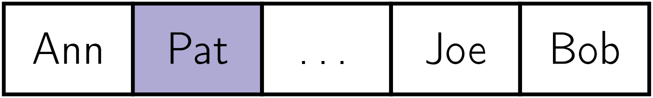
Linear Search in Array

Ann	Pat	...	Joe	Bob

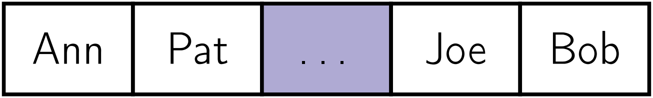
Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array



Linear Search in Array



Linear Search in Array

Ann	Pat	. . .	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
car	voiture	auto	Auto	auto
table	table	tavola	Tabelle	mesa

Searching in an array

Input: An array A with n elements.
A key k.

Output: An index, i, where $A[i] = k$.
If there is no such i, then NOT_FOUND.

Recursive Solution

`LinearSearch`$(A, low, high, key)$

Recursive Solution

LinearSearch$(A, low, high, key)$

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
```

Recursive Solution

LinearSearch$(A, low, high, key)$

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recursive Solution

$\texttt{LinearSearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

if $\mathit{high} < \mathit{low}$:
  return NOT_FOUND
if $A[\mathit{low}] = \mathit{key}$:
  return low
return $\texttt{LinearSearch}(A, \mathit{low} + 1, \mathit{high}, \mathit{key})$

Definition

A recurrence relation is an equation recursively defining a sequence of values.

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Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

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$$0, 1, 1, 2, 3, 5, 8, \ldots$$

LinearSearch($A, low, high, key$)

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

`LinearSearch`$(A, low, high, key)$

```
if  high < low :
  return NOT_FOUND
if  A[low] = key :
  return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

$$T(n) = T(n-1) + c$$

LinearSearch(A, low, $high$, key)

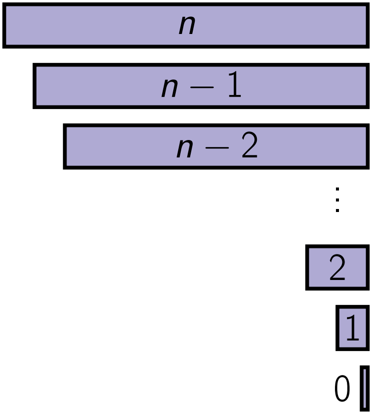
```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

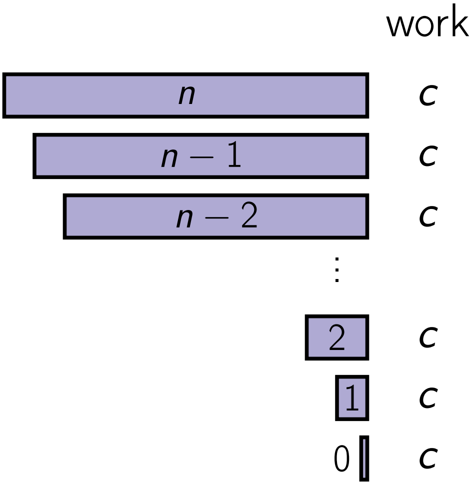
$$T(n) = T(n-1) + c$$

$$T(0) = c$$

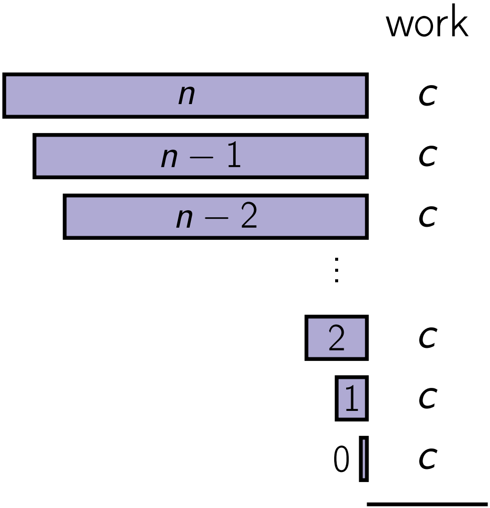
Runtime of Linear Search



Runtime of Linear Search



Runtime of Linear Search



$$\text{Total: } \sum_{i=0}^{n} c = \Theta(n)$$

Iterative Version

$\texttt{LinearSearchIt}(A, low, high, key)$

```
for i from low to high:
   if A[i] = key:
      return i
return NOT_FOUND
```

Summary

- Create a recursive solution

Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T

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- Determine $T(n)$: worst-case runtime

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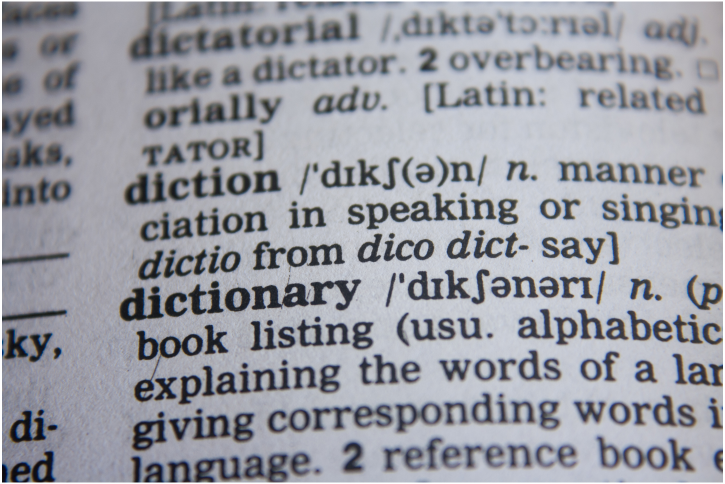
- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime
- Optionally, create iterative solution

Outline

Searching Sorted Data



Searching in a sorted array

Input: A sorted array $A[low \dots high]$
$(\forall low \leq i < high\colon A[i] \leq A[i+1])$.
A key k.

Output: An index, i, ($low \leq i \leq high$) where
$A[i] = k$.
Otherwise, the greatest index i,
where $A[i] < k$.
Otherwise ($k < A[low]$), the result is
$low - 1$.

Searching in a Sorted Array

Example

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$
$search(3) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$
$search(3) \to 1$
$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$ $search(20) \to 4$

$search(3) \to 1$

$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$ $\quad search(20) \to 4$

$search(3) \to 1$ $\quad search(20) \to 5$

$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$$\begin{array}{ll} \textit{search}(2) \to 0 & \textit{search}(20) \to 4 \\ \textit{search}(3) \to 1 & \textit{search}(20) \to 5 \\ \textit{search}(4) \to 1 & \textit{search}(60) \to 7 \end{array}$$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$$
\begin{array}{ll}
\mathit{search}(2) \to 0 & \mathit{search}(20) \to 4 \\
\mathit{search}(3) \to 1 & \mathit{search}(20) \to 5 \\
\mathit{search}(4) \to 1 & \mathit{search}(60) \to 7 \\
 & \mathit{search}(70) \to 7
\end{array}
$$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

BinarySearch(A, *low*, *high*, *key*)

BinarySearch(A, low, $high$, key)

```
if high < low:
    return low − 1
```

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
    return low − 1
```

$\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$

BinarySearch$(A, low, high, key)$

```
if high < low:
  return low − 1
mid ← ⌊low + (high − low)/2⌋
if key = A[mid]:
  return mid
```

`BinarySearch`$(A, low, high, key)$

```
if  high < low :
  return  low − 1
mid ← ⌊low + (high−low)/2⌋
if  key = A[mid] :
  return  mid
else if  key < A[mid] :
  return  BinarySearch(A, low, mid − 1, key)
```

BinarySearch(A, low, $high$, key)

```
if  high < low:
  return  low − 1
mid ← ⌊low + (high−low)/2⌋
if  key = A[mid]:
  return  mid
else if  key < A[mid]:
  return BinarySearch(A, low, mid − 1, key)
else:
  return BinarySearch(A, mid + 1, high, key)
```

$$\text{if } high < low:$$
$$\quad \text{return } low - 1$$
$$mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$$

Example: Searching for the key 50

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Example: Searching for the key 50

`BinarySearch`$(A, 1, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *high*

Example: Searching for the key 50

`BinarySearch`$(A, 1, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

`BinarySearch`$(A, 1, 11, 50)$
`BinarySearch`$(A, 7, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

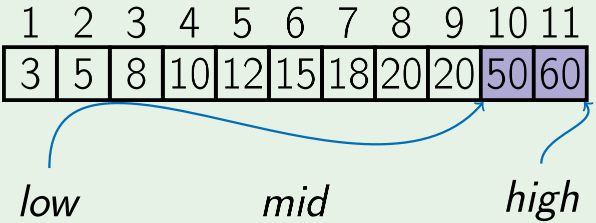
$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$



Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50) \rightarrow 10$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Summary

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- Break problem into non-overlapping subproblems of the same type.

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- Recursively solve those subproblems.

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- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.
- Combine results of subproblems.

$\texttt{BinarySearch}(A, low, high, key)$

```
if high < low:
  return low − 1
mid ← ⌊low + (high − low)/2⌋
if key = A[mid]:
  return mid
else if key < A[mid]:
  return BinarySearch(A, low, mid − 1, key)
else:
  return BinarySearch(A, mid + 1, high, key)
```

Binary Search Recurrence Relation

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$

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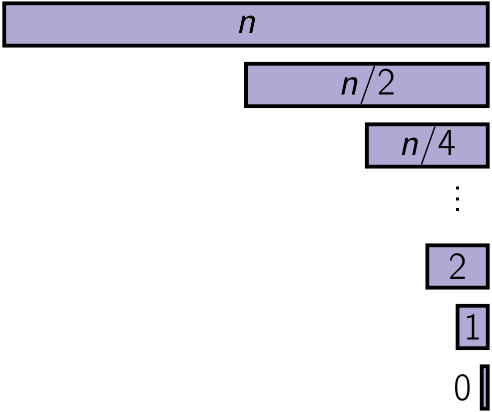
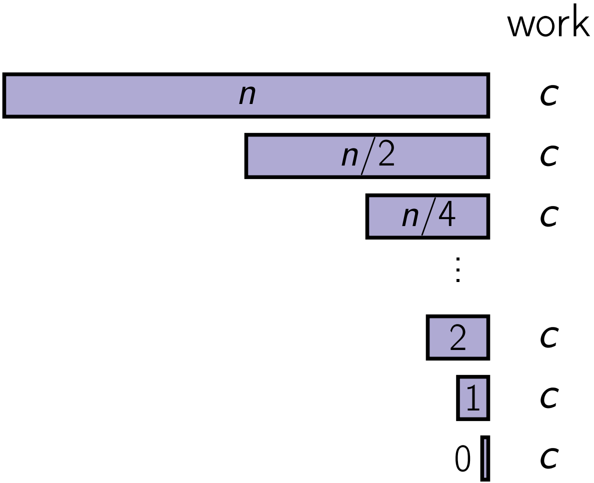
$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$
$$T(0) = c$$

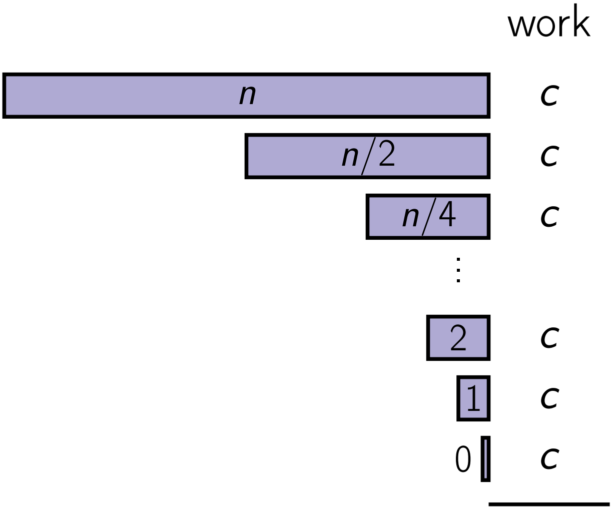
Runtime of Binary Search



Runtime of Binary Search



Runtime of Binary Search



Total: $\sum_{i=0}^{\log_2 n} c = \Theta(\log_2 n)$

Iterative Version

`BinarySearchIt`$(A, low, high, key)$

`while` $low \le high$:

$$mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$$

Iterative Version

$\texttt{BinarySearchIt}(A, \mathit{low}, \mathit{high}, \mathit{key})$

$\texttt{while } \mathit{low} \leq \mathit{high}:$

$\quad \mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$

$\quad \texttt{if } \mathit{key} = A[\mathit{mid}]:$

$\quad\quad \texttt{return } \mathit{mid}$

Iterative Version

```
BinarySearchIt(A, low, high, key)
```

$\texttt{BinarySearchIt}(A, low, high, key)$

$\texttt{while}$ $low \le high$:
  $mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$
  $\texttt{if}$ $key = A[mid]$:
    $\texttt{return}$ mid
  $\texttt{else if}$ $key < A[mid]$:
    $high = mid - 1$

Iterative Version

$\texttt{BinarySearchIt}(A, low, high, key)$

```
while low ≤ high:
    mid ← ⌊low + (high−low)/2⌋
    if key = A[mid]:
        return mid
    else if key < A[mid]:
        high = mid − 1
    else:
        low = mid + 1
```

Iterative Version

```
BinarySearchIt(A, low, high, key)
while low ≤ high:
    mid ← ⌊low + (high − low)/2⌋
    if key = A[mid]:
        return mid
    else if key < A[mid]:
        high = mid − 1
    else:
        low = mid + 1
return low − 1
```

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

Real-life Example

english (sorted)	**french** (sorted)	**italian** (sorted)	**german** (sorted)	**spanish** (sorted)
chair	chaise	casa	Haus	casa
house	bouton	foruncolo	Pickel	espenilla
pimple	maison	sedia	Sessel	silla

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted	**spanish** sorted
2	1
1	3
3	2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english
sorted

2
1
3

spanish
sorted

1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
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english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
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english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Summary

Summary

The runtime of binary search is $\Theta(\log n)$.