

Introduction: Greatest Common Divisors II

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Algorithmic Toolbox
Data Structures and Algorithms

Learning Objectives

- Implement the Euclidean Algorithm.
- Approximate the runtime.

GCDs

Definition

For integers, a and b , their **greatest common divisor** or $\gcd(a, b)$ is the largest integer d so that d divides both a and b .

GCDs

Definition

For integers, a and b , their **greatest common divisor** or $\gcd(a, b)$ is the largest integer d so that d divides both a and b .

Compute GCD

Input: Integers $a, b \geq 0$.

Output: $\gcd(a, b)$.

Key Lemma

Lemma

Let a' be the remainder when a is divided by b , then

$$\gcd(a, b) = \gcd(a', b) = \gcd(b, a').$$

Proof

Proof (sketch)

- $a = a' + bq$ for some q
- d divides a and b if and only if it divides a' and b

Euclidean Algorithm

Function EuclidGCD(a, b)

if $b = 0$:

 return a

$a' \leftarrow$ the remainder when a is
 divided by b

return EuclidGCD(b, a')

Euclidean Algorithm

Function EuclidGCD(a, b)

```
if  $b = 0$ :  
    return  $a$   
 $a' \leftarrow$  the remainder when  $a$  is  
    divided by  $b$   
return EuclidGCD( $b, a'$ )
```

Produces correct result by Lemma.

Example

$\text{gcd}(3918848, 1653264)$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \end{aligned}$$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \\ &= \gcd(612320, 428624) \end{aligned}$$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \\ &= \gcd(612320, 428624) \\ &= \gcd(428624, 183696) \end{aligned}$$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \\ &= \gcd(612320, 428624) \\ &= \gcd(428624, 183696) \\ &= \gcd(183696, 61232) \end{aligned}$$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \\ &= \gcd(612320, 428624) \\ &= \gcd(428624, 183696) \\ &= \gcd(183696, 61232) \\ &= \gcd(61232, 0) \end{aligned}$$

Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ &= \gcd(1653264, 612320) \\ &= \gcd(612320, 428624) \\ &= \gcd(428624, 183696) \\ &= \gcd(183696, 61232) \\ &= \gcd(61232, 0) \\ &= 61232. \end{aligned}$$

Runtime

- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.

Runtime

- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.
- GCDs of 100 digit numbers takes about 600 steps.
- Each step a single division.

Summary

- Naive algorithm is too slow.
- The correct algorithm is much better.
- Finding the correct algorithm requires knowing something interesting about the problem.