

Dynamic Programming: Placing Parentheses

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Algorithmic Toolbox
Data Structures and Algorithms

Outline

- 1 Problem Overview
- 2 Subproblems
- 3 Algorithm
- 4 Reconstructing a Solution

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

Example

- $((((1 + 2) - 3) \times 4) - 5) = -5$

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

Example

- $((((1 + 2) - 3) \times 4) - 5) = -5$
- $((1 + 2) - ((3 \times 4) - 5)) = -4$

Answer

$$((1 + 2) - (3 \times (4 - 5))) = 6$$

Another example

What about

$$5 - 8 + 7 \times 4 - 8 + 9?$$

Another example

What about

$$5 - 8 + 7 \times 4 - 8 + 9?$$

Soon

We'll design an efficient dynamic programming algorithm to find the answer.

Outline

- 1 Problem Overview
- 2 Subproblems
- 3 Algorithm
- 4 Reconstructing a Solution

Placing parentheses

Input: A sequence of digits d_1, \dots, d_n and a sequence of operations $op_1, \dots, op_{n-1} \in \{+, -, \times\}$.

Output: An order of applying these operations that maximizes the value of the expression

$$d_1 \ op_1 \ d_2 \ op_2 \ \cdots \ op_{n-1} \ d_n .$$

Intuition

- Assume that the last operation in an optimal parenthesizing of $5 - 8 + 7 \times 4 - 8 + 9$ is \times :

$$(5 - 8 + 7) \times (4 - 8 + 9).$$

Intuition

- Assume that the last operation in an optimal parenthesizing of $5 - 8 + 7 \times 4 - 8 + 9$ is \times :

$$(5 - 8 + 7) \times (4 - 8 + 9).$$

- It would help to know optimal values for **subexpressions** $5 - 8 + 7$ and $4 - 8 + 9$.

However

We need to keep track for both the minimal and the maximal values of subexpressions!

Example: $(5 - 8 + 7) \times (4 - 8 + 9)$

$$\min(5 - 8 + 7) = (5 - (8 + 7)) = -10$$

$$\max(5 - 8 + 7) = ((5 - 8) + 7) = 4$$

$$\min(4 - 8 + 9) = (4 - (8 + 9)) = -13$$

$$\max(4 - 8 + 9) = ((4 - 8) + 9) = 5$$

Example: $(5 - 8 + 7) \times (4 - 8 + 9)$

$$\min(5 - 8 + 7) = (5 - (8 + 7)) = -10$$

$$\max(5 - 8 + 7) = ((5 - 8) + 7) = 4$$

$$\min(4 - 8 + 9) = (4 - (8 + 9)) = -13$$

$$\max(4 - 8 + 9) = ((4 - 8) + 9) = 5$$

$$\max((5 - 8 + 7) \times (4 - 8 + 9)) = 130$$

Subproblems

- Let $E_{i,j}$ be the subexpression

$$d_i \ op_i \ \cdots \ op_{j-1} \ d_j$$

Subproblems

- Let $E_{i,j}$ be the subexpression

$$d_i \ op_i \cdots \ op_{j-1} \ d_j$$

- Subproblems:

$M(i, j)$ = maximum value of $E_{i,j}$

$m(i, j)$ = minimum value of $E_{i,j}$

Recurrence Relation

$$M(i, j) = \max_{i \leq k \leq j-1} \begin{cases} M(i, k) & op_k & M(k+1, j) \\ M(i, k) & op_k & m(k+1, j) \\ m(i, k) & op_k & M(k+1, j) \\ m(i, k) & op_k & m(k+1, j) \end{cases}$$

$$m(i, j) = \min_{i \leq k \leq j-1} \begin{cases} M(i, k) & op_k & M(k+1, j) \\ M(i, k) & op_k & m(k+1, j) \\ m(i, k) & op_k & M(k+1, j) \\ m(i, k) & op_k & m(k+1, j) \end{cases}$$

Outline

- 1 Problem Overview
- 2 Subproblems
- 3 Algorithm
- 4 Reconstructing a Solution

MinAndMax(i, j)

$min \leftarrow +\infty$

$max \leftarrow -\infty$

for k from i to $j - 1$:

$a \leftarrow M(i, k) \quad op_k \quad M(k + 1, j)$

$b \leftarrow M(i, k) \quad op_k \quad m(k + 1, j)$

$c \leftarrow m(i, k) \quad op_k \quad M(k + 1, j)$

$d \leftarrow m(i, k) \quad op_k \quad m(k + 1, j)$

$min \leftarrow \min(min, a, b, c, d)$

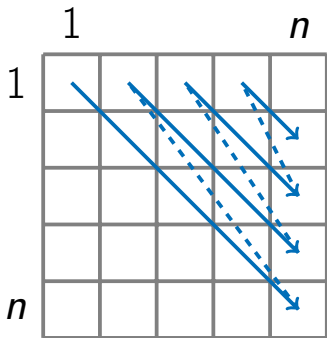
$max \leftarrow \max(max, a, b, c, d)$

return (min, max)

Order of Subproblems

- When computing $M(i, j)$, the values of $M(i, k)$ and $M(k + 1, j)$ should be already computed.
- Solve all subproblems in order of increasing $(j - i)$.

Possible Order



Parentheses($d_1 \text{ op}_1 d_2 \text{ op}_2 \dots d_n$)

for i from 1 to n :

$m(i, i) \leftarrow d_i, M(i, i) \leftarrow d_i$

for s from 1 to $n - 1$:

for i from 1 to $n - s$:

$j \leftarrow i + s$

$m(i, j), M(i, j) \leftarrow \text{MinAndMax}(i, j)$

return $M(1, n)$

Example: $5 - 8 + 7 \times 4 - 8 + 9$

5	-3	-10	-55	-63	-94
	8	15	36	-60	-195
		7	28	-28	-91
			4	-4	-13
				8	17
					9

m

5	-3	4	25	65	200
	8	15	60	52	75
		7	28	20	35
			4	-4	5
				8	17
					9

M

Outline

- 1 Problem Overview
- 2 Subproblems
- 3 Algorithm
- 4 Reconstructing a Solution

Example: $5 - 8 + 7 \times 4 - 8 + 9$

5	-3	-10	-55	-63	-94
	8	15	36	-60	-195
		7	28	-28	-91
			4	-4	-13
				8	17
					9

m

5	-3	4	25	65	200
	8	15	60	52	75
		7	28	20	35
			4	-4	5
				8	17
					9

M