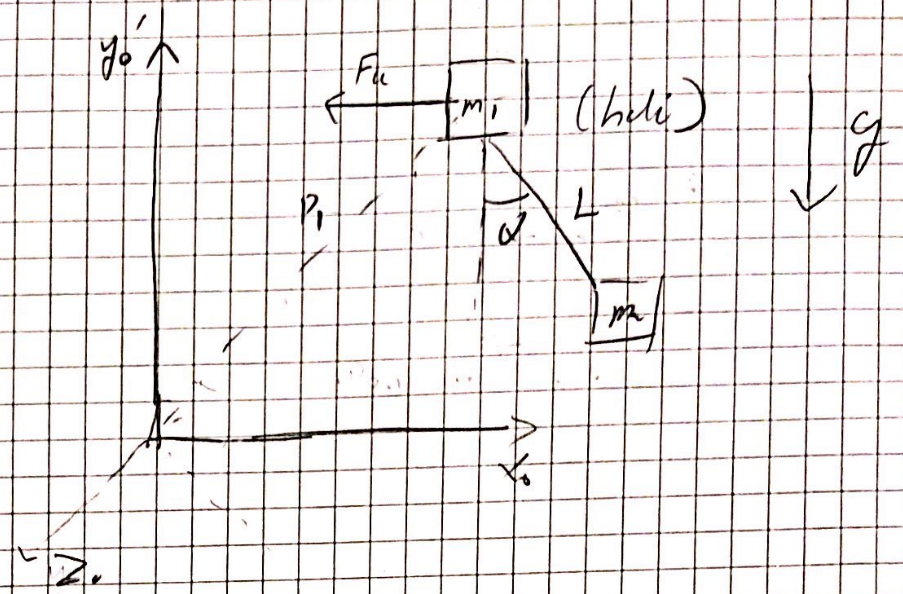
Modsim øving 5

- Problem 1
  - Helicopter hovering a mass
    - Two point masses: m_1, m_2

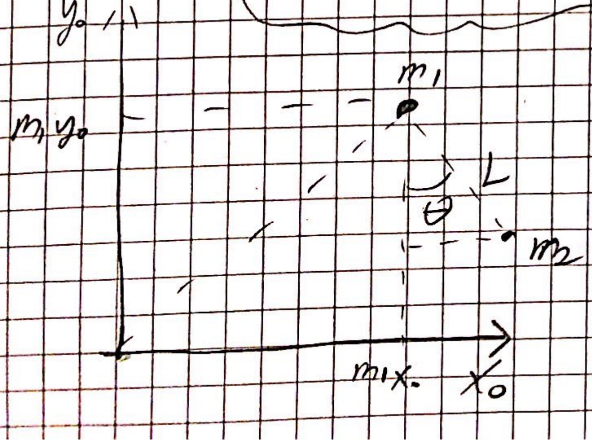


a) Model this system by using classical lagrange
- Number of coordinates = 3 dof
- The position of helicopter: $p_1 \in \mathbb{R}^3$
- Position of the hovering mass: θ, ϕ

$$q = \begin{bmatrix} p_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5$$

z-axis out of paper

$\phi = 0$

1) 

$$P_{m_2 x_0} = P_{m_1 x_0} + \cos\phi L \sin\theta$$

$$P_{m_2 y_0} = P_{m_1 y_0} - L\cos\theta$$

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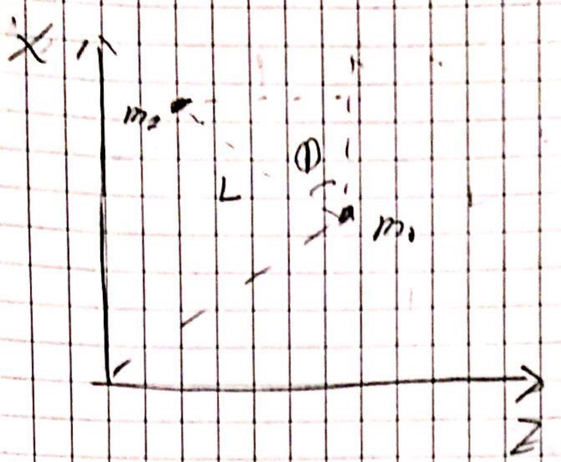

(X-axis out of paper)

$$P_{m_{1z}} = P_{m_{2z_0}} - \sin\phi \, L \sin\theta$$

$$P_{m_2} = \begin{bmatrix} \cos\phi \, L\sin\theta \\ -L\cos\theta \\ -\sin\phi \, L\sin\theta \end{bmatrix} + P_{m_1}$$

$$Q =$$

$$T = \frac{1}{2} m_1 V_1^T V_1 + \frac{1}{2} m_2 V_2^T V_2$$

$$= \frac{1}{2} m_1 \dot{P}_1^T P_1 + \frac{1}{2} m_2 \dot{P}_2^T P_2$$

$$U = mgh = m_1 g \cdot P_{1y_0} + m g P_{2y_0}$$

$$L = T - V$$

b) Use constrained Lagrange approach to model the dynamics of the system

$$\longrightarrow \quad q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in R^6$$

$$c = \frac{1}{2}\left(e^T e - L^2\right) \qquad e = p_1 - p_2$$

The dynamics of the system

- $\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) - z \nabla C(q) = Q$

- $C(q) = 0$

The fully assembled form should have the form

- $\dot{q} = v$
- $M(q)\, \dot{v} = b(q, z, u)$
- $0 = C(q)$

$$Q = \begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix}$$

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$$T = \frac{1}{2} m_1 \dot{p}_1^T \dot{p}_1 + \frac{1}{2} m_2 \dot{p}_2^T \dot{p}_2 = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$$= \frac{1}{2} m_1 \left(\dot{q}^T \frac{\partial p_1}{\partial q}^T \right) \frac{\partial p}{\partial q} \dot{q} + \frac{1}{2} m_2 \dot{q}^T \frac{\partial p_2}{\partial q}^T \frac{\partial p_2}{\partial q} \dot{q}$$

$$= \frac{1}{2} \dot{q}^T \left(m_1 \frac{\partial p_1}{\partial q}^T \frac{\partial p}{\partial q} + m_2 \frac{\partial p_2}{\partial q}^T \frac{\partial p_2}{\partial q} \right) \dot{q}$$

$$= \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$$M(q) = m_1 \frac{\partial p_1}{\partial q}^T \frac{\partial p}{\partial q} + m_2 \frac{\partial p_2}{\partial q}^T \frac{\partial p_2}{\partial q}$$

$$q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\frac{\partial p_1}{\partial q} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \frac{\partial p_2}{\partial q} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\frac{\partial p_1}{\partial q}^T \frac{\partial p_1}{\partial q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \frac{\partial p_2}{\partial q}^T \frac{\partial p_2}{\partial q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{M(q) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}$$

$$b(q, \dot{q}, u) = M(q) \ddot{q} = \tau - C(q, \dot{q}) \dot{q} - g(q)$$

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$$\begin{bmatrix} M(q) & \nabla_q C^T \\ \nabla_q C & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = \begin{bmatrix} Q - \nabla_q \mathcal{L} \\ -\frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\dot{q}\right)\dot{q} \end{bmatrix}$$

1b is much more compact as everything becomes standardized inside ~~a mat~~ matrices. This makes analysis easier. That said 1a seems more intuitive and more straight forward

Problem 2: Explicit vs. implicit model

a) $a = \nabla_q C$

$$c(q, \dot{q}, v) = \begin{bmatrix} Q - \nabla_q \mathcal{L} \\ -\frac{\partial}{\partial q} \cdot \left(\frac{\partial C}{\partial q}\dot{q}\right)\dot{q} \end{bmatrix}$$

M is symmetric and positive definite

The matrix $= \begin{bmatrix} M & a(q) \\ a(q)^T & 0 \end{bmatrix}$ is non singular

and ~~is~~ invertible. Thus $\begin{bmatrix} \ddot{q} \\ z \end{bmatrix}$ can be computed

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Problem 3 (Δ-robot)

The positions of the pivots of the yellow arm in this cartesian frame $P_{1,2,3}$ are given by

$$P_k = R_k \begin{bmatrix} d + L\cos\alpha_k \\ 0 \\ -L\sin\alpha_k \end{bmatrix}, \quad R_k = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\gamma_{1,2,3} = \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}$, d is the constant distance from the centre of the upper platform to the axis of the motors

a) Only using $\alpha_{1,2,3}$ would yield two possible points for the ~~apelle~~ needle

b) $q = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ p \end{bmatrix}$

$$V = mg \, [0 \;\; 0 \;\; 1] \, \vec{p}$$

$$T = \frac{1}{2} J \, \dot{\alpha}_k^T \dot{\alpha}_k + \frac{1}{2} m \, \dot{p}^T \dot{p}$$

$$\mathcal{L} = T - V$$

$$C_1(q) = 0 = \|\vec{p} - \vec{P}_1\|^2 - L^2 = 0$$

$$C_2(q) = 0 = \|P - P_2\|^2 - L^2 = 0$$

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$$C_3(q) = 0 = \|\vec{P} - \vec{P_3}\|^2 - L^2 = 0$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q \qquad Q = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix}$$

$$M(q) = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \begin{bmatrix} y & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -mg \end{bmatrix}$$

c) differential index of 3. As this is a constrained Lagrange problem.

d)

```
clear all
clc

% Parameters
syms m1 m2 L g real
% Force
u = sym('u',[3,1]);

% Position point mass 1
pm1 = sym('p1',[3,1]);
dpm1 = sym('dp1',[3,1]);
ddpm1 = sym('d2p1',[3,1]);
% Angles for point mass 2
a  = sym('a',[2,1]);
da = sym('da',[2,1]);
dda = sym('d2a',[2,1]);
% Generalized coordinates
q  = [pm1;a];
dq = [dpm1;da];
ddq = [ddpm1;dda];

% Position of point mass 2
pm2  = pm1 + [L*cos(a(2))*sin(a(1)), -L*cos(a(1)), -L*sin(a(2))*sin(a(1))]';
% Velocity of point mass 2
dpm2 = jacobian(pm2,q)*dq;
% Generalized forces
Q = [u; 0; 0];
% Kinetic energy
T = (1/2)*m2*(dpm2)'*dpm2 + (1/2)*m1*(dpm1)'*dpm1;
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```
T = (1/2)*m2*(dpm2)'*dpm2 + (1/2)*m1*(dpm1)'*dpm1;
T = simplify(T);
% Potential energy
%
V = m1*g*pm1(2) + m2*g*pm2(2);
% Lagrangian
Lag =  T - V;

% Derivatives of the Lagrangian
Lag_q = simplify(jacobian(Lag,q)).';
Lag_qdq = simplify(jacobian(Lag_q.',dq));
Lag_dq = simplify(jacobian(Lag,dq)).';
Lag_dqdq = simplify(jacobian(Lag_dq.',dq)); % W

% Matrices for problem 1
M = Lag_dqdq;
b = Q + simplify(Lag_q - Lag_qdq*dq);
```

```
clear all
clc

% Parameters
syms m1 m2 L g real
% Force
u = sym('u',[3,1]);

% Positions of point masses
pm1 = sym('pm1',[3,1]);
pm2 = sym('pm2',[3,1]);
dpm1 = sym('dpm1',[3,1]);
dpm2 = sym('dpm2',[3,1]);
ddpm1 = sym('d2pm1',[3,1]);
ddpm2 = sym('d2pm2',[3,1]);
% Generalized coordinates
q  = [pm1;pm2];
dq = [dpm1;dpm2];
ddq = [ddpm1;ddpm2];
% Algebraic variable
z = sym('z');

% Generalized forces
Q = [u; 0; 0];
% Kinetic energy (function of q and dq)
dpm1 = jacobian(pm1,q)*dq;
dpm2 = jacobian(pm2,q)*dq;
T =0.5*m1*(dpm1)'*dpm1 + 0.5*m2*(dpm2)'*dpm2;
% Potential energy
```

```
% Potential energy
V = m1*g*pm1(2) + m2*g*pm2(2);
% Lagrangian (function of q and dq)
Lag = T - V;
% Constraint
dpm  = pm1 - pm2; % difference of positions
C =0.5*((dpm)'*dpm - L*L);

% Derivatives of constrained Lagrangian
Lag_q = simplify(jacobian(Lag,q)).';
Lag_qdq = simplify(jacobian(Lag_q.',dq));
Lag_dq = simplify(jacobian(Lag,dq)).';
Lag_dqdq = simplify(jacobian(Lag_dq.',dq)); % W
C_q = simplify(jacobian(C,q)).';

% Matrices for problem 1
M = Lag_dqdq;
b = Q - Lag_q - z*C_q;

% Matrices for problem 2
Mimplicit = [Lag_dqdq C_q; C_q.' 0];
c = [Q-Lag_q; -dq.'*C_q];
Mexplicit = simplify(inv(Mimplicit));
rhs = simplify(Mexplicit*c);
```