0.0.1 Question 1.3

M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and corresponding eigenvectors

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

What are the eigenvalues of the matrix $M + 2I$ (where I is the identity matrix)?

Please prove this or hand calculate it, don't use libraries to do it for you numerically.

Points: 0.3

We can find out what will be the eigenvalues by finding out what M is first and then recalculate the eigenvalues and eigenvectors for $M + 2I$.

$$M = P^{-1} \Lambda P$$

Where P is the matrix with the eigenvectors of M and Λ is a diagonal with λ_1, λ_2, ... being the corresponding eigenvalues in the diagonal. Below is P, P^{-1}, Λ, M and $M + 2I$:

$$\begin{aligned}
P &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\
P^{-1} &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\
\Lambda &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\
M = P^{-1} \Lambda P &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\
M &= \begin{bmatrix} \frac{7}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{-2}{5} \end{bmatrix} \\
M + 2I &= \begin{bmatrix} \frac{17}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{8}{5} \end{bmatrix}
\end{aligned}$$

Now we can calculate the new eigenvalues and vectors:

$$
\begin{aligned}
M + 2I &= \begin{bmatrix} \frac{17}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{8}{5} \end{bmatrix} \\
det(M + 2I) &= (\frac{17}{5} - \lambda)(\frac{8}{5} - \lambda) - \frac{36}{25} \\
&= (\lambda - 1)(\lambda - 4) \\
\lambda_1, \lambda_2 &= 4, 1
\end{aligned}
$$

After finding the eigenvalues, we can find the eigenvector by finding the vectors u_i that satisfy the equation $Mu_i = \lambda_i u_i$ for the corresponding eigenvalue λ_i. The eigenvectors we obtain are:

$$
u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
$$

0.0.2 Question 1.4

M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and corresponding eigenvectors

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

What are the eigenvalues of the matrix $M^2 = MM$?

Please prove this or hand calculate it, don't use libraries to do it for you numerically.

Points: 0.3

$$M^2 = MM = \begin{bmatrix} \frac{7}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{-2}{5} \end{bmatrix} \begin{bmatrix} \frac{7}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{-2}{5} \end{bmatrix}$$
$$M^2 = \begin{bmatrix} \frac{17}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{8}{5} \end{bmatrix}$$

Here the $M^2 = M + 2I$ so the M^2 will have the same eigenvalues and eigenvectors of $M + 2I$:

$$\lambda_1 = 4, u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \lambda_2 = 1, u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

0.0.3 Question 1.5

If $\mathbf{v}$ is an eigenvector of the matrix $A^T A$ with eigenvalue λ, show that $A\mathbf{v}$ is an eigenvector of AA^T with the same eigenvalue λ.

Remember to obey the rules for *matrix* multiplication.

BTW, this is a cool linear algebra trick we will use later to make calculations easier when A has a huge number of rows and a reasonable number of columns.

Points: 0.8

Let's say that the eigenvalue of $A^T A$ is λ_1 and the eigenvalue of AA^T is λ_2. Then we can follow this derivation to prove $\lambda_1 = \lambda_2$

$$A^T Av = \lambda_1 vAv = bAA^T(Av) = AA^T b = \lambda_2 b = \lambda_2 AvA(A^T Av) = A\lambda_1 v = \lambda_1 Av = \lambda_1 b\lambda_1 b = \lambda_2 b\lambda_1 = \lambda_2$$

0.0.4 Q3.4 Interpret Eigenvalues and Eigenvectors

Please explain what these eigenvectors represent, and what the associated eigenvalues represent as well.

Points: 0.3

0.1 My Answer

The eigenvectors represent the direction of maximum variance in a certain direction. Since the eigenvectors are orthogonal to eachother, then we can say that the eigenvectors can span the space the dataset lives in. So a dataset full of 2 dimensional vectors will have 2 eigenvectors that are orthogonal to each other and when linearly combine can span the entire 2 dimensional space. These directions are the "principal components" of the clustering that describe the directions of maximum variance, orthogonal to each other. The eigenvalues represent how much the eigenvector explains the data's variance. The higher the eigenvalue, the more the eigenvector represents the data's variance.

0.1.1 Q3.5 PCA Projection and Plot

1. PLOT #1: You will project the mean centered data onto its principal component basis. Make a scatter plot of the projected data. The graph should have a title ("Data projected onto principal components") and axis labels ("PC1", "PC2").

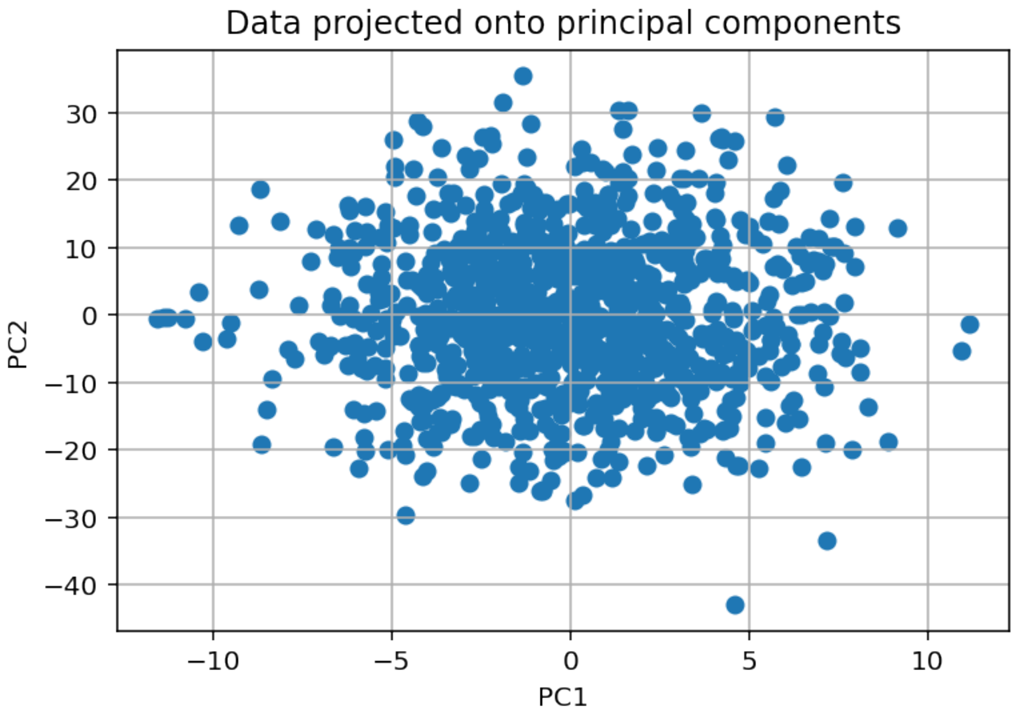
2. PLOT #2: You will visualize the PCs in the data's original coordinates. The graph should have a title ("PC directions in the original data basis") and axis labels ("x", "y"). Lay to following plots on top of each other to form a single graph:

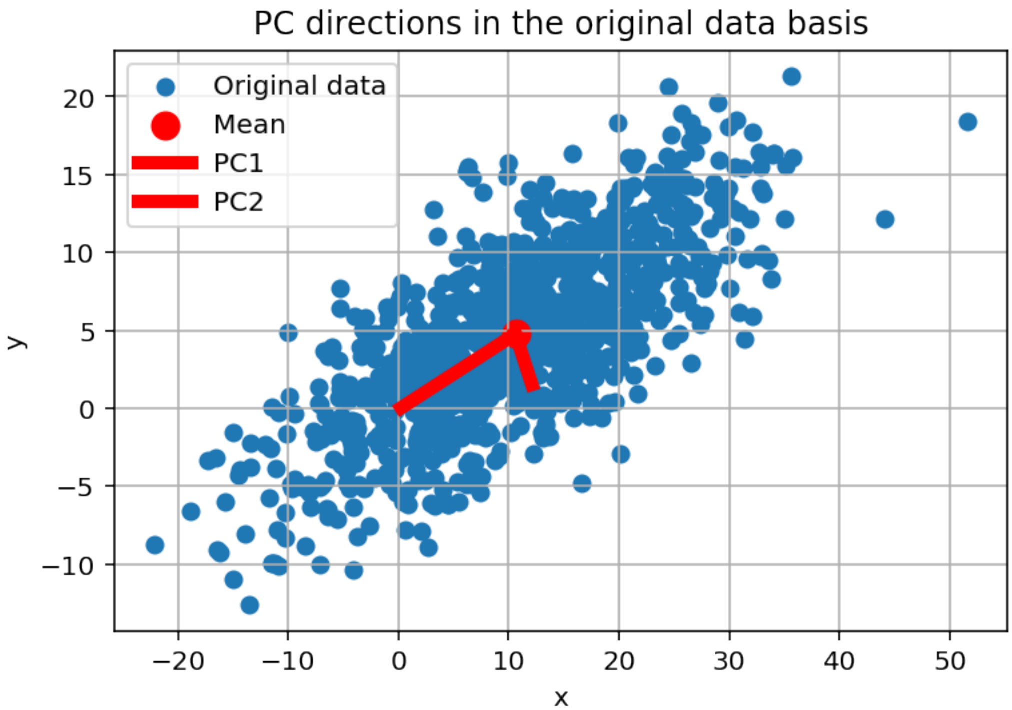
   - Scatter plot the original data
   - Plot a large red dot (use `c='r', s=100`) at the mean of the data
   - using commands of the form `plt.plot([vec_x_start, vec_x_end],[vec_y_start, vec_y_end], c='r', linewidth=5)` plot both PC1 and PC2 in wide, red lines. Note that
     - PCs should start at the mean of the data
     - PCs should be scaled by the square root of their eigenvalues (remember we squared the data to get the covariance matrix)

Points: 0.2

In [24]:
```python
# plot those two graphs here.
# 1: Plot mean centered data on the PC basis
projection = np.dot(eigenvectors, Z).T # (1000, 2)
plt.scatter(projection[:, 0], projection[:, 1])
plt.title('Data projected onto principal components')
plt.xlabel('PC1')
plt.ylabel('PC2')
plt.grid(True)
plt.show()

# 2: Plot PC directions on data's original coordinates
# Plot original data
plt.scatter(X[0, :], X[1, :], label='Original data')

# Plot the mean
plt.scatter(mu.flatten()[0], mu.flatten()[1], c='r', s=100, label='Mean')

# Plotting PC1
pc1_start = mu.T.flatten()
pc1_end = mu.T.flatten() + eigenvectors.T[0] * np.sqrt(eigenvalues[0])  # Scaling by the squar
plt.plot([pc1_start[0], pc1_end[0]], [pc1_start[1], pc1_end[1]], c='r', linewidth=5, label='PC

# Plotting PC2
pc2_start = mu.T.flatten()
pc2_end = mu.T.flatten() + eigenvectors.T[1] * np.sqrt(eigenvalues[1])  # Scaling by the squar
plt.plot([pc2_start[0], pc2_end[0]], [pc2_start[1], pc2_end[1]], c='r', linewidth=5, label='PC

plt.title('PC directions in the original data basis')
plt.xlabel('x')
```

```
plt.ylabel('y')
plt.legend()
plt.grid(True)
plt.show()
```




PC directions in the original data basis
Original data
Mean
PC1
PC2
y
x
20
15
10
5
0
−5
−10
−20
−10
0
10
20
30
40
50

0.1.2 Question 4.6 Non-face Image Projection

Does the reconstructed dog look like the original picture? Explain why the reconstruction looks the way it does.

Points: 0.4

0.2 My Answer

The reconstructed dog doesn't look like the original picture. The reason is because of the eigenvectors and mean. The eigenvectors and mean were based off of another dataset. The eigenvectors and mean were the result of matrix multiplying the faces dataset to get a lower dimension representation of the faces data. When we multiplied the dog by the set of eigenvectors to get the dog's projection, we created a projection of the dog on the direction of variances used to represent a completely different dataset. So the reconstruction was done based on the lower dimension representation of the faces dataset, which is what gave us a frankenstein like picture instead of the original dog.