1 Complexity Zoo

1.1 TIME[f(n)]

Informally: problems that can be solved in f(n) time.

Definition 1.1. Given some function $f : \mathbb{N} \to \mathbb{N}$, TIME$[f(n)]$ are the set of problems solvable within $O(f(n))$ atomic steps on a deterministic Turing machine. Where n is the size of the input.

1.2 NTIME[f(n)]

Informally: problems that can be solved nondeterministically in f(n) time.

Definition 1.2. Given some function $f : \mathbb{N} \to \mathbb{N}$, NTIME$[f(n)]$ are the set of problems solvable within $O(f(n))$ atomic steps on a nondeterministic Turing machine.

1.3 SPACE[f(n)]

Informally: problems that can be solved in f(n) space.

Definition 1.3. Given some function $f : \mathbb{N} \to \mathbb{N}$, SPACE$[f(n)]$ are the set of problems solvable using a tape of length $O(f(n))$ on a deterministic Turing machine. Where n is the size of the input.

1.4 NSPACE[f(n)]

Informally: problems that can be solved non-deterministically in f(n) space.

Definition 1.4. Given some function $f : \mathbb{N} \to \mathbb{N}$, NSPACE$[f(n)]$ are the set of problems solvable using a tape of length $O(f(n))$ on a non-deterministic Turing machine. Where n is the size of the input.

1.5 P

Informally: all problems that can be solved in polynomial time.

Definition 1.5.

$$\mathbf{P} = \bigcup_{k \geq 0} \text{TIME}[n^k]$$

Descriptive Complexity definitions:

Definition 1.6.

$$\mathbf{P} = \text{FO(LFP)}$$

(First Order logic extended with the Least Fixed Point operator, with successor. A high level, handwavy description of the LFP operator is the added ability to recursively define FO formulas.)

Definition 1.7.

$$\mathbf{P} = \text{SO(Horn)}$$

(Second Order logic restricted with Horn. SO logic allows you to quantify over subsets/relations/functions on the domain, and Horn means all 'clauses' are really implications with literal in the conclusion and all literals positive.)

Circuit Complexity definition:

Definition 1.8.

P = Set of problems that can be solved by a polynomial-time uniform family of boolean circuits

Notable Problems in **P**:

- 2-SAT
- 2-Colourability
- Reachability

1.6 NP

Informally: all problems that can be solved in nondeterministic polynomial time.

Definition 1.9.

$$\mathbf{NP} = \bigcup_{k \geq 0} \text{NTIME}[n^k]$$

In terms of a verifier:
Informally: The set of decision problems where a solution can be verified in polynomial time.

Descriptive Complexity Definition:

Definition 1.10.

$$\mathbf{NP} = \text{SO}\exists$$

(Existential Second Order)

Notable Problems in **NP**:

- SAT
- 3-Colourability
- TSP
- Subset sum

1.7 FPT

1.8 W[1]

1.9 FPTAS

1.10 PTAS

1.11 L

1.12 NL

1.13 PSPACE

1.14 coNP

1.15 Σ_2^p

1.16 Σ_i^p

1.17 Π_2^p

1.18 Π_i^p

1.19 PH

1.20 P^{SAT}

1.21 NP^{SAT}

1.22 P/poly

1.23 P-Uniform

1.24 EXP

1.25 NC

1.26 NC_0

1.27 NC_1

1.28 NC_2

1.29 NC_i

1.30 AC_i

1.31 AC_0

1.32 AC_1

1.33 BPP

1.34 RP

1.35 co-RP

1.36 ZPP

1.37 APX

1.38 PO

1.39 PCP