1 Complexity Zoo

1.1 General time and space complexity classes

Central to complexity theory is identifying the 'complexity' of a problem with the amount of time needed to solve it.

Definition 1.1. Given some function $f : \mathbb{N} \to \mathbb{N}$, $\texttt{TIME}[f(n)]$ are the set of problems solvable within $O(f(n))$ atomic steps on a deterministic Turing machine. Here n is the size of the input.

For most conventional time-complexity classes, the big-oh in the definition is superficial.

Definition 1.2. Given some function $f : \mathbb{N} \to \mathbb{N}$, $\texttt{NTIME}[f(n)]$ are the set of problems solvable within $O(f(n))$ atomic steps on a nondeterministic Turing machine.

`SPACE` and `NSPACE` are defined similarly.

1.2 P

Informally: all problems that can be solved in polynomial time.

Definition 1.3.

$$\mathbf{P} = \bigcup_{k \geq 0} \text{TIME}[n^k]$$

Descriptive Complexity definitions:

Definition 1.4.

$$\mathbf{P} = \text{FO(LFP)}$$

(First Order logic extended with the Least Fixed Point operator, with successor. A high level, handwavy description of the LFP operator is the added ability to recursively define FO formulas.)

Definition 1.5.

$$\mathbf{P} = \text{SO(Horn)}$$

(Second Order logic restricted with Horn. SO logic allows you to quantify over subsets/relations/functions on the domain, and Horn means all 'clauses' are really implications with literal in the conclusion and all literals positive.)

Circuit Complexity definition:

Definition 1.6.

$\mathbf{P}$ = Set of problems that can be solved by a polynomial-time uniform family of boolean circuits

Notable Problems in **P**:

- 2-SAT
- 2-Colourability
- Reachability

1.3 NP

Informally: all problems that can be solved in nondeterministic polynomial time.

Definition 1.7.

$$\mathbf{NP} = \bigcup_{k \geq 0} \text{NTIME}[n^k]$$

In terms of a verifier:
Informally: The set of decision problems where a solution can be verified in polynomial time.

Descriptive Complexity Definition:

Definition 1.8.

$$\mathbf{NP} = \text{SO}\exists$$

(Existential Second Order)

Notable Problems in **NP**:

- SAT
- 3-Colourability
- TSP
- Subset sum

1.4 FPT

1.5 W[1]

1.6 FPTAS

1.7 PTAS

1.8 L

1.9 NL

1.10 PSPACE

1.11 coNP

1.12 Σ_2^p

1.13 Σ_i^p

1.14 Π_2^p

1.15 Π_i^p

1.16 PH

1.17 P^{SAT}

1.18 NP^{SAT}

1.19 P/poly

1.20 P-Uniform

1.21 EXP

1.22 NC

1.23 NC_0

1.24 NC_1

1.25 NC_2

1.26 NC_i

1.27 AC_i

1.28 AC_0

1.29 AC_1

1.30 BPP

1.31 RP

1.32 co-RP

1.33 ZPP

1.34 APX

1.35 PO

1.36 PCP