1 Complexity Zoo

1.1 TIME[f(n)]

Informally: problems that can be solved in f(n) time.

Definition 1.1. Given some function $f : \mathbb{N} \to \mathbb{N}$, TIME[$f(n)$] are the set of problems solvable within $O(f(n))$ atomic steps on a deterministic Turing machine. Where n is the size of the input.

1.2 NTIME[f(n)]

Informally: problems that can be solved nondeterministically in f(n) time.

Definition 1.2. Given some function $f : \mathbb{N} \to \mathbb{N}$, NTIME[$f(n)$] are the set of problems solvable within $O(f(n))$ atomic steps on a nondeterministic Turing machine.

1.3 SPACE[f(n)]

Informally: problems that can be solved in f(n) space.

Definition 1.3. Given some function $f : \mathbb{N} \to \mathbb{N}$, SPACE[$f(n)$] are the set of problems solvable using a tape of length $O(f(n))$ on a deterministic Turing machine. Where n is the size of the input.

1.4 NSPACE[f(n)]

Informally: problems that can be solved non-deterministically in f(n) space.

Definition 1.4. Given some function $f : \mathbb{N} \to \mathbb{N}$, NSPACE[$f(n)$] are the set of problems solvable using a tape of length $O(f(n))$ on a non-deterministic Turing machine. Where n is the size of the input.

1.5 P

Informally: all problems that can be solved in polynomial time.

Definition 1.5.

$$\mathbf{P} = \bigcup_{k \geq 0} \mathrm{TIME}[n^k]$$

Descriptive Complexity definitions:

Definition 1.6.

$$\mathbf{P} = \mathrm{FO(LFP)}$$

(First Order logic extended with the Least Fixed Point operator, with successor. A high level, handwavy description of the LFP operator is the added ability to recursively define FO formulas.)

Definition 1.7.

$$\mathbf{P} = \mathrm{SO(Horn)}$$

(Second Order logic restricted with Horn. SO logic allows you to quantify over subsets/relations/functions on the domain, and Horn means all 'clauses' are really implications with literal in the conclusion and all literals positive.)

Circuit Complexity definition:

Definition 1.8.

P = Set of problems that can be solved by a polynomial-time uniform family of boolean circuits

Notable Problems in **P**:

- 2-SAT
- 2-Colourability
- Reachability

1.6 NP

Informally: all problems that can be solved in nondeterministic polynomial time.

Definition 1.9.

$$\mathbf{NP} = \bigcup_{k \geq 0} \mathrm{NTIME}[n^k]$$

Turing Machine definition:

Definition 1.10.

$$x \in \mathbf{NP} \iff \exists w : \|w\| \leq p(\|x\|) \text{s.t.} M(x, w) = 1$$

In terms of a verifier:
Informally: The set of decision problems where a solution can be verified in polynomial time.

Descriptive Complexity Definition:

Definition 1.11.

$$\mathbf{NP} = \mathrm{SO}\exists$$

(Existential Second Order)

Notable Problems in **NP**:

- SAT
- 3-Colourability
- TSP
- Subset sum

1.7 coNP

Turing Machine definition:

Definition 1.12.

$$x \in \mathbf{coNP} \iff \forall w : \|w\| \leq p(\|x\|)\text{s.t.}M(x,w) = 1$$

In terms of a verifier:
Informally: The set of decision problems where a solution can be refuted in polynomial time.

1.8 FPT

Informally, the set of problems that can be solved in polynomial time for some fixed parameter.

Definition 1.13. The set of problems that can be parameterised by k and can be solved in $f(k)n^c$, where $f(x)$ is only dependent on k, and c is an independent constant.

P is contained within **FPT**.
If a problem is in **FPT**, then for any fixed k that problem is in **P**.
FPT is also known as $\mathbf{W}[\mathbf{0}]$

Notable Problems in **FPT**:

- Vertex Cover

1.9 W[1]

Definition 1.14. The class of parametrized problems that admit a parametrized reduction to the following problem: Given a nondeterministic single-tape Turing machine, decide if it accepts within k steps.

N.B This is short acceptance

Definition 1.15. The class of parametrized problems that admit a parametrized reduction to the following problem: Given a Boolean circuit C, with a mixture of fanin-2 and unbounded-fanin gates. There is at most 1 unbounded-fanin gate along any path to the root, and the total depth (fanin-2 and unbounded-fanin) is constant. Does C have a satisfying assignment of Hamming weight k?

N.B This is Weighted 3-SAT.

Notable Problems in $\mathbf{W[1]}$:

- Short Acceptance
- Weighted 3-SAT
- Clique (of size k)
- Independent set (of size k)

1.10 W[2]

1.11 W[i]

1.12 FPTAS

1.13 PTAS

1.14 L

Informally: all problems that can be solved using logarithmic space (excluding the input)

Definition 1.16.

$$\mathbf{L} = \text{SPACE}[\log n]$$

This means you effectively have the input and then a fixed number of counters/pointers (up to the size of the input)

Notable Problems in **L**:

- Planar Graph Isomorphism

1.15 NL

Informally: all problems that can be solved using nondeterministic logarithmic space (excluding the input)

Definition 1.17.

$$\mathbf{NL} = \mathrm{NSPACE}[\log n]$$

This means you effectively have the input and then a fixed number of counters/pointers (up to the size of the input)

Definition 1.18.

$$\mathbf{NL} = \mathrm{SO}(\mathrm{Krom})$$

Definition 1.19.

$$\mathbf{NL} = \mathbf{coNL}$$

Notable Problems in **NL**:

- Reachability
- Unreachability

1.16 PSPACE

1.17 Σ_2^p

Definition 1.20.

$$\Sigma_2^P = \mathbf{NP}^{\mathbf{NP}}$$

Turing Machine definition:

Definition 1.21.

$$x \in \Sigma_2^P \iff \exists w : \|w\| \leq p(\|x\|) \forall u : \|u\| \leq p(\|x\|) \text{s.t.} M(x, w, u) = 1$$

1.18 Σ_i^p

Definition 1.22.

$$\Sigma_i^P = \mathbf{NP}^{\Sigma_{i-1}^P}$$

Turing Machine definition:

Definition 1.23.

$$x \in \Sigma_i^P \iff \exists u_1 \forall u_2 ... Q_i u_i M(x, u_1, ..., u_i) = 1$$

$|u_j| \leq p(x)$ and $Q_i = \forall/\exists$ if i is even/odd.

Definition 1.24.

$$\Sigma_i^p = co - \Pi_i^p$$

1.19 Π_2^p

Definition 1.25.

$$\Pi_2^P = \mathbf{coNP^{NP}}$$

Turing Machine definition:

Definition 1.26.

$$x \in \Pi_2^P \iff \forall w : \|w\| \leq p(\|x\|) \exists u : \|u\| \leq p(\|x\|) \text{s.t.} M(x, w, u) = 1$$

1.20 Π_i^p

Definition 1.27.

$$\Pi_i^P = \mathbf{coNP}^{\Sigma_{i-1}^P}$$

Turing Machine definition:

Definition 1.28.

$$x \in \Pi_i^P \iff \forall u_1 \exists u_2 ... Q_i u_i M(x, u_1, ..., u_i) = 1$$

$|u_j| \leq p(x)$ and $Q_i = \exists/\forall$ if i is even/odd.

Definition 1.29.

$$\Pi_i^p = co - \Sigma_i^p$$

1.21 PH

Definition 1.30.

$$\mathbf{PH} = \bigcup_i \Sigma_i^p$$

1.22 P^{SAT}

1.23 NP^{SAT}

1.24 P/poly

1.25 P-Uniform

1.26 EXP

1.27 NC

1.28 NC_0

1.29 NC_1

1.30 NC_2

1.31 NC_i

1.32 AC_i

1.33 AC_0

1.34 AC_1

1.35 BPP

1.36 RP

1.37 co-RP

1.38 ZPP

1.39 APX

1.40 PO

1.41 PCP

1.42 BQP

1.43 $\#P$

1.44 PPAD