Question 1

Given Set-up

$$Y_i \sim N(\mu, \tau^2)$$
$$\mu_i = f(x_i) = \alpha - \beta \gamma^{x_i}$$

where

$$\alpha \sim N(0, \sigma_a^2) \qquad \alpha \in (1, \infty)$$
$$\beta \sim N(0, \sigma_\beta^2) \qquad \beta \in (1, \infty)$$
$$\gamma \sim \cancel{N} \text{ Unif}(0,1) \qquad \gamma \in (0, 1)$$
$$\tau^2 \sim IG(a, b) \qquad \tau^2 \in (0, \infty)$$

ⓐ Characteristics

- Since the hyper parmaters α, β, τ^2 are from Normal Distribution, and our statistical model is also Normal. By Conjugate Properties, posterior of α, β, τ^2 must also be Normal

- As for γ, it can't follow those properties of Conjugacy and can't thus be represented in any of the distribution of exponential family..

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(b) Corresponding Likelihood function

Since $Y_i \sim N(\mu_i, \tau^2)$

$$P(y_1) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(y_1-\mu_1)^2}{2\tau^2}\right)$$

$\vdots$

$$P(y_n) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(y_n-\mu_n)^2}{2\tau^2}\right)$$

$$\Rightarrow P(y_1, y_2, \dots y_n) = P(y_1) \times P(y_2) \times \dots P(y_n)$$

$$\Rightarrow P(y_1, y_2, \dots y_n) = \frac{1}{\sqrt{2\pi\tau^2}} \times \frac{1}{\sqrt{2\pi\tau_2}} \times \dots \frac{1}{\sqrt{2\pi\tau^2}} \times \exp\left(\frac{-1}{2\tau^2}\left([y_1-\mu_1]^2 + [y_2-\mu_2]^2 + \dots + [y_n-\mu_n]^2\right)\right)$$

$$\Rightarrow P(y_1, \dots y_n) = \left(\frac{1}{\sqrt{2\pi\tau^2}}\right)^n \times \exp\left(\frac{-1}{2\tau^2} \sum_{i=1}^{n} (y_i - \mu_i)^2\right)$$

Then

$$L(y_1, \dots y_n \mid \alpha, \beta, \gamma, x, y, \tau^2) = \frac{1}{\sqrt{(2\pi\tau^2)}^n} \times \exp\left(\frac{-\sum_{i=1}^{n} (y_i - (\alpha-\beta\gamma^{x_i}))^2}{2\tau^2}\right)$$

$$= \frac{1}{(\sqrt{2\pi\tau^2})^n} \exp\left(-\frac{\sum_{i=1}^{n} (y_i-(\alpha-\beta\gamma^{x_i}))^2}{2\tau^2}\right)$$

Ans

(c) Expression of joint prior distribution of Parameters and choose hyperparameters

$$P(\alpha, \beta, \gamma, \tau^2) = P(\alpha)\, P(\beta)\, P(\gamma)\, P(\tau^2)$$

But

$$P(\alpha) = \frac{1}{\sqrt{2\pi\Omega_a^2}} \exp\left(-\frac{\alpha^2}{2\Omega_a^2}\right)$$

$$P(\beta) = \frac{1}{\sqrt{2\pi\Omega_b^2}} \exp\left(\frac{-\beta^2}{2\Omega_\beta^2}\right)$$

$$P(\gamma) = 1$$

$$P(\tau^2) = (\tau^2)^{-a-1} \exp\left(\frac{-b}{\tau^2}\right)$$

$$\Rightarrow P(\alpha, \beta, \gamma, \tau^2) = \frac{1}{\sqrt{2\pi\Omega_a^2}} \frac{1}{\sqrt{2\pi\Omega_b^2}} \exp\left(\frac{-\alpha^2}{2\Omega_a^2} - \frac{\beta^2}{2\Omega_b^2} - \frac{b}{\tau^2}\right) (\tau^2)^{-a-1}$$

Ans

The only concern for the choice of Ω_a^2, Ω_b^2 is that distribution should have enough variance in it so that

Sampling is fair and we have a lot of options to choose from.

Furthermor we always have to make sure that

$\alpha \in (1, \infty)$ $\quad \gamma \in (0,1)$

$\beta \in (1, \infty)$ $\quad \tau^2 \in (0,1)$

As for a, b in Inverse Gamma, a and b both should be positive...

Some options for distribution of $\alpha, \beta, \gamma, \tau^2$ ~~ea~~ alongside their graphs are given in the code ---

$\therefore$ See the Code

(d)

See the Code

Q2

(a) Analytic Expression of acceptance Probability

$\Rightarrow$ We have to simulate $\theta \sim \Pi_\theta(\theta)$ where $\Pi_\theta(\theta)$ satisfies this boundness condition

$$\Pi_\theta(\theta) \le K = K\,\Pi_y(\theta)$$

For each $Y \sim \Pi_y(.)$ we use an auxiliary experiment

$$y \rightarrow \begin{cases} y^A & \text{if } E = A \\ y^R & \text{if } E = R \end{cases}$$

The distribution of this auxiliary experiment is Bernoulli such that

$$E = A = 1 \quad \text{and} \quad E = R = 0$$

so

$$P_r(E=1 \mid Y=y) = \frac{\Pi_\theta(y)}{K\,\Pi_y(y)} = \frac{\Pi_\theta(y)}{K} \in [0,1]$$

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But the distribution is in fact, a conditional distribution

$$\pi_\theta(\theta) = Pr(\theta \le \theta) = Pr\{Y \le \theta \mid Y = X^A\}$$

$$= Pr\{Y \le \theta \mid E = 1\}$$

To get unconditional distribution

$$Pr\{E=1\} = \int_{(0,1)} Pr\{E=1 \mid Y=y\}\, \pi_Y(y)\, dy$$

$$= \int_{0,1} \frac{\pi_\theta(y)}{K\pi_Y(y)}\, \pi_Y(y)\, dy$$

$$= \frac{1}{K} \int_{0,1} \pi_\theta(y)\, dy$$

$$= \boxed{1/K}$$

Ans

(b) Prove

We have to make sure that our target distribution $\Pi_\theta(.)\ \theta \equiv Y^A$

So

$$\Pi_\theta(\theta) = P_r\{\theta \le \theta\}$$

$$= P_r\{Y \le \theta \mid E=1\}$$

$$= \frac{P_r\{Y \le \theta, E=1\}}{P_r\{E=1\}}$$

$$= \frac{P_r\{Y \varepsilon [0,\theta], E=1\}}{P_r\{E=1, Y \varepsilon [0,1]\}}$$

$$= \frac{\int_0^1 \mathbb{1}_{(0,\theta)}(y) P_r\{E=1 / Y=y\}\ \Pi_Y(y)\,dy}{\int_0^1 P_r\{E=1 / Y=y\}\ \Pi_Y(y)\,dy}$$

$$= \frac{\int_0^\theta \frac{\Pi_\theta(y)}{K\Pi_Y(y)}\ \Pi_Y(y)\,dy}{\int_0^1 \frac{\Pi_\theta(y)}{K\Pi_Y(y)}\ \Pi_y(y)\,dy} = \frac{1/K \int_0^\theta \Pi_\theta(y)\,dy}{1/K \int_0^1 \Pi_\theta(y)\,dy}$$

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$$= \int_0^\theta \pi_\theta(y)\,dy$$

$$\Rightarrow \quad \theta \sim \pi_\theta(\cdot)$$

So θ is from target distribution....

(c)

We need to understand two Cases:

Case 1: We don't know normalizing factor of Posterior

Case 2: We Konw normalizing factor for Posterior

I'd be discussing Case 2 ...

so let

$$f(\theta) = \frac{f(y_1 \ldots y_n / \theta)\, P(\theta)}{f(y_1 \ldots y_n)}$$

$$= K f(y_1 \ldots y_n / \theta)\, g(\theta)$$

$$\Rightarrow K \bar{f}(\theta)$$

where $f'(\theta) = f(y_1 \ldots y_n | \theta)\, g(\theta)$

Let C be a constant which satisfies the boundness condition

Posterior $\leq$ C * Prior

Now, Any rv drawn from

Unif [0, 1] must be scaled to the level $C * g(\theta)$

This rv would be accepted if boundness condition is met

So

$\Rightarrow U * C g(\theta) \leq$ target

Then, We might have two cases

$U * C g(\theta) < K \bar{f}(\theta)$

when complete Posterior is known

Case1

$U * C g(\theta) < K \bar{f}(\theta)$

when complete Posterior isn't known

Case2

Case1 will be discussed here

Let $X \sim g(\theta)$

A rv

Prior

$$P(Y \leq y) = P(X \leq y \mid U * c\, g(\theta) \leq K \bar{f}(\theta))$$

$$= \frac{P(x \leq y,\ U * c\, g(\theta) \leq K \bar{f}(\theta))}{P\left(U \leq \dfrac{K \bar{f}(\theta)}{c\, g(\theta)}\right)}$$

$$= \frac{\displaystyle\int_{-\infty}^{y} \int_{0}^{K\bar{f}(\theta)/c\,g(\theta)} 1 \, du \; g(\theta)\, d\theta}{\displaystyle\int_{-\infty}^{\infty} \int_{0}^{K\bar{f}(\theta)/c\,g(\theta)} 1 \, du \; g(\theta)\, d\theta}$$

$$= \int_{-\infty}^{y} u \Big|_{0}^{K\bar{f}(\theta)/c\,g(\theta)} g(\theta)\, d\theta \Big/ \int_{-\infty}^{\infty} u \Big|_{0}^{K\bar{f}(\theta)/c\,g(\theta)} g(\theta)\, d\theta$$

$$= \int_{-\infty}^{y} \frac{K \bar{f}(\theta)}{c\, g(\theta)}\, g(\theta)\, d\theta \Big/ \int_{-\infty}^{\infty} \frac{K \bar{f}(\theta)}{c\, g(\theta)}\, g(\theta)\, d\theta$$

$$= \int_{-\infty}^{y} \frac{K}{c} \bar{f}(\theta)\, d\theta \Big/ \int_{-\infty}^{\infty} \frac{K}{c} \bar{f}(\theta)\, d\theta$$

$$\leq \int_{-\infty}^{y} \bar{f}(\theta)\, d\theta$$

(K get cancelled so)

So we can say the generated rv will have a distribution of $\bar{f}(\theta)$

$\therefore$ We don't actually need to Know the proportionality constant, as shown in the derivation, it gets cancelled ----.

(d)

Imagine the following likelihood and prior

$$\text{Prior}(\theta) = \Pi(\theta) \sim \text{dbeta}(a, b) \ \{a \neq 0, b \neq 1\}$$

$$P(x_1 \dots x_n \mid \theta) \sim \text{Binomial}(n, \theta)$$

Then

$$\text{Posterior } P(\theta \mid x_1 \dots x_n) = \frac{P(x_1 \dots x_n \mid \theta)\, P(\theta)}{P(x_1 \dots x_n)}$$

$$= \text{dbeta}(\theta_1, a + \Sigma x_i, b + n - \Sigma x_i)$$

$\therefore$ from properties of Conjugate model

Difficulties:

* It is not possible to draw from Prior Predictive... The reason for that is

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Sampling function doesn't exist for beta density (analytically)...

Because the integral over density of beta is not possible analytically

Prior

* The tail of $g(\theta)$ has to cove the tails of $f'(\theta)$ which is not always possible

(e)

$\therefore$ See the Code

Question 3

Standard Cauchy

$$f(x;0,1) = \frac{1}{\pi(1+x^2)}$$

Standard Normal

$$f(x;0,1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{x^2}{2}\right)\right)$$

Acceptance - Probability =
$$\frac{1/\sqrt{2\pi}\ \exp\left(-\left(\frac{x^2}{2}\right)\right)}{K\left(\frac{1}{\pi(1+x^2)}\right)}$$

Acceptance Prob = $Pr(E=1 \mid X=x)$ =
$$\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \div \frac{K}{\frac{1}{\pi(1+x^2)}}$$

∴ Also, See the code

Finally

$\Rightarrow$ Acceptance - Prob = $1/K$

for $K = 2$

Accep- Prob = $1/2$

Question 4

ⓐ See the code for Simulation ...

ⓑ Empirical relative frequency achieved are:

State1	State2	state 3
0.390	0.345	0.266

ⓒ Relative frequency of final states

State1	State2	State 3
0.374	0.352	0.274

Since The overall distribution is Stationary or time invariant, It does not matter which 't' are we considering...

So the distribution that we are approximating is our target distribution or Stationary distribution π

as

$X_1, X_2, X_3 \sim \pi \Rightarrow X_{t+1}, X_{t+2}, X_{t+3} \sim \pi$

so final_state 1, final_state 2, final_state 3 $\sim \pi$

(d) We have a transition Probability Matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Where $\Pi = \{\Pi_1, \Pi_2, \Pi_3\}^T$ $S = \{1, 2, 3\}$ be state space 'S'

and the stationary distribution Π must satisfy the equations

$$\Pi_1 P_{11} + \Pi_2 P_{21} + \Pi_3 P_{31} = \Pi_1$$

$$\Pi_1 P_{12} + \Pi_2 P_{22} + \Pi_3 P_{32} = \Pi_2$$

$$\Pi_1 P_{13} + \Pi_2 P_{23} + \Pi_3 P_{33} = \Pi_3$$

which is equiv to

$$P^T \Pi = P$$

$$\Rightarrow \quad (P^T - \lambda I)\Pi = 0$$

corresponding to $\lambda = 1$, So Π must be in eigenspace corresponding to eigenvalue $\boxed{\lambda = 1}$

$\therefore$ See the code for actual implementation/calculation

(e) Yes, π is well approximated frequencies in part b and c.

∴ See the code

(f) The starting position does not matter if we choose $X_0=2$

If we repeat part a, b, c, e this time by $X_0=2$, we again get the same results as before.

It is quite intuitive following the properties of finite state Markov chain

∴ See the code

Q5...

$$y_i \sim N(\mu_i, \tau^2)$$

$$L_y(\alpha, \beta, \gamma, \tau^2) = \frac{1}{(2\pi\tau^2)^n} \exp\left(-\frac{\sum_{i=1}^{n}(y_i - \mu_i)^2}{2\tau^2}\right)$$

$$\text{Prior}(\alpha, \beta, \gamma, \tau^2) = \exp\left(\frac{-\alpha^2}{2\Omega_\alpha^2}\right) \exp\left(\frac{-\beta^2}{2\Omega_\beta^2}\right) (\tau^2)^{-a-1}$$

$$\text{Exp}\left(\frac{-b}{\tau^2}\right)$$

$$\text{Joint} = L_y(\alpha, \beta, \gamma, \tau^2, x_i) \times \text{Prior}(\alpha, \beta, \gamma, \tau^2)$$

a) conditional for α

$$\pi(\alpha \mid \beta, \gamma, \tau^2, y_i) = \frac{\pi(\alpha, \beta, \gamma, \tau^2 \mid y_i)}{\pi(\gamma, \beta, \tau^2 \mid y_i)}$$

$$= \exp\left(-\sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{2\tau^2} - \frac{\alpha^2}{2\Omega_\alpha^2}\right)$$

$$= \exp\left(-\frac{(\alpha^2 n + \sum_{i=1}^{n}(y_i + \beta\gamma^{x_i})^2 - 2\alpha\sum_{i=1}^{n}(y_i + \beta\gamma^{x_i})}{2\tau^2} - \frac{\alpha^2}{2\Omega^2}\right)$$

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$$\doteq \exp\left(- \frac{[\sigma_a^2 \alpha^2 n - 2\alpha\sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i}) + \sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i})^2 + \tau^2\alpha^2}{2\tau^2\sigma_a^2} \right)$$

$$\doteq \exp\left(- \frac{[\alpha^2(\sigma_a^2 n + \tau^2) - 2\alpha\sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i}) + \sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i})^2]}{2\tau^2\sigma_a^2} \right)$$

$$= \exp\left(- \frac{[(\alpha - (\sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i})))^2}{\frac{2\tau^2\sigma_a^2}{\sigma_a^2 n + \tau^2}} \right)$$

$$\sim N\left(\frac{\sigma_a^2 \sum_{i=1}^{n} (y_i + \beta\gamma^{x_i})}{\tau^2 + n\sigma_a^2}, \frac{\tau^2\sigma_a^2}{\sigma_a^2 n + \tau^2} \right)$$

(b) Conditional for β

$$\pi(\beta|\alpha,\gamma,\tau^2,y_i) = L_\alpha(\alpha,\beta,\gamma,\tau^2) \times \text{Prior}(\beta)$$

$$= \exp\left(\frac{\sum_{i=1}^{n}(y_i-\mu_i)^2}{2\tau^2} - \frac{\beta^2}{2\Omega_b^2} \right)$$

$$\sum_{i=1}^{n}(y_i-\mu_i)^2 = \beta^2 + \frac{\sum_{i=1}^{n}(y_i-\alpha)^2}{\sum_{i=1}^{n}\gamma^{2x_i}} + 2b\sum_{i=1}^{n}(y_i-\alpha)\gamma^{-n_i}$$

put the values of $\sum_{i=1}^{n}(y_i-\mu_i)^2$

$$= \exp\left(\frac{-[B^2(\Omega_\beta^2+\tau^2) + 2b\Omega_B^2\sum_{i=1}^{n}(y_i-\alpha)\gamma^{-n_i} + \Omega_b^2\sum_{i=1}^{n}\frac{(y_i-\alpha)^2}{\sum_{i=1}^{n}\gamma^{2x_i}}}{2\tau^2\Omega_b^2} \right)$$

$$= \exp\left(\frac{-\left[B - \frac{\Omega_B^2(\sum_{i=1}^{n}(\alpha-y_i)\gamma^{-n_i})}{\tau^2+\Omega_b^2}\right]^2}{\frac{2\tau^2\Omega_b^2}{\tau^2+\Omega_b^2}} \right)$$

$$\sim N\left(\frac{\Omega_B^2\sum_{i=1}^{n}(\alpha-y_i)\gamma^{-n_i}}{\Omega_\beta^2+\tau^2}, \frac{\tau^2\Omega_B^2}{\tau^2+\Omega_B^2} \right)$$

c Conditional of γ

$$\pi(\gamma \mid \alpha, \beta, \tau^2) = L_x(\alpha, \beta, \gamma, \tau^2) \times p(\gamma)$$

$$= \exp\left(\frac{-\left[\beta^2 \sum_{i=1}^{n} \gamma^{2x_i} + 2\beta \sum_{i=1}^{n} \gamma^{x_i}(y_i-\alpha) + \sum_{i=1}^{n}(y_i-\alpha)^2\right]}{2\tau^2} \right)$$

$$= \exp\left(\frac{-\left[\sum_{i=1}^{n} (\gamma^{x_i})^2 \beta^2 + 2\sum_{i=1}^{n} \gamma^{x_i}(y_i-\alpha)\beta + \sum_{i=1}^{n}(y_i-\alpha)^2\right]}{2\tau^2} \right)$$

$\therefore$ We can't complet the square here, So we can't write it in Normal form, or any other exponential distribution

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d. Conditional of τ^2

$$\pi(\tau^2 \mid \alpha, \beta, \gamma) = L_y(\alpha, \beta, \gamma, y) \propto p(\tau^2)$$

$$= \frac{1}{(\sqrt{2\pi\tau^2})^n} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{\tau^2}\right) \times (\tau^2)^{-a-1} \times \exp(-b/\tau^2)$$

$$= (\tau^2)^{-a-n-1} \exp\left(-\frac{(\sum_{i=1}^{n} (y_i - \mu_i)^2 + 2b)}{2\tau^2}\right)$$

$$= (\tau^2)^{-a-n-1} \exp\left(-\frac{(\sum_{i=1}^{n} (y_i - \mu_i)^2 + 2b)/2}{\tau^2}\right)$$

$$\approx IG\left(a+n, \frac{2b + \sum_{i=1}^{n} (y_i - \mu_i)^2}{2}\right)$$