ECE 5984-Introduction to Modern Optical Microscopy

Homework-II

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In this report, the Fresnel and Fraunhofer diffraction integrals are implemented numerically in Matlab. To illustrate the diffraction phenomenon, a circular aperture is used.

DIFFRACTION

Diffraction is a general characteristic of wave phenomenon occurring whenever a portion of a wavefront, be it sound, a matter wave (or) light is obstructed in some way. The various segments of the wavefront that propagate beyond the obstacle interfere, causing the energy density distribution referred as the diffraction pattern.

As per the Huygens principle, the aperture emits spherical waves. After a considerable distance of propagation, the spherical waves can be approximated as planar waves and Fraunhofer diffraction obtains. On the other hand, if the observation screen is too close to the aperture, the planar approximation cannot be made and a Fresnel diffraction pattern would exist.

The Fresnel number (F) can be used to identify the type of diffraction that would occur for a given setup. The Fresnel number is given by,

$$F = \frac{w^2}{\lambda z} \tag{1}$$

Where 'w' is the characteristic size (e.g. radius) of the aperture, λ is the wavelength and 'z' is the distance between the screen and the aperture. As a rule of thumb, Fresnel diffraction will occur if F is greater than or equal to one. On the other hand, for 'F' much smaller than one, Fraunhofer approximation can be used.

Apart from the distance between the source and observation plane, the diffraction pattern also depends on the wavelength, geometry and dimension of the aperture, shape of the incident wavefront and the number of samples used.

FRESNEL DIFFRACTION

To begin with, the diffraction geometry is as shown in the following figure,

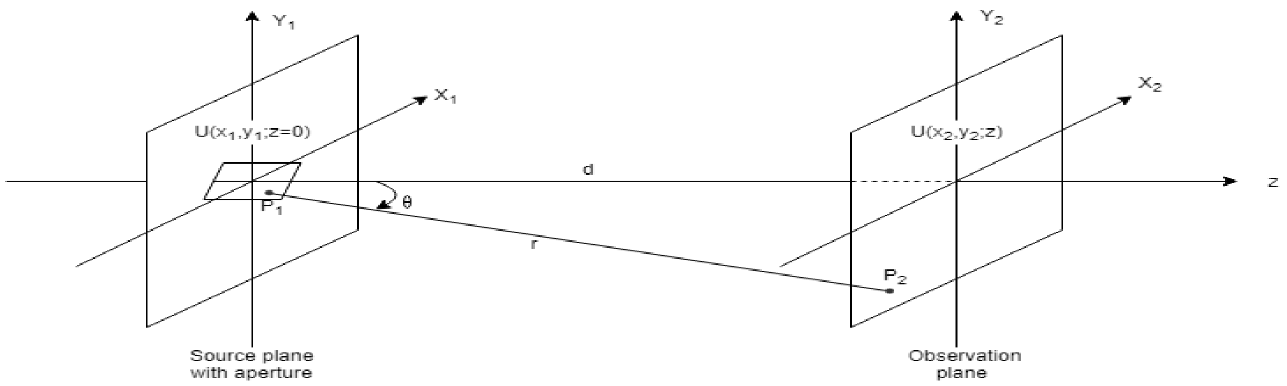

Figure 1: Diffraction geometry.

The source plane co-ordinates are represented as $r_1 = (x_1, y_1)$ and the observation plane co-ordinates are represented as $r_2 = (x_2, y_2)$. The aperture placed at $z = 0$ plane is uniformly illuminated by plane waves in the positive z direction. The source plane with aperture has N sample points and δ_1 is the grid spacing. The spacing in the frequency domain is $\delta_f = 1/(N\delta_1)$. The optical field in the observation plane, parallel to and at a distance z from the source plane is determined using the diffraction integral.

The Fourier transform representation of Fresnel Diffraction integral is,

$$U(x_2,y_2) = \frac{exp(jkz)}{j\lambda z} exp\left[\frac{jk}{2z}(x_2^2+y_2^2)\right] \iint_{-\infty}^{\infty} U(x_1,y_1) exp\left[\frac{jk}{2z}(x_1^2+y_1^2)\right] exp\left[-j2\pi(\frac{x_2}{\lambda z}x_1 + \frac{y_2}{\lambda z}y_1)\right] dx_1\, dy_1 \qquad (2)$$

The above equation is a two dimensional Fourier transform of the expression $U(x_1,y_1)exp\left[\frac{jk}{2z}(x_1^2+y_1^2)\right]$ evaluated at spatial frequencies given by,

$$f_x = \frac{x_2}{\lambda z} \qquad f_y = \frac{y_2}{\lambda z}$$

In the Fourier transform representation, the observation plane size is dependent on the wavelength, distance of propagation and the spatial frequency spectrum as shown in the above equations. Thus, the observation plane size is determined by

$$x_2 = f_x \lambda z \qquad y_2 = f_y \lambda z$$

Thus, for a source plane with N sample points and grid spacing δ_1, illuminated by a wavelength λ, the observation plane at a distance 'z' has a grid spacing given by,

$$\delta_2 = \frac{\lambda z}{N\delta_1}$$

It can be observed that the observation plane grid spacing δ_2 increases with increase in propagation distance z. Thus, at larger propagation distance, the observation plane will be larger in size compared to the source plane.

FRAUNHOFER DIFFRACTION

It was seen in Eq.(2) that, in the region of Fresnel diffraction, the observed field strength U(x, y) can be found from a Fourier transform of the product of the aperture distribution U(x_1,y_1) and a quadratic phase function $exp\big[\frac{ik}{2z}(x_1^2+y_1^2)\big]$. If in addition to the Fresnel approximation the stronger (Fraunhofer) approximation

$$z > \frac{k(x_1^2+y_1^2)_{max}}{2}$$

is satisfied, then the quadratic phase factor is approximately unity over the entire aperture, and the observed field strength can be found directly from a Fourier transform of the aperture distribution itself. Thus in the region of Fraunhofer diffraction (or equivalently, in the farfield),

$$U(x_2,y_2) = \frac{exp(jkz)}{j\lambda z} exp\left[\frac{jk}{2z}(x_2^2+y_2^2)\right] \iint\limits_{-\infty}^{\infty} U(x_1,y_1) exp\left[-j2\pi\left(\frac{x_2}{\lambda z}x_1 + \frac{y_2}{\lambda z}y_1\right)\right] dx_1\, dy_1 \tag{3}$$

The observation plane grid spacing is determined using the same principle explained in the previous section.

SIMULATION RESULTS

In this section, Fresnel and Fraunhofer diffraction patterns for a circular aperture are simulated and analyzed. The specifications used for the numerical simulation are as follows,

Radius of aperture, a	1 mm
Number of sample points, N	512
Dimension of source plane	10 mm $\times$ 10 mm
Source plane grid spacing, δ_1	0.1953125 μm
Illumination wavelength λ	1 μm
Propagation distance for F=10	0.1 m
Propagation distance for F=4	0.25 m
Propagation distance for F=1.0	1 m
Propagation distance for F=0.01	100 m

Figure 2: Simulation parameters for the computer model

In the question at hand, the cross-sectional intensity distribution of Fresnel diffraction at four distances corresponding to F= 10, 4, 1.0,and 0.01 is to be determined. To compute the distance of propagation corresponding to a Fresnel number, equation (1) is used.

$$z = \frac{a^2}{\lambda F} = \frac{10^{-6}}{10^{-6}\times F} = \frac{1}{F}$$

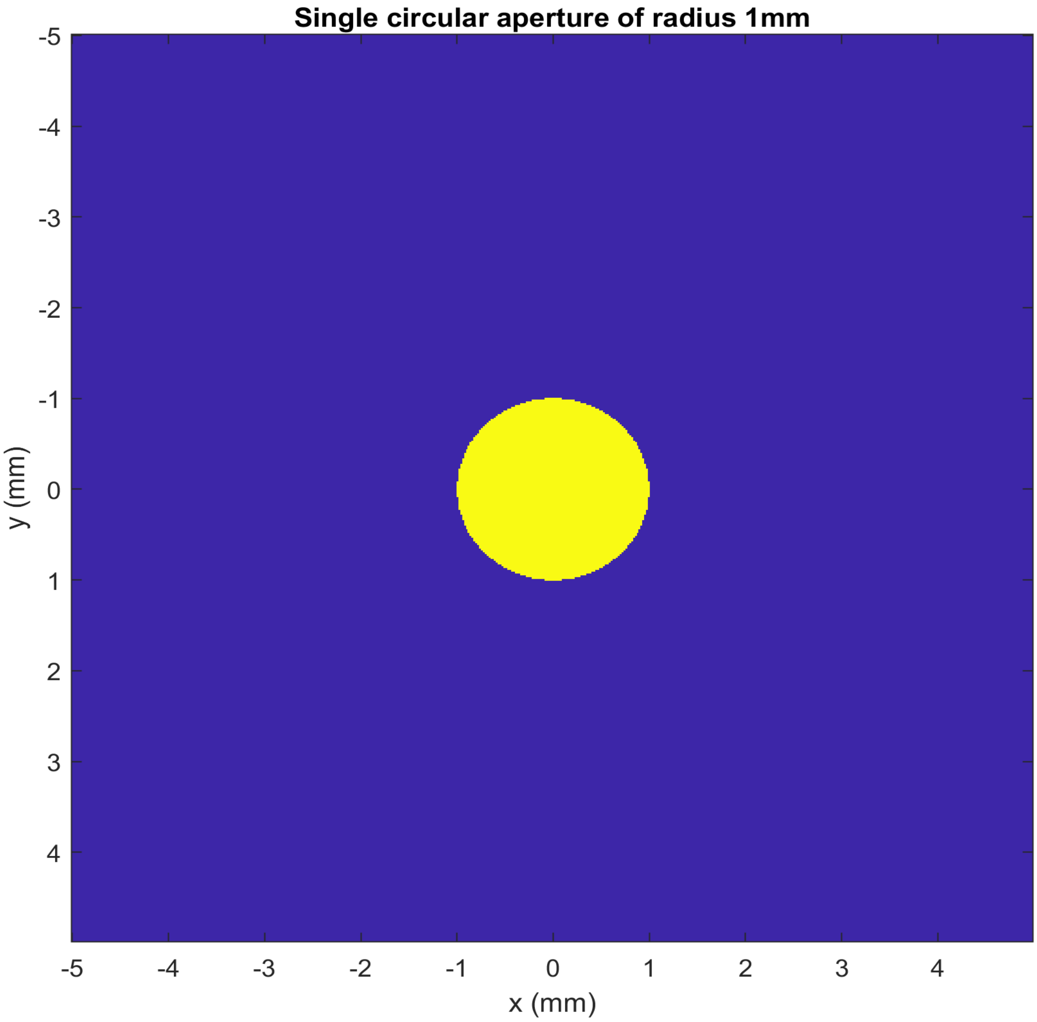

Figure 3: 2D source plane with circular aperture

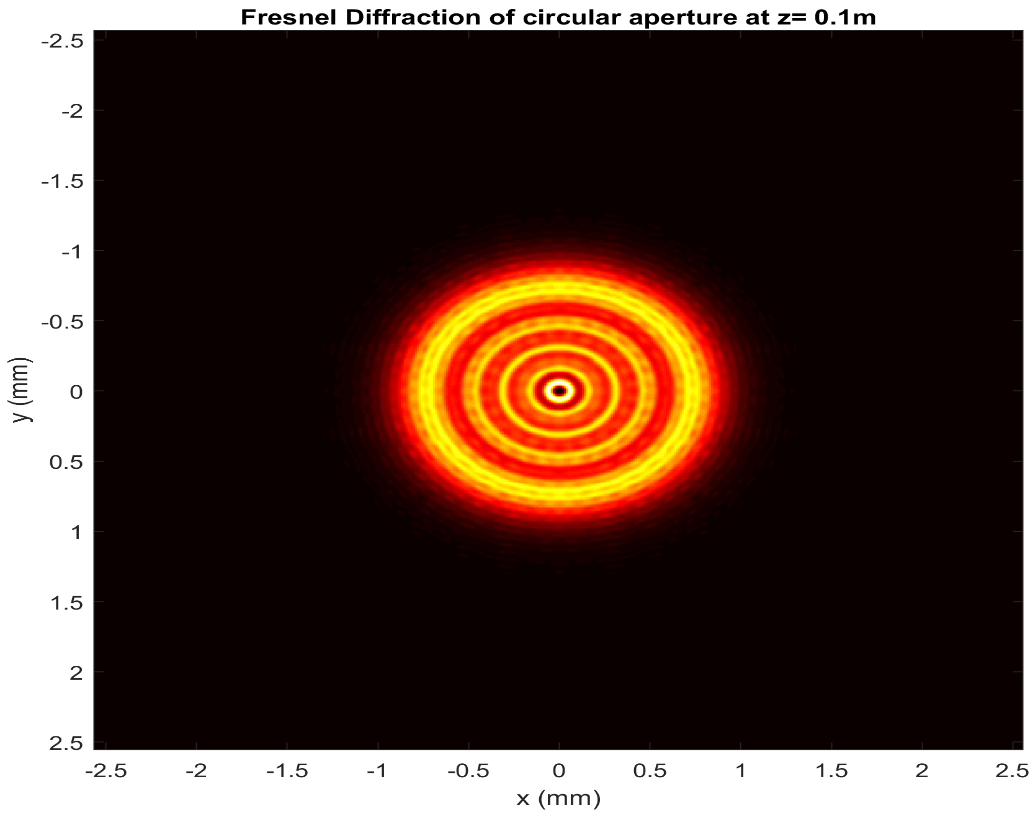

Figure 4: Fresnel Intensity pattern at z = 0.1m (F=10)

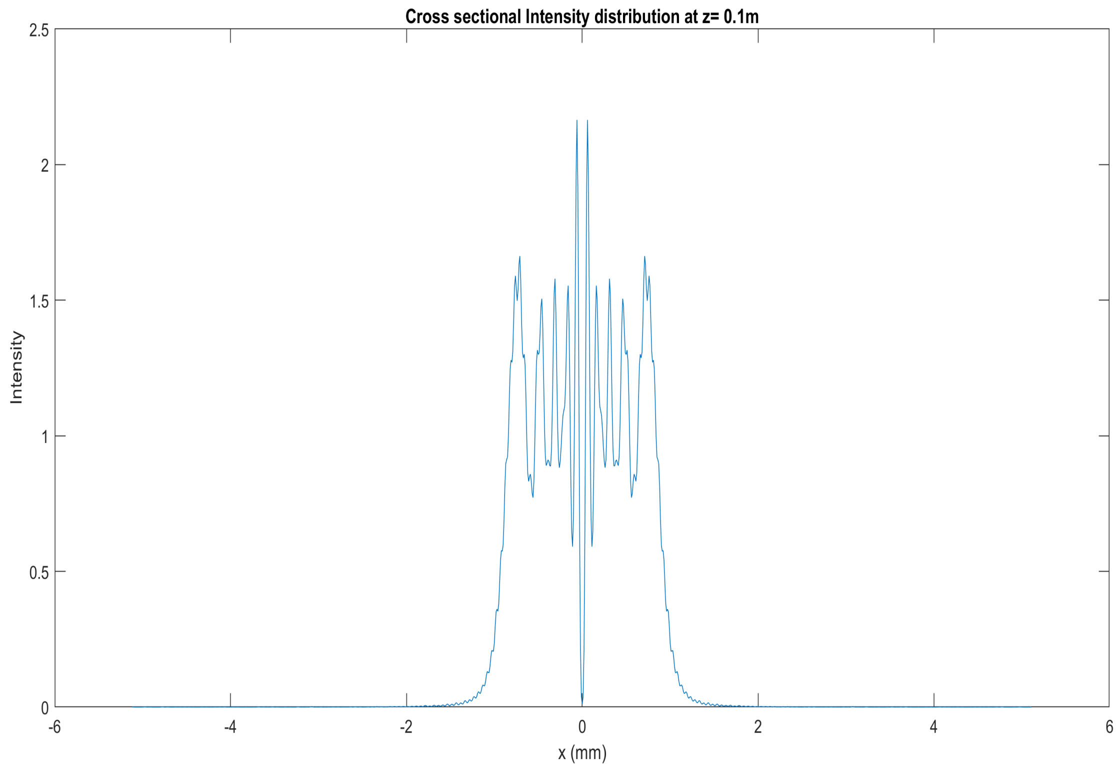

Figure 5: Cross-sectional Fresnel Intensity pattern at z = 0.1m (F=10)

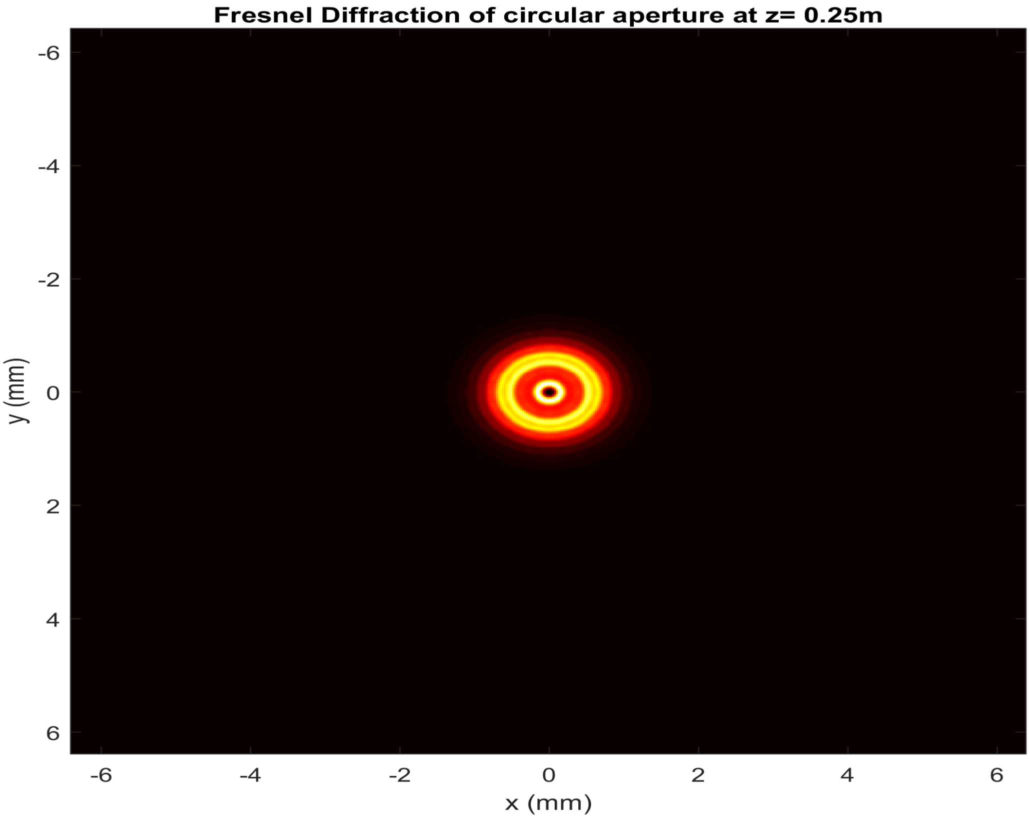

Figure 6: Fresnel Intensity pattern at z = 0.25m (F=4)

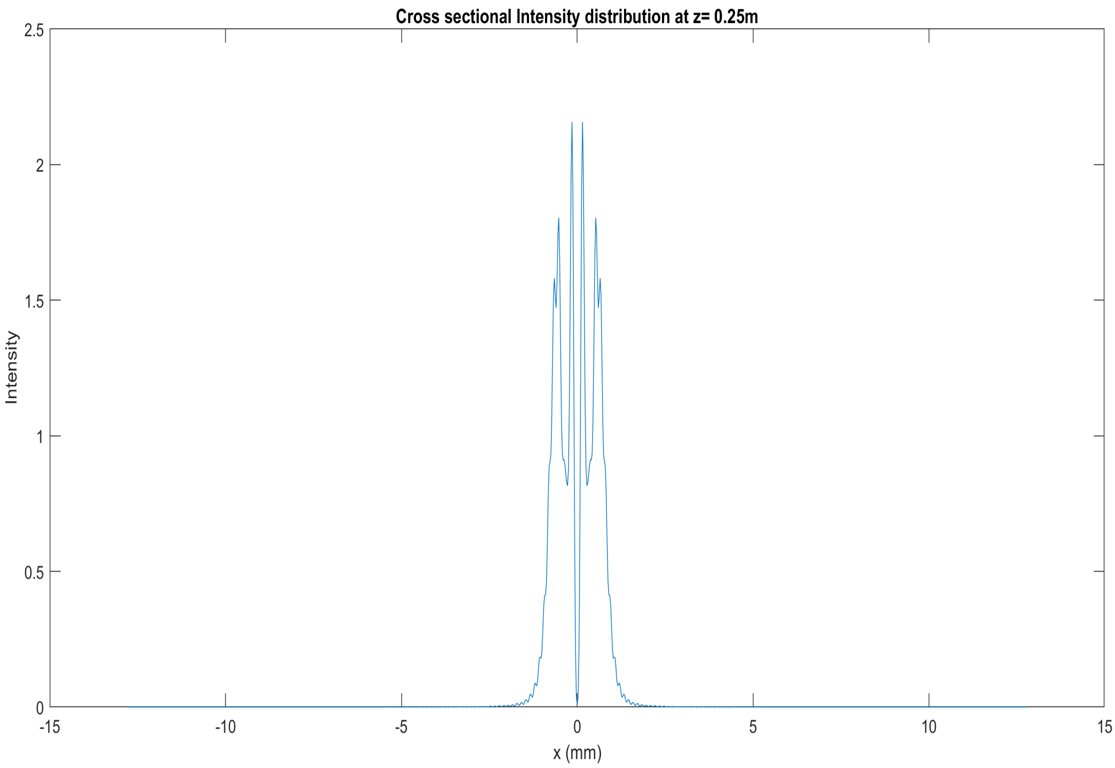

Figure 7: Cross-sectional Fresnel Intensity pattern at z = 0.25m (F=4)

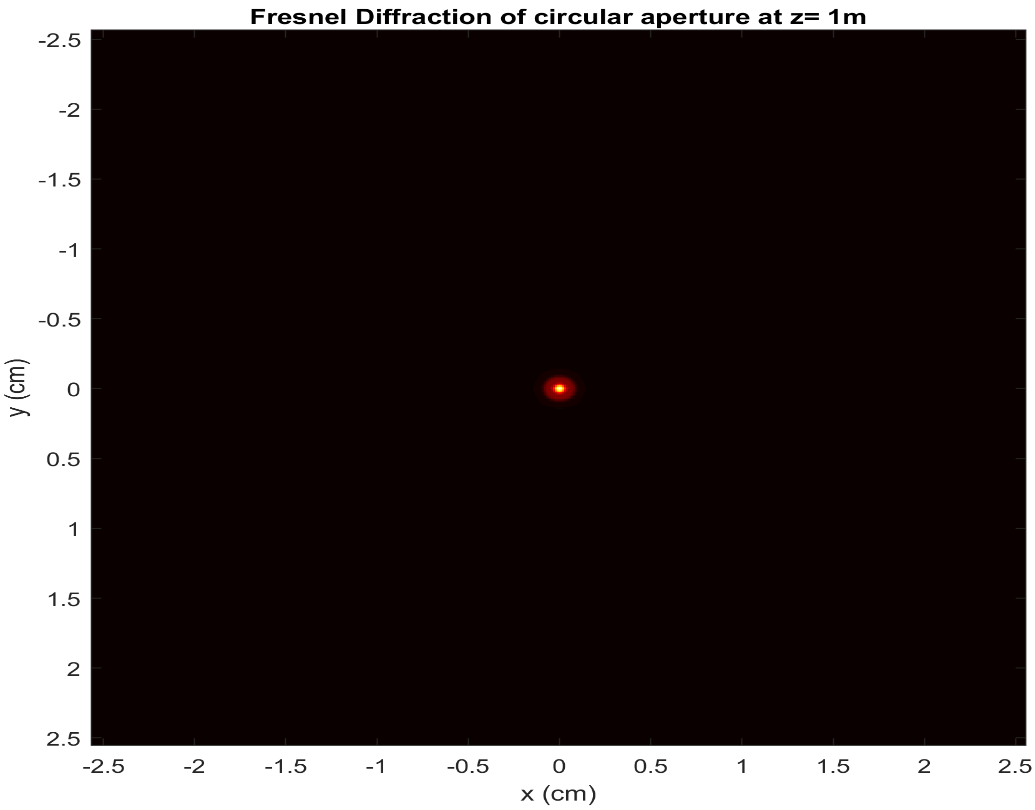

Figure 8: Fresnel Intensity pattern at z = 1m (F=1)

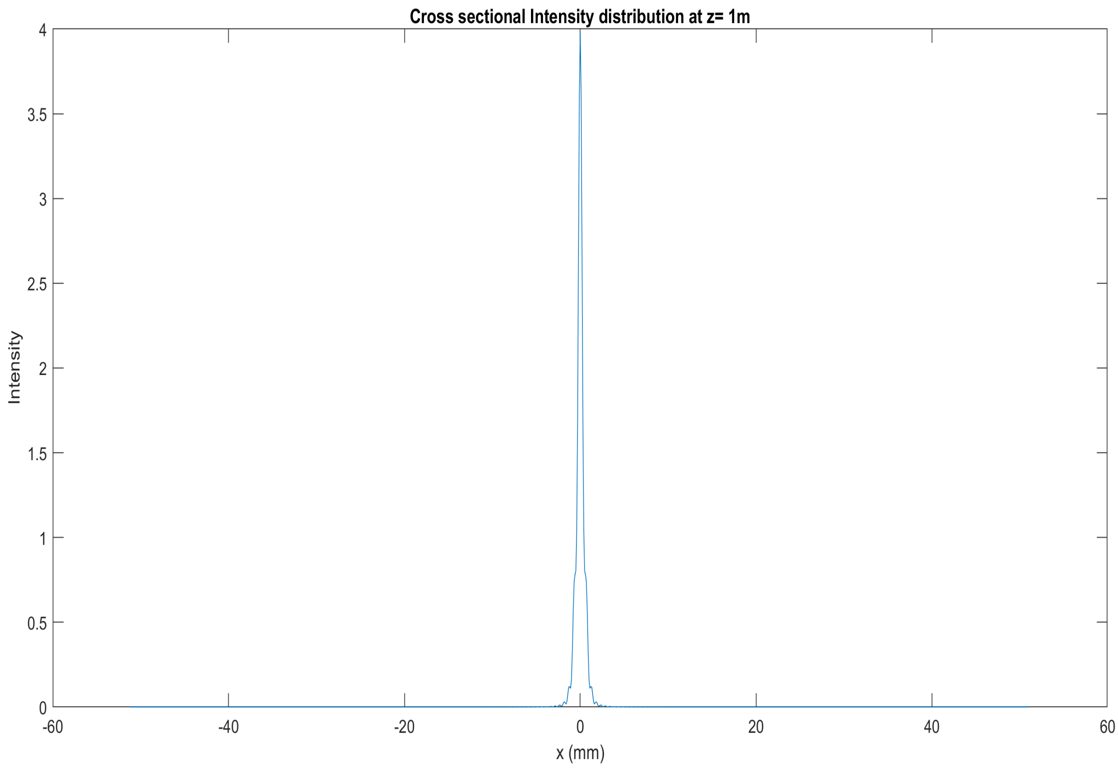

Figure 9: Cross-sectional Fresnel Intensity pattern at z = 1m (F=1)

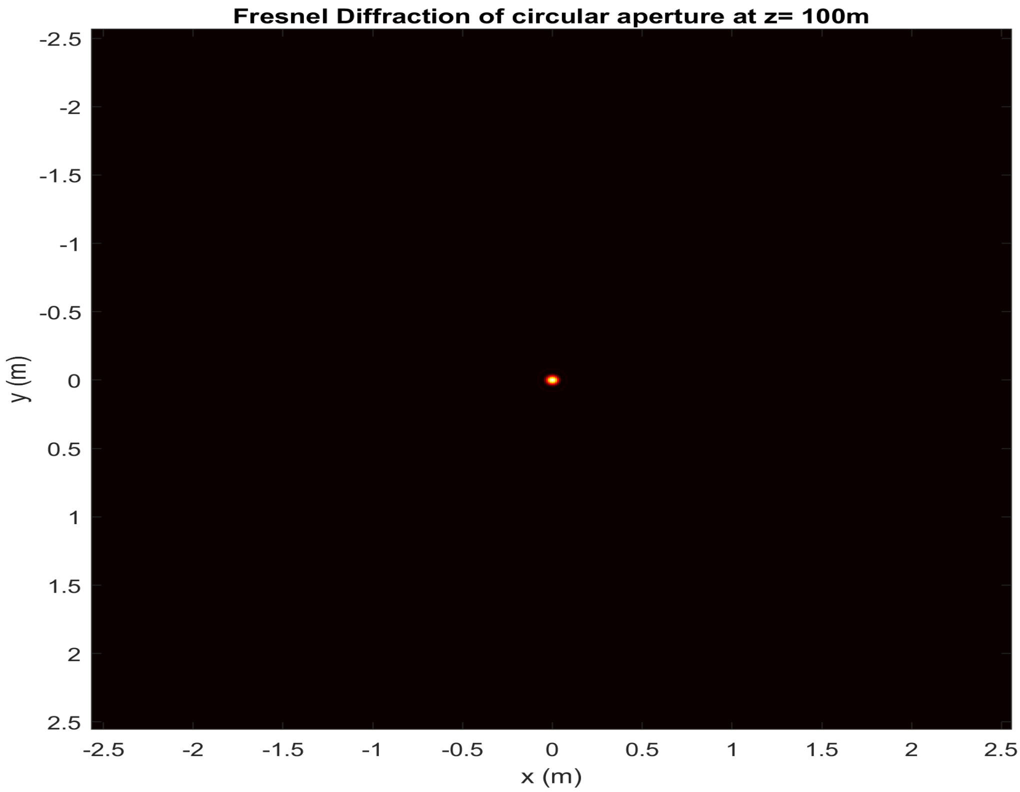

Figure 10: Fresnel Intensity pattern at z = 100 m (F=0.01)

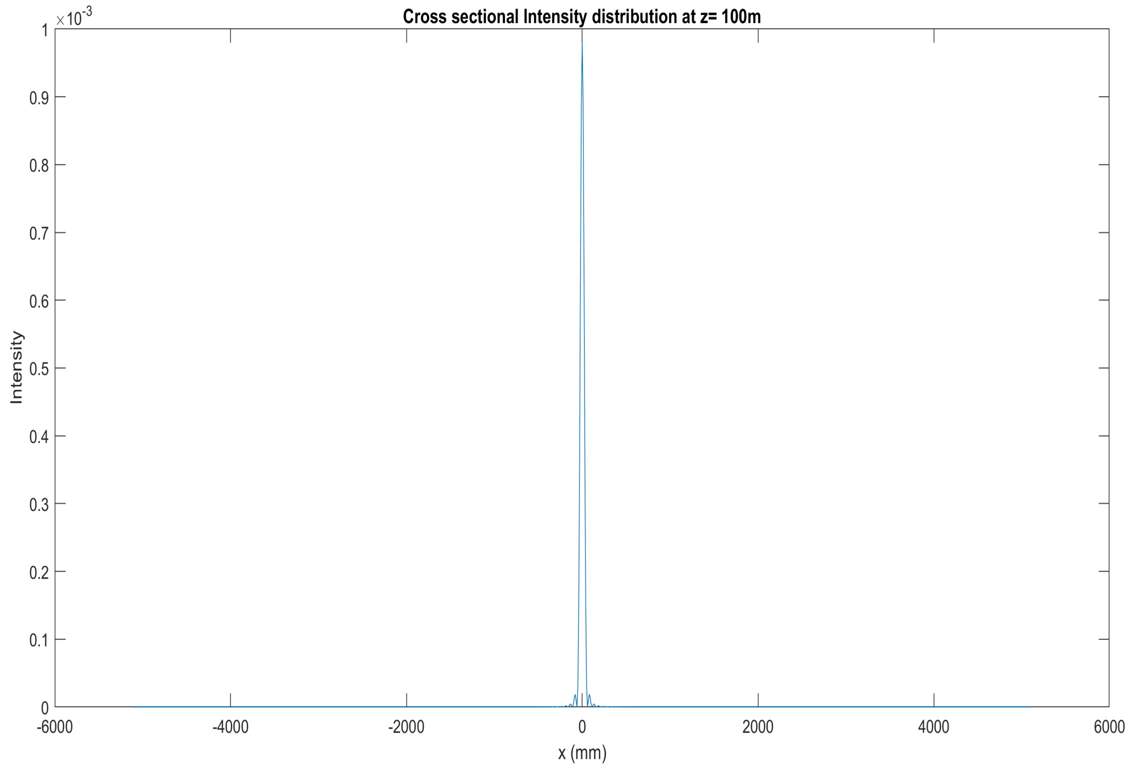

Figure 11: Cross-sectional Fresnel Intensity pattern at z = 100 m (F=0.01)

The above figures shows the intensity pattern, under Fresnel approximation, observed for a circular aperture of radius 1 mm illuminated by 1 μm light for a range of observation distances z, which are displayed on the images, as well as the Fresnel number F.

Using the model, it is possible to observe the transition from the Fresnel regime into the region of validity of the Fraunhofer approximation, as the distance to the observation plane is increased. The images clearly show a transition from Fresnel to Fraunhofer behaviour that occurs somewhere between F = 0.1 and F = 0.01, which is what we would expect from the conditions mentioned. The Fresnel diffraction pattern at z= 100m can be compared with the Fraunhofer diffraction pattern at z= 100m. As per the Fraunhofer diffraction theory, the intensity pattern for a circular aperture is a bessel function given below,

$$I(r) = \left(\frac{A}{\lambda z}\right)^2 \left[2\frac{J_1(kwr/z)}{kwr/z}\right]^2 \tag{4}$$

Where,
w is the radius of aperture
A is the πw^2 area of the aperture
k is the wave number $(2\pi/\lambda)$
r is the distance from origin $\sqrt{x^2+y^2}$
z is the distance of propagation
λ is the wavelength
J_1 is the first order Bessel function

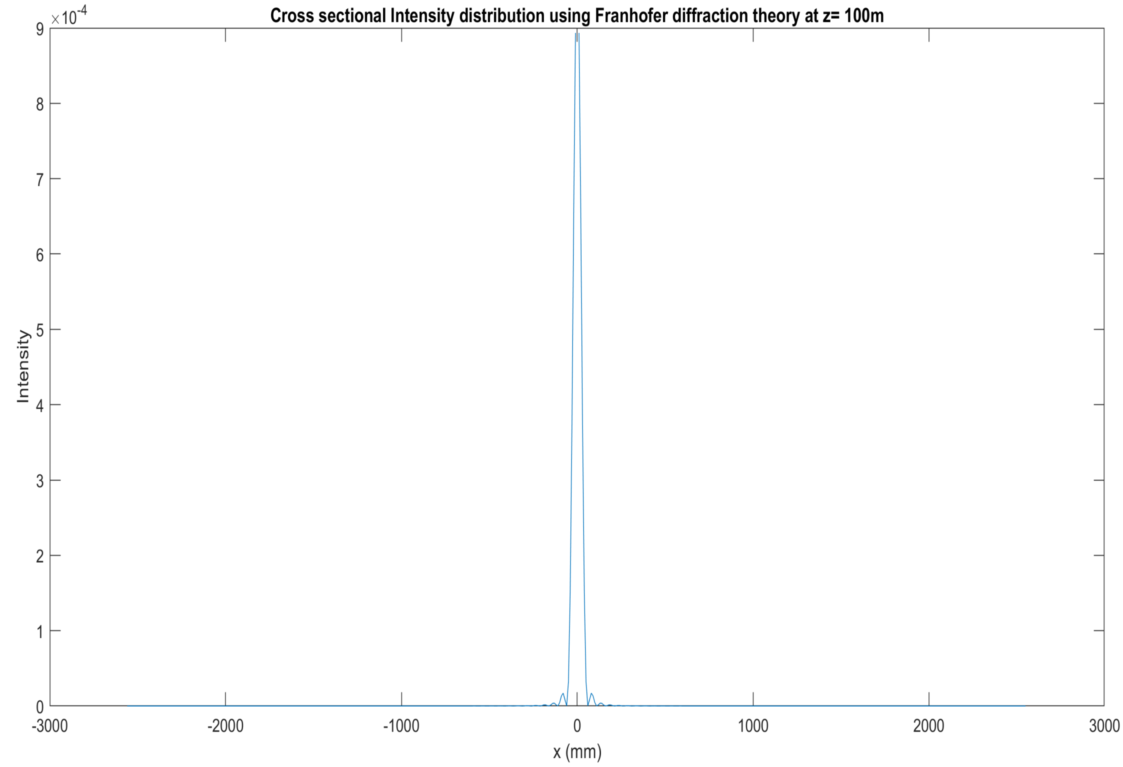

Figure 12: Cross-sectional Fraunhofer Intensity pattern at z = 100 m (F=0.01) using Fraunhofer diffraction theory in Equation 4

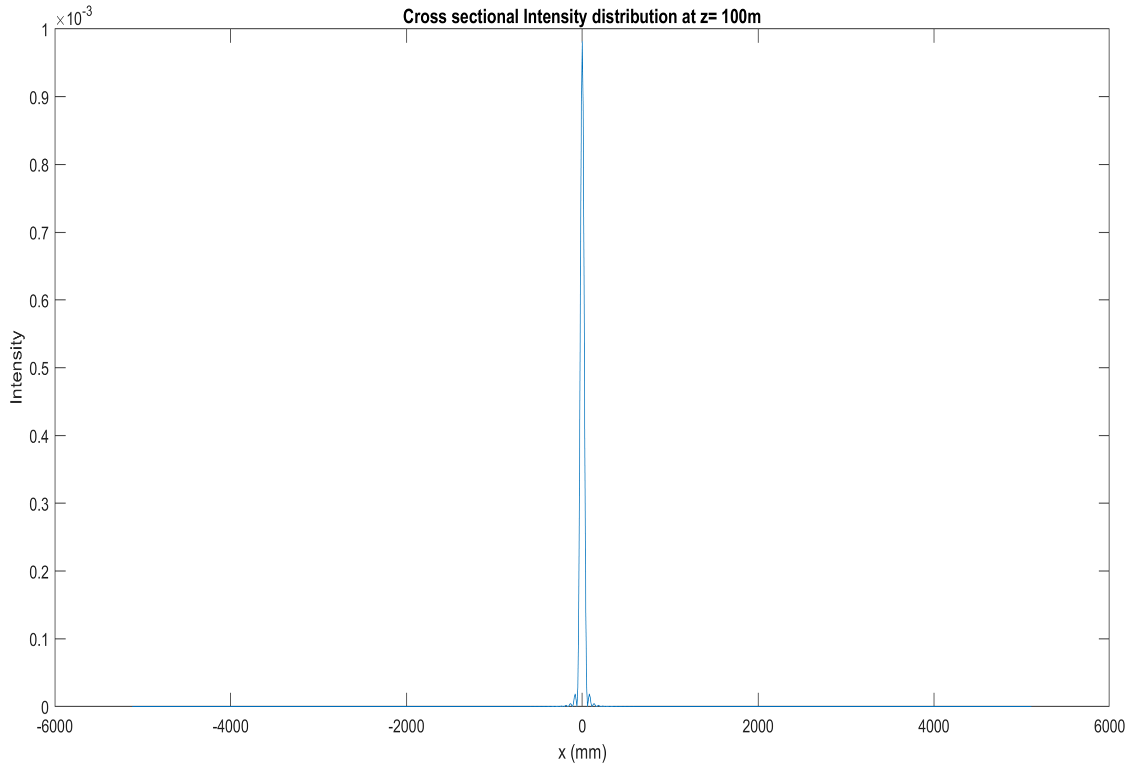

Figure 13: Simulated Cross-sectional Fresnel Intensity pattern at z = 100 m (F=0.01)

By comparing figures 12 and 13, an interesting observation can be made. The intensity pattern obtained using numerical simulation of Fresnel diffraction theory is very similar to the analytical expression for Fraunhofer diffraction given in equation 4. Thus, we can expect the Fresnel diffraction to behave similarly to Fraunhofer diffraction at large propagation distances.

2D Fresnel diffraction code

```
clc;
clear all;
close all;
Wavelength= 10^-6;
z=100;
N=2^10;
k=(2*pi)/Wavelength;
R=1*10^-3;
Dimesnion=10*10^-3;
del1=Dimesnion/N
del2=(Wavelength*z)/(N*del1);
ddf=1/(N*del1);
M=zeros(N);
x1=((-N/2):((N/2)-1))*del1;
y1=((-N/2):((N/2)-1))*del1;
[X1,Y1]=meshgrid(x1,y1);
A=X1.^2+ Y1.^2<=R.^2;
M(A)=1;
F=(R)^2/(z*Wavelength)
%Aperture
figure
imagesc(x1.*10^3,y1.*10^3,M);
axis image
title(['Single circular aperture of radius ',num2str(R*10^3),'mm']);
xlabel('x (mm)')
ylabel('y (mm)')

x2=((-N/2):((N/2)-1))*del2;
y2=((-N/2):((N/2)-1))*del2;

exp_term= exp(((1j*k)/(2*z)).* (X1.^2+Y1.^2));
fft_in= M.*exp_term;
fft_ou= fftshift(fft2(fft_in));
[X2,Y2]=meshgrid(x2,y2);
expo=exp(((1j*k)/(2*z)).* (X2.^2+Y2.^2)) ;
irre=exp(1j.*k.*z)/(1j*Wavelength*z);
yscal=(sum(sum((abs(M).^2))*del1*del1))/(sum(sum(abs(fftshift(fft2(M))).^2)*ddf*ddf));
Output=irre*expo.*fft_ou;
I=abs(Output).^2.*10^-20;
```

```
figure
imagesc(x2*10^3,y2*10^3,I);
axis image
title(['Diffraction pattern at z= ',num2str(z),'m']);
colormap(hot)
axis on
xlabel('x (mm)')
ylabel('y (mm)')
colorbar

figure
x = [0 size(I,2)];
y = [size(I,1)/2 size(I,1)/2  ];
c = improfile(I,x,y);
plot(x2*10^3, c((2:N+1),1));
xlabel('x (mm)')
ylabel('Intensity')
axis on
title(['Cross sectional Intensity distribution at z= ',num2str(z),'m']);


%%%Fraunhofer
A= pi*R*R;
r=sqrt(x2.^2);
ji=k*R*r/z;
Bes=besselj(1,ji)./ji;
IF=((A/(Wavelength*z)).*2.*(besselj(1,ji,1)./ji)).^2;
figure
plot(x2*10^3,IF);
xlabel('x (mm)')
ylabel('Intensity')
axis on
title(['Cross sectional Intensity distribution using Fraunhofer diffraction theory at z= ',n
```

2D Fraunhofer diffraction code

```
clc;
clear all;
close all;
Wavelength= 10^-6;
z=100;
N=2^9;
k=(2*pi)/Wavelength;
R=1*10^-3;
Dimesnion=10*10^-3;
del1=Dimesnion/N;
del2=(Wavelength*z)/(N*del1)
M=zeros(N);
x1=((-N/2):((N/2)-1))*del1;
y1=((-N/2):((N/2)-1))*del1;
[X1,Y1]=meshgrid(x1,y1);
A=X1.^2+ Y1.^2<=R.^2;
M(A)=1;
F=R^2/(z*Wavelength)
%Aperture
figure
imagesc(x1.*10^3,y1.*10^3,M);
axis image
title(['Single circular aperture of radius ',num2str(R*10^3),'mm']);
axis on
xlabel('x (mm)')
ylabel('y (mm)')

x2=((-N/2):((N/2)-1))*del2;
y2=((-N/2):((N/2)-1))*del2;
fft_in= M;
fft_ou= fftshift(fft2(fft_in));
[X2,Y2]=meshgrid(x2,y2);
expo=exp(((1j*k)/(2*z)).* (X2.^2+Y2.^2)) ;
irre=exp(1j.*k.*z)/(1j*Wavelength*z);
Output=irre*expo.*fft_ou;
I=abs(Output).^2;

figure
imagesc(x2,y2,I);
axis image
title(['Franhoufer Diffraction of circular aperture at z= ',num2str(z),'m']);
colormap(hot)
axis on
xlabel('x (m)')
ylabel('y (m)')
```