Discrete Applied Mathematics 24 (1989) 63–82
North-Holland

KÖNIG–EGERVÁRY GRAPHS, 2-BICRITICAL GRAPHS AND FRACTIONAL MATCHINGS*

Jean-Marie BOURJOLLY

Concordia University, Department of Decision Sciences and MIS, Montréal, Québec, Canada H3G 1M8

William R. PULLEYBLANK

University of Waterloo, Department of Combinatorics and Optimization, Waterloo, Ontario, Canada

Received 27 October 1988

A *fractional node cover* of a graph is an assignment of the values $0, \frac{1}{2}, 1$ to the nodes, so that for each edge, the sum of the values assigned to its two ends is at least one. Such a cover is *minimum* if the sum of the assigned values is minimized. A *König–Egerváry graph* is a graph for which there exists a minimum fractional cover in which all nodes receive the value 0 or 1. A 2-*bicritical graph* is one for which the unique minimum fractional cover is obtained by assigning $\frac{1}{2}$ to all the nodes. We describe a polynomial method for decomposing a graph into 2-bicritical components and König–Egerváry components. This decomposition yields a minimum fractional node cover in which the number of nodes receiving the value $\frac{1}{2}$ is minimized. We also show how excluded minor characterizations by Deming, Sterboul and Lovász of König–Egerváry graphs can be used to obtain a structural characterization of 2-bicritical graphs.

1. Introduction

Let $G=(V,E,\psi)$ be a simple graph where V is the set of nodes, E is the set of edges and ψ is the incidence function such that for any $j\in E$, $\psi(j)$ is the set of two nodes incident with j. A *node cover* of G is a set $X\subseteq V$ such that every edge is incident with at least one member of X. The *minimum node covering problem* is to find a node cover of minimum cardinality. This problem is well known to be NP-complete [6].

For any vector $x=(x_j: j\in J)$ and any $K\subseteq J$ we let $x(K)$ denote $\sum_{j\in K} x_j$. The minimum node covering problem can be formulated as the following integer program.

$$\text{(NC)}\qquad \text{minimize}\quad y(V)=\sum_{i\in V} y_i,$$
$$\text{subject to}\quad y_i\in\{0,1\}\quad \text{for all } i\in V,$$
$$y(\psi(j))\ge 1\quad \text{for each edge } j\in E.$$

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada.

If we replace the bivalent constraint on y_i by a nonnegativity constraint then we obtain the linear program:

$$\text{(LP)} \qquad \text{minimize} \quad y(V),$$
$$\text{subject to} \quad y_i \geq 0 \quad \text{for all } i \in V,$$
$$y(\psi(j)) \geq 1 \quad \text{for each edge } j \in E.$$

For any $S \subseteq V$ we let $\delta(S)$ denote the set of edges having exactly one end in S. We abbreviate $\delta(\{i\})$ by $\delta(i)$ for any $i \in V$.

The dual of (LP) is the problem:

$$\text{(MP)} \qquad \text{maximize} \quad x(E) - \sum_{j \in E} x_j,$$
$$\text{subject to} \quad x_j \geq 0 \quad \text{for all } j \in E,$$
$$x(\delta(i)) \leq 1 \quad \text{for all } i \in V.$$

(MP) is the *maximum fractional matching problem*. It is well known (see, e.g. [15]) that if x is a basic feasible solution to (MP), then $x_j \in \{0, \frac{1}{2}, 1\}$ for all $j \in E$, and moreover, the edges j for which $x_j = \frac{1}{2}$ form node disjoint odd cycles. If we replace the constraints $x_j \geq 0$ with $x_j \in \{0, 1\}$ for all $j \in E$, then we obtain the *maximum (integer) matching problem*. An equivalent form of (MP) which is also studied is the so-called 2-*matching problem*. In this case, the constraints $x(\delta(i)) \leq 1$ are replaced by $x(\delta(i)) \leq 2$, for all $i \in V$, and x is a 2-matching if and only if $x_j \in \{0, 1, 2\}$ for all $j \in E$ and $\frac{1}{2}x$ is a feasible solution to the corresponding fractional matching problem.

Our main subject of study here is the problem (LP). When G is bipartite, it follows from König's Theorem that there exists an optimum solution in which $y_i \in \{0, 1\}$ for all $i \in V$. Using this, and a fairly standard node splitting argument (see Nemhauser and Trotter [12]) it is easy to see that for general G, there always exist optimum solutions to (LP) in which $y_i \in \{0, \frac{1}{2}, 1\}$ for all $i \in V$. If there exists an integer optimum solution to (LP), i.e., one in which $y_i \in \{0, 1\}$ for all $i \in V$, then G is said to be a *König–Egerváry graph* (KEG). Lovász [7] showed that if G was a KEG, then there also existed an integer optimum solution to (MP).

By contrast, a 2-*bicritical graph* [15] is a graph for which the unique optimum solution to (LP) has $y_i = \frac{1}{2}$ for all $i \in V$. In general there will be many different optimum solutions to (LP), and from a point of view of using this linear program as a relaxation of the minimum node covering problem, what is of particular interest is finding one for which the set of nodes i having $y_i = \frac{1}{2}$ is empty, or at least as small as possible. In fact, Nemhauser and Trotter [12] showed that in any optimal solution to (LP), all nodes receiving integer values could have these values fixed, and there exists an optimum solution to the corresponding integer problem, consistent with these values.

The problem of obtaining an optimal solution to (LP) with as few fractional variables as possible was posed by Nemhauser and Trotter [12] and solved by Picard and Queyranne [13, 14] (see also [15]). In Section 5 we describe an algorithm, of the same complexity as that of solving (MP), which decomposes a connected graph into some number of 2-bicritical components and König–Egerváry components, with the property that a feasible solution to (LP) is minimum if and only if it induces an optimum solution on each of these components. We thereby obtain an optimum solution to (LP) for which the number of fractional variables is minimized.

We also show that 2-bicritical graphs have an ear decomposition, analogous to the ones known for several other classes of graphs. This enables them to be constructed, starting from the forbidden subgraphs of the Deming–Sterboul characterization of KEGs and then performing certain simple operations.

2. König–Egerváry graphs

Recall that a König–Egerváry graph (KEG) is a graph for which (LP) has an integer optimum solution. Lovász [7] showed that for such a graph, (MP) has an integer optimum solution. Therefore these are the graphs for which the size of a largest (integer) matching equals the size of a minimum node cover. When studying KEGs, we can restrict our attention to graphs having *perfect matchings* (i.e., matchings that saturate every node) for the following reason. The *Edmonds–Gallai partition* of a graph $G=(V,E,\psi)$ is the partition of V into $I\cup O\cup P$ defined as follows:

$$\begin{aligned} O &= \{v\in V\text{: some maximum matching of } G \text{ leaves } v \text{ exposed}\},\\ I &= \{v\in V\setminus O\text{: } v \text{ is adjacent to a member of } O\},\\ P &= V\setminus(I\cup O). \end{aligned}$$

Lemma 2.1 (Lovász [9, Lemma 3.3]). *A graph G is a KEG, if and only if no edge joins two members of O, and $G[P]$ is a KEG.*

Let us mention that the Edmonds–Gallai partition can be found with a single application of Edmonds' maximum matching algorithm.

Let M be a matching of G. A *blossom B* with respect to M is an odd simple cycle of length $2k+1$ for which exactly k edges belong to M. The *tip* of the blossom is the unique node of B not incident with an edge of M which belongs to B. A *blossom pair* is a graph consisting of the union of blossoms B_1 and B_2 plus an odd length path P joining the tips of B_1 and B_2, but which contains no other nodes of B_1 or B_2, and for which the first, third, ... edges belong to M. (Note that B_1 and B_2 need not be disjoint but the tips must be distinct.) (See Fig. 1.)

Deming and Sterboul proved the following characterization.

Theorem 2.2 (Deming [3], Sterboul [17]). *Let G be a graph with a perfect matching M. Then G is a KEG if and only if there exists no blossom pair with respect to M.*

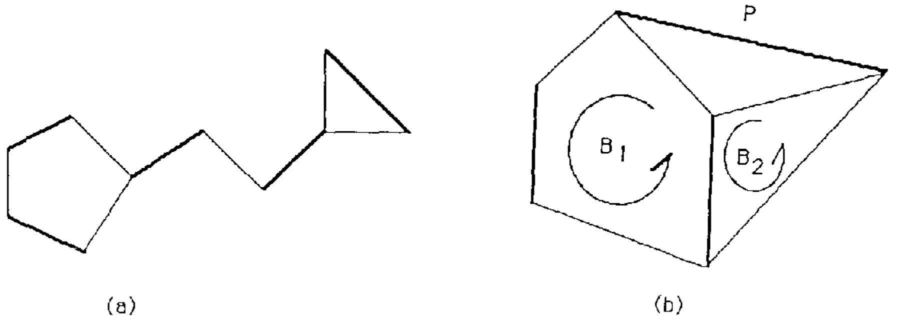

Fig. 1. Two blossom pairs.

A simpler, but similar characterization was obtained by Lovász. We say that G' is an *odd homeomorph* of G if G' is obtained by replacing each edge of G with a path of odd length, such that these paths remain internally disjoint.

Theorem 2.3 (Lovász [9]). *A connected graph G which admits a perfect matching is a KEG if and only if there exists no subgraph G' which is an odd homeomorph of one of the graphs of Fig. 2, and such that $G \setminus G'$ has a perfect matching.*

(Here $G \setminus G'$ is the graph obtained from G by deleting all nodes of G', plus incident edges.)

Notice that if G contains a subgraph G' satisfying the conditions of Theorem 2.3, then it is easy to define a perfect matching of G which makes G' a blossom pair. The converse is much less obvious. In Section 5 we prove this directly; we show how to obtain a graph G' as required by Theorem 2.3 from a blossom pair.

Korach [11] also proved a similar, but different characterization of KEGs.

Notice that both Theorems 2.2 and 2.3 characterize KEGs by making use of integer matchings. Bourjolly et al. [2] showed that KEGs can also be characterized using fractional matchings, i.e., solutions to (MP). This has some appeal, for solving (MP) is equivalent to solving a bipartite matching problem on a graph in which each node has been split in two (see [10, Theorem 6.1.4]). Solving maximum

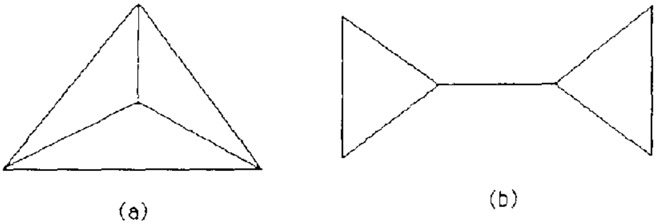

Fig. 2. Forbidden "odd" minors.

bipartite matching problems is easier (in a practical, if not theoretical sense) than solving nonbipartite problems.

Let x be a fractional matching, i.e., a feasible solution to (MP). We say that node i is *saturated* if $x(\delta(i))=1$, and otherwise *unsaturated*. If every node is saturated, then x is *perfect*. An edge j is called *active* if $x_j>0$, and otherwise *passive*. Note that if x is basic, then for any unsaturated node i, $x(\delta(i))=0$ and for every active edge j, $x_j=\frac{1}{2}$ or 1.

The complementary slackness conditions for optimality of feasible solutions $\hat{x}$ to (MP) and $\hat{y}$ to (LP) are the following:

$$\text{for each edge } j, \text{ if } \hat{x}_j>0 \text{ then } \hat{y}(\psi(j))=1; \tag{2.1}$$

$$\text{for each node } i, \text{ if } \hat{y}_i>0 \text{ then } \hat{x}(\delta(i))=1. \tag{2.2}$$

Condition (2.2) has a useful consequence. If $\hat{x}$ is a maximum fractional matching, i.e., an optimum solution to (MP), then for any node i unsaturated by $\hat{x}$, we must have $\hat{y}_i=0$ in every optimum solution to (LP). In order to obtain feasibility therefore we must also have $\hat{y}_v=1$ for any neighbour v of such a node i.

Let us define the sets

$$V_0=\{i\in V\colon \hat{y}_i=0 \text{ in every optimum solution to (LP)}\},$$

$$V_1=\{i\in V\colon \hat{y}_i=1 \text{ in every optimum solution to (LP)}\}.$$

(In [5], V_0 (respectively V_1) is called V_1 (respectively V_0) and is defined as the set of nodes i for which $\hat{z}_i$ takes value 1 (respectively 0) in every optimum solution $\hat{z}$ to the fractional stable set problem.)

Hammer et al. [5] showed that V_0 and V_1 (as defined in the present paper) are such that $i\in V_0$ if and only if i is left unsaturated by some maximum fractional matching $\hat{x}$, and V_1 is the neighbour set of V_0. They also showed that every minimum node cover, i.e., integer optimum solution to (LP), contained all nodes of V_1, and no nodes of V_0. These results are discussed further in Section 4.

Now we summarize how a maximum fractional matching $\hat{x}$ can be used to determine if G is a KEG [1,2]. First, any unsaturated node i must have y_i fixed at 0 and all neighbours v must have y_v fixed at 1. (Note that maximality of $\hat{x}$ guarantees that two adjacent nodes cannot be left unsaturated, so the preceding cannot give rise to a contradiction.) All such nodes are deleted, a new maximum fractional matching is obtained and the process repeated.

When the above terminates, $\hat{x}$ is a perfect fractional matching of the remaining graph G. If $\hat{x}$ has any components with $\hat{x}_j=\frac{1}{2}$, then these components form the edge sets of disjoint odd cycles. By (2.1) we must have $\hat{y}(\psi(j))=1$ for every such edge in every optimal solution, and the only feasible solution has $\hat{y}_i=\frac{1}{2}$ for all nodes i of each cycle. Hence G is not a KEG.

Therefore $\hat{x}$ must in fact be a perfect integer valued matching of G, and so we can apply a labelling type algorithm such as that of Deming [3] to determine whether

G is a KEG. We describe such an algorithm in Section 5 as part of the general decomposition process.

Finally, suppose that (LP) has an integer optimal solution. Then by the preceding argument, every optimal basic solution to the fractional matching problem must be integer valued. So not only do KEGs have integral optimal solutions to (MP), but they have only integral optimal basic solutions.

3. 2-bicritical graphs

Recall that $G=(V,E,\psi)$ is 2-bicritical if (LP) has a unique optimal solution obtained by defining $y_i=\frac{1}{2}$ for all $i\in V$. Note that 2-bicritical graphs need not be connected; a graph is 2-bicritical if and only if each component is 2-bicritical. The following characterizes these graphs in terms of fractional matchings:

Theorem 3.1 (Pulleyblank [15]). *The graph G is 2-bicritical if and only if $|V|\geq 3$ and for each $v\in V$, $G\setminus\{v\}$ has a perfect fractional matching.*

(In fact, in [15], graphs were defined to be 2-bicritical if these perfect fractional matchings existed, and the fact that this was equivalent to the existence of a unique optimal solution to (LP) which was completely fractional was proved. Also, they were defined in terms of 2-matchings, rather than fractional matchings, which accounts for the name.)

Corollary 3.2. *Every edge of a 2-bicritical graph $G=(V,E,\psi)$ is active in some perfect fractional matching of G.*

Proof. Let $j\in E$ and let $\{u,v\}=\psi(j)$. By Theorem 3.1, there exist fractional matchings x^u and x^v of G, which saturate all nodes of G except u and v respectively and have $x^u(\delta(u))=x^v(\delta(v))=0$. Define $\hat{x}_k=\frac{1}{2}(x_k^u+x_k^v)$ for $k\in E\setminus\{j\}$ and $\hat{x}_j=\frac{1}{2}$. Then $\hat{x}$ is the required perfect fractional matching. □

It is easy to see that an odd cycle is 2-bicritical. So too are the blossom pairs described in the previous section.

Proposition 3.3. *Let $\hat{x}$ be a matching in $G=(V,E,\psi)$ and let B be a blossom pair. Then B is 2-bicritical.*

Proof. Consider the problem (LP) defined for B. We show that the unique optimum solution is obtained by letting $y_i=\frac{1}{2}$ for every node i. If we let $\bar{x}$ be the restriction of $\hat{x}$ to the edges of B, then $\bar{x}$ is a perfect matching and so an optimum solution to (MP) for B. Let y be an optimum solution to (LP). Let v be the tip of one of the two blossoms contained in B. For each passive edge j of this blossom we have the inequality

$$y(\psi(j)) \geq 1. \tag{3.1}$$

By complementary slackness (2.1), for each active edge j of this blossom, we have the equation

$$y(\psi(j)) = 1. \tag{3.2}$$

If we subtract the sum of these equations from the sum of the above inequalities, we obtain

$$2y_v \geq 1. \tag{3.3}$$

Let v' be the tip of the other blossom and let P be the path of B joining v and v'. If we add to (3.3) twice the inequalities (3.1) corresponding to passive edges of P and subtract twice the equations (3.2) corresponding to active edges of P, we obtain

$$2y_{v'} \leq 1. \tag{3.4}$$

If we repeat these arguments, with the two blossoms interchanged, we obtain $2y_{v'} \geq 1$ and $2y_v \leq 1$. Therefore each of the above inequalities and (3.3) and (3.4) must hold with equality, implying that all the inequalities summed to obtain them must also hold with equality, which in turn implies $y_i = \frac{1}{2}$ for all nodes i of B. □

Let B be a subgraph of G. An *ear* in G with respect to B is an odd length path in G, for which the endpoints are in B but all other nodes are not in B, and for which all nodes except possibly the endpoints are distinct. A *pendant* in G with respect to B consists of an odd length simple cycle C in G, node disjoint from B, plus a positive length simple path with one end in C, the other in B and all other nodes not in B or C. See Fig. 3. The node in B is called the *end* of the pendant.

Proposition 3.4. *Let B be a 2-bicritical subgraph of a graph G. Then if we add an ear to B or add a pendant to B, the result is 2-bicritical.*

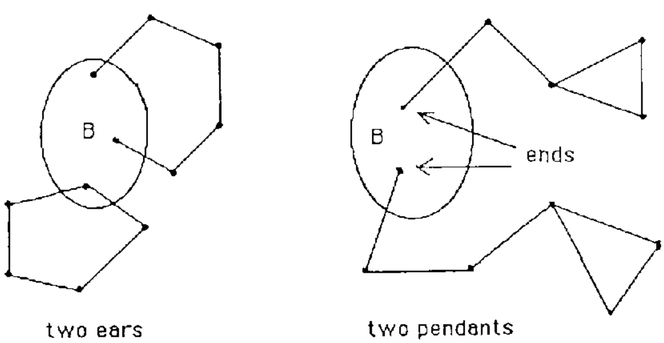

Fig. 3.

Proof. This result follows easily from Theorem 3.1 and from [15, Theorem 2.3] which states that every 2-bicritical graph has a perfect fractional matching. □

An *ear-pendant decomposition* of a graph G is a sequence $G_0, G_1, G_2, \ldots, G_p$ of graphs where G_0 is either an odd cycle or a blossom pair of G and for each $i \in \{1, 2, \ldots, p\}$, G_i is obtained from G_{i-1} by adding either an ear or a pendant. It follows from Propositions 3.3 and 3.4 that any graph having an ear-pendant decomposition is 2-bicritical. In Section 5 we will show that every 2-bicritical graph has such a decomposition. In fact we can refine the result somewhat by making use of Theorem 2.3. It will allow us to show that every 2-bicritical graph has an ear-pendant decomposition which starts with an odd homeomorph of K_3 or K_4.

Note that these results parallel earlier results concerning ear decompositions of critical graphs. A graph $G=(V,E)$ is *critical* if, for each $v \in V$, $G \setminus \{v\}$ has a perfect (integer) matching. Lovász [8], and Pulleyblank and Edmonds [16] showed that G is critical if and only if there is a sequence $G_0, G_1, G_2, \ldots, G_p = G$ of graphs where G_0 is an odd length cycle and for $i = 1, 2, \ldots, p$, G_i is obtained from G_{i-1} by the addition of an ear.

4. Fractional matchings

In this section we describe a polynomially bounded algorithm which constructs an optimum solution to (MP) and (LP). Although this algorithm can easily be deduced from the (integer) matching algorithm of Edmonds [4] (see also Lovász and Plummer [10]), we include a description here. This will be frequently referred to in Section 5.

At each stage of the algorithm, we have a basic fractional matching x. That is, for all $j \in E$, $x_j \in \{0, \frac{1}{2}, 1\}$ and the edges j for which $x_j = \frac{1}{2}$ form a collection of node disjoint odd cardinality cycles. As usual, the algorithm either finds an augmenting path which enables the number of saturated nodes to be increased or else obtains a structure which shows that x is maximum.

There are three types of augmenting paths used by the algorithm. The first type and simplest is an odd-length path which joins unsaturated nodes u and v and for which the second, fourth, sixth, ... edges j have $x_j = 1$. Performing the augmentation consists of redefining x_j to be $1 - x_j$ for all edges of this path. See Fig. 4.

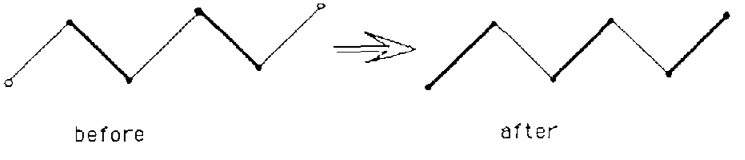

Fig. 4. First type of augmenting path.

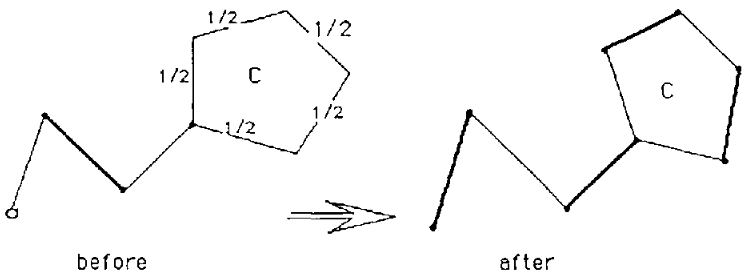

Fig. 5. Second type of augmentation.

The second type is similar, except v is not an unsaturated node, but a node of an odd cycle C for which all edges j have $x_j = \frac{1}{2}$. (This implies that C is disjoint from the path.) The augmentation consists of first making v unsaturated, by redefining x_j to be alternately 0 and 1 for the edges of C, then augmenting between u and v exactly as in the first case. See Fig. 5.

The third type of augmentation is essentially the reverse operation. An even-length path is found joining an unsaturated node to a node v of an odd cycle, such that the second, fourth, etc. edges of the path have $x_j = 1$ and the second, fourth, etc. edges of the cycle have $x_j = 1$. The augmentation consists of redefining x_j to be $\frac{1}{2}$ for the edges of the cycle, then redefining x_j to be $1 - x_j$ for the edges of the path. See Fig. 6.

Note that in all three cases, this augmentation results in u becoming saturated, without any other node becoming unsaturated.

Now we can describe the algorithm. In order to find augmenting paths if they exist, a labelling process is carried out. A node may receive the label "+", "−" or be unlabelled. If a node v is labelled "+", then we know an even length simple path in G from v to some unsaturated node u, such that the edges alternately have $x_j = 1$ and $x_j = 0$. If v has the label "−", then we know an odd length simple path in G from v to an unsaturated node u, such that the edges alternately have $x_j = 0$ and $x_j = 1$. If a node v has a label "+" or "−", then we have a predecessor node $p(v)$ defined, which is the next node in the path to the unsaturated node. If v is itself unsaturated, and hence has label "+", then $p(v)$ is null.

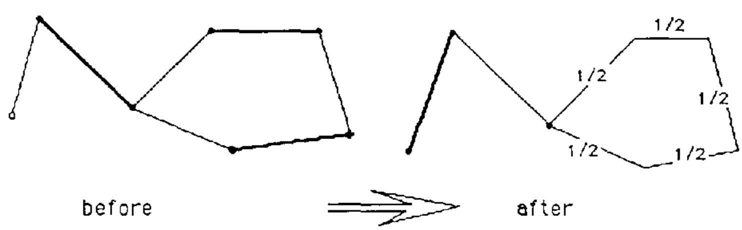

Fig. 6. Third type of augmentation.

Fractional Matching Algorithm

Input. A graph $G=(V,E)$; a basic solution to (MP), i.e., a fractional matching for which each edge j has $x_j \in \{0, \frac{1}{2}, 1\}$ and the edges with $x_j = \frac{1}{2}$ form node disjoint odd cycles. (Initially, x can be the 0 vector.)

Output. Optimum solutions to (MP) and (LP); a final labelling.

Step 1 (Initialization). All unsaturated nodes are given the label "+". All saturated nodes are unlabelled.

Step 2 (Edge scan). If every edge incident with a node having label "+" has as its other end a node with label "–", then stop. The matching is maximum. Otherwise, let $[u,v]$ be an edge joining a node u having label "+" to a node v which either has label "+" or is unlabelled. If v has label "+", go to Step 3. If v is unlabelled, go to Step 4.

Step 3 (Augment). Tracing back the predecessor pointers from v and u, we reach unsaturated nodes r_1 and r_2. If they are different then these paths together with the edge $[u,v]$ comprise an augmenting path of the first type. If $r_1=r_2$ then these paths plus $[u,v]$ comprise an augmenting path of the third type. In either case we perform the augmentation then return to Step 1.

Step 4 (Label or augment). Since v is unlabelled, it must be saturated. If there exists an edge $j=[v,w]$ incident with v having $x_j=1$, then we give v the label "–" and give w the label "+" and define $p(w)=v$, $p(v)=u$. Then we return to Step 2. Otherwise, v is a node of an odd cycle C for which every edge j has $x_j=\frac{1}{2}$. We perform an augmentation of the second kind and return to Step 1.

End

Proving the correctness of this algorithm involves establishing two things. First, the augmenting paths found by the algorithm have the asserted properties. Second, when the algorithm terminates, the fractional matching is maximum. These follow from the fact that the labellings produced by the algorithm satisfy the following:

every unsaturated node has the label "+"; (4.1)

every node with the label "–" is joined by an edge j having $x_j=1$ to a node with label "+"; (4.2)

upon termination, all neighbours of any node with label "+" have label "–". (4.3)

The correctness of the augmenting paths follows from (4.1) and (4.2). Now suppose the algorithm has terminated.

Define

$$y_i = \begin{cases} 0, & \text{if } i \text{ has the label ``+''}, \\ 1, & \text{if } i \text{ has the label ``}-\text{''}, \\ \frac{1}{2}, & \text{if } i \text{ is unlabelled.} \end{cases}$$

It follows from (4.3) that y is a feasible solution to (LP) and from (4.1) and (4.2) that x and y satisfy the complementary slackness conditions for optimality. Hence x and y are optimum solutions to (MP) and (LP) respectively. Finally, note that using standard methods we can implement this algorithm so that it runs in time $O(|V| \cdot |E|)$. (See [10].)

Recall that V_0 and V_1 are the sets of nodes which take on the values 0 and 1 respectively in every optimum solution to (LP).

Proposition 4.1. *V_0 and V_1 are the sets of nodes with labels "+" and "−" respectively in the final labelling produced by the algorithm. For each edge j which joins a node of V_0 and a node of V_1 there is some maximum fractional matching $\hat{x}$ which satisfies $\hat{x}_j = 1$.*

Proof. Each node v with label "+" is joined to an unsaturated node by an even length path whose edges alternately have $x_j = 1$ and $x_j = 0$. By redefining x_j to be $1 - x_j$ we obtain another optimum solution to (MP) which leaves v unsaturated. By (2.2), we have therefore that $v \in V_0$. Therefore, by (4.2) and (4.3) we must have each node labelled "−" in V_1. Since our solution to (LP) defined above is fractional for all unlabelled nodes, no other nodes belong to V_0 or V_1. Finally, suppose j joins $u \in V_0$ and $v \in V_1$. Let $\hat{x}$ be a maximum fractional matching which leaves u unsaturated. Redefining $\hat{x}_j = 1$ and $\hat{x}_k = 0$ for all $k \in \delta(v) \setminus \{j\}$, we obtain the desired matching. □

If G has a perfect fractional matching, then $V_0 = V_1 = \emptyset$. If not, then the above shows that V_0 is an independent set of nodes, V_1 is the neighbour set and $|V_0| > |V_1|$. This inequality holds because initially some nodes have the label "+", and none have the label "−". Moreover, any time a node is given the label "−" (Step 4) another node is given the label "+".

For any $S \subseteq V$, we let $\Gamma(S)$ denote the neighbour set of S, i.e., the set of all nodes not in S but adjacent to a node of S. It follows from properties (4.1)–(4.3) plus the existence of the paths implied by the labellings that for any $X \subseteq V_1$, $|\Gamma(X) \cap V_0| > |X|$. It is easy to see that this is sufficient to show that V_0 is contained in every maximum cardinality stable set.

Proposition 4.2. *Let I be any stable set of a graph G such that $|\Gamma(X) \cap I| > |X|$ for all $X \subseteq \Gamma(I)$. Then every maximum cardinality stable set contains I.*

Proof. Let S be a stable set which does not contain I. Then $I' = S \cup (I \setminus S) \setminus (\Gamma(I) \cap S)$ is a stable set and since $|I \setminus S| \geq |\Gamma(\Gamma(I) \cap S) \cap I| > |\Gamma(I) \cap S|$, we have $|I'| > |S|$. □

Corollary 4.3. (Hammer et al. [5]). *V_0 is contained in every maximum cardinality stable set.*

5. Decompositions

We have already seen that the Fractional Matching Algorithm of the previous section also finds the sets V_0 and V_1. We let F (for "fractional") denote the set of nodes i having $y_i=\frac{1}{2}$ in every optimum solution to (LP). In this section we describe an algorithm which finds F and constructs an optimum solution to (LP) which is fractional only for the nodes of F. In fact the algorithm will also prove that each component of $G[F]$ is 2-bicritical by giving an ear-pendant decomposition, and hence will prove that every 2-bicritical graph has such a decomposition.

Let $E^+=\{j\in E\colon j$ is active in some optimum solution to (MP)$\}$. Let G^+ be the graph obtained from G by deleting all edges not in E^+. Note that it is easy to check whether an edge j belongs to E^+. We can define a cost function c^j by

$$c_k^j=\begin{cases}1, & \text{if } k\in E\setminus\{j\},\\ 1+\delta\ \ (0<\delta<\frac{1}{2}), & \text{if } k=j.\end{cases}$$

Then if we find a fractional matching x which maximizes c^jx and such that $x_j>0$, it will be a maximum fractional matching with j active, if such a matching exists.

Alternatively, suppose $x^*(E)=\eta$ for a maximum fractional matching x^*. We can determine whether any optimum solution x to (MP) has $x_j=1$ by first deleting j and its two ends, then finding a maximum fractional matching $\hat{x}$ in the resulting graph. If the sum of the components of $\hat{x}$ is $\eta-1$, then such an x exists. We can find whether there exists an optimum solution x to (MP) with $x_j=\frac{1}{2}$ by deleting each end of j in turn and finding a maximum fractional matching in each of the resulting graphs. Let x^u and x^v be these two matchings. If the sum of the components of both x^u and x^v is $\eta-\frac{1}{2}$, then such an x exists. It is defined by $\hat{x}=\frac{1}{2}(x^u+x^v)$ and $\hat{x}_j=\frac{1}{2}$.

Lovász and Plummer [10, Theorem 6.3.5] showed that G is a KEG if and only if G^+ is bipartite. In fact a stronger result is true, which also extends [15, Theorem 2.9].

Theorem 5.1. *Let G be a graph and let G^+ be as defined above. Then each nonbipartite component of G^+ is 2-bicritical. G is a KEG if and only if every component of G^+ is bipartite. G is 2-bicritical if and only if no component of G^+ is bipartite. The set F of nodes receiving fractional values in every optimum solution to (LP) is the nodeset of the nonbipartite components.*

Proof. By complementary slackness condition (2.1), every optimum solution $\hat{y}$ to (LP) satisfies $\hat{y}_u+\hat{y}_v=1$ for every pair of nodes u,v joined by an edge of E^+. Therefore, for any component K of G^+, either all nodes will have integer or all nodes will have fractional values. If K is nonbipartite, then considering the equations (2.1) corresponding to the edges of an odd cycle, we see that we must have $\hat{y}_i=\frac{1}{2}$ for all nodes of K, for every optimum solution $\hat{y}$ of (LP). Therefore by [15, Theorem 2.4], K must be 2-bicritical. Since a bipartite graph cannot be 2-bicritical, it also follows from [15, Theorem 2.4] that there is an optimum solution y^* to (LP) which is integer for all nodes of bipartite components of G^+. Therefore if G^+ has no nonbipartite components, y^* will be integer valued and so G is a KEG. If G^+ has no bipartite components, then $y_v^* = \frac{1}{2}$ for all $v \in V$ and G^+ is 2-bicritical. $\square$

Note that this theorem would not be true if we let E^+ be the set of edges active in maximum *integer* matchings. For example, the blossom pair of Fig. 1(a) has a unique maximum integer matching but every edge is active in a perfect fractional matching and, indeed, the graph is 2-bicritical.

We now describe an efficient labelling algorithm which will construct an optimal solution y^* to (LP) with the property that the nodes receiving fractional values will be the nodes of 2-bicritical components of G^+. It will show that these components are 2-bicritical by constructing ear-pendant decompositions as described in Section 3.

This algorithm will provide a second proof of Theorem 5.1. For by complementary slackness, the only edges which can belong to E^+ are those which join two nodes u, v such that $y_u^* = y_v^* = \frac{1}{2}$ or $y_u^* = 0$ and $y_v^* = 1$. By Corollary 3.2, all of the first class of edges is in G^+. Trivially, the second class of edges forms a bipartite subgraph.

We say that a subgraph $G' = (V', E')$ of G is *insulated* if there exists a maximum fractional matching x of G satisfying $x_j = 0$ for all $j \in \delta(V')$, and which induces a perfect fractional matching of G'. (This does *not* imply $\delta(V') \subseteq E \setminus E^+$.) Also, if G' is any graph whose edge set is a subset of E and x is a fractional matching of G, we will mildly abuse notation by also letting "x" refer to the restriction of this vector to the edge set of G'.

Decomposition Algorithm

Input. A graph $G = (V, E, \psi)$.

Output.

The sets $W_0 \cup V_0$, $W_1 \cup V_1$ and F of nodes receiving values 0, 1 and $\frac{1}{2}$ respectively in every optimal solution to (LP); (5.1)

an optimal solution to (LP) which is integer valued for all $i \in V \setminus F$; (5.2)

an ear-pendant decomposition of each component of $G[F]$. (5.3)

Step 0 (Initialization). Apply the maximum fractional matching algorithm of Section 4 to G to obtain an optimum solution $\hat{x}$ to (MP) as well as the sets V_0 and V_1. Delete these nodes and all incident edges. Now $\hat{x}$ is a perfect fractional matching of the remaining graph G. Let $W_0 = W_1 = \emptyset$.

Step 1. If $\hat{x}$ is integer valued, i.e., $\hat{x}_j \in \{0, 1\}$ for all $j \in E$, then go to Step 3. Otherwise, let C be an odd cycle of G for which j has $\hat{x}_j = \frac{1}{2}$. Note that C is a 2-bicritical insulated subgraph of G. Shrink C to form a pseudonode p. Now p is the only unsaturated node of G.

Step 2 (Grow 2-bicritical component). Apply the fractional matching algorithm to G starting with $\hat{x}$. Since G has only one unsaturated node, there are only three possible outcomes: an augmenting path of type 2 or 3 is found or $\hat{x}$ is maximum. If we find an augmenting path, this gives us an ear or pendant we can add to the graph shrunk to form p, thereby by Proposition 3.4 obtaining a larger insulated 2-bicritical graph C'. Shrink C' to form p and return to Step 2. If $\hat{x}$ is maximum, then the graph C shrunk to form p is a 2-bicritical component of the (LP)-decomposition. Let $\hat{W}_0$ be the set of nodes of G other than p with label "+" and let $\hat{W}_1$ be the set of nodes with label "−". Add $\hat{W}_0$ to W_0 and add $\hat{W}_1$ to W_1. Delete C, $\hat{W}_0$ and $\hat{W}_1$ and if G is nonempty, return to Step 1. Otherwise terminate.

Step 3 (König–Egerváry test). ($\hat{x}$ is a perfect integer valued matching of G.) Choose an edge j for which $\hat{x}_j = 1$ and let $\{u, v\} = \psi(j)$. Temporarily delete u and all incident edges from G. Now v is the only node unsaturated by $\hat{x}$. Apply the fractional matching algorithm. If $\hat{x}$ is maximum, then let $\hat{W}_0$ be the set of nodes (including v) with label "+" and let $\hat{W}_1$ be the set of nodes with label "−", plus u. Add $\hat{W}_0$ to W_0 and add $\hat{W}_1$ to W_1. Delete $\hat{W}_1$ and $\hat{W}_0$ from G and if G is nonempty, return to Step 3; if G is now empty, stop.

Otherwise, an augmenting path of the third type has been found, consisting of an even length path π from v to a node w plus an odd cycle C which contains w, but is otherwise disjoint from π. Let k be the edge of $\delta(w)$ with $\hat{x}_k = 1$ and let t be the end of k different from w. We now repeat Step 3, but this time replace u and delete w. Again, if $\hat{x}$ is optimal we increase W_0 and W_1, delete all labelled nodes and return to Step 3, if the resulting G is nonempty.

However, suppose an augmenting path is found consisting of an even length path π' from t to a node w' and an odd cycle C' containing w'. Go to Step 4 where we construct an insulated blossom pair.

Step 4 (Construct blossom pair). Let $\bar{\pi}$ be the concatenation of the edge k and the path π'. Then $\bar{\pi}$ is an odd length path joining node w of C with node w' of C'. For each of these cycles, the edges l alternately have $\hat{x}_l = 0$ and $\hat{x}_l = 1$, starting with w and w' respectively. Moreover, $\bar{\pi}$ has only w' in common with C'. If the only node of C belonging to $\bar{\pi}$ is w, then we are done; C, C' and $\bar{\pi}$ form a blossom pair. Suppose this is not the case. (See Fig. 7.)

Note that w and w' are distinct. Let $\bar{w}$ be the first node of $\bar{\pi}$ other than w which belongs to C. (If w' is in C, then so too is the preceding node of $\bar{\pi}$, so $\bar{w} \neq w'$.) Then the portion of $\bar{\pi}$ from w to $\bar{w}$ plus the odd length path in C from $\bar{w}$ to w forms an odd cycle $\bar{C}$.

The portion of $\bar{\pi}$ from $\bar{w}$ to w' is an odd length path $\hat{\pi}$ joining the node $\bar{w}$ of $\bar{C}$ to the node w' of C'. Moreover, $\hat{x}$ induces a perfect matching of the union of $\bar{C}$, C' and $\hat{\pi}$. We can now repeat this construction as necessary with $\bar{C}$, C', and $\hat{\pi}$ until an insulated blossom pair is obtained. Since $\hat{\pi}$ always has at least two fewer edges than $\bar{\pi}$, this happens after at most $\frac{1}{2}(l-1)$ iterations, where l is the length of $\bar{\pi}$.

We now have an insulated blossom pair, which by Proposition 3.3 is 2-bicritical. Shrink this graph to form a pseudonode p and go to Step 2.

End.

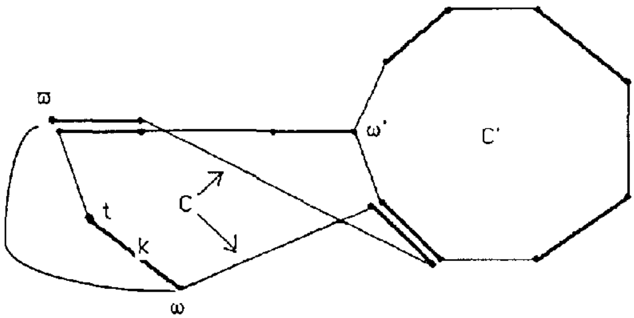

Fig. 7.

Now we prove that this algorithm produces output satisfying (5.1)–(5.3). Let F be the set of nodes belonging to the 2-bicritical components constructed, and let W_0 and W_1 be as constructed by the algorithm. Define $\hat{y}$ by

$$\hat{y}_i = \begin{cases} \frac{1}{2} & \text{for } i \in F, \\ 1 & \text{for } i \in W_1 \cup V_1, \\ 0 & \text{for } i \in W_0 \cup V_0. \end{cases}$$

First we verify that $\hat{y}$ is a feasible solution to (LP). We must show that each node in $W_0 \cup V_0$ is only adjacent to nodes of $W_1 \cup V_1$. Since $V_1 = \Gamma(V_0)$ and no two nodes of V_0 are adjacent, this follows for V_0. (See (4.1)–(4.3).) Nodes are added to W_0 in Steps 2 and 3 of the algorithm. In both cases, properties (4.1)–(4.3) of the labelling algorithm assure that all nodes of $\hat{W}_0$ are either adjacent to nodes of $\hat{W}_1$ or to nodes which were added to W_1 at an earlier step. So $\hat{y}$ is feasible. Also, if we let $E^= = \{j \in E: \hat{y}(\psi(j)) = 1\}$ then $\hat{x}_j$ is active only for $j \in E^=$ and every node of $F \cup W_1 \cup V_1$ is saturated. Therefore by complementary slackness (2.1) and (2.2), $\hat{y}$ is an optimum solution to (LP).

The algorithm has constructed an ear-pendant decomposition of each component of $G[F]$, so these graphs are 2-bicritical, and hence nonbipartite. By Corollary 3.2 each edge of $G[F]$ is in E^+, so as we saw in the proof of Theorem 5.1, every node $i \in F$ has $y_i = \frac{1}{2}$ in every optimal solution to (LP).

Now we note some consequences of this algorithm. First, it generalizes an earlier labelling algorithm of Deming [3] for determining whether a graph is a KEG. His algorithm would stop if an insulated blossom pair was found; ours proceeds to grow it into a 2-bicritical component of G^+.

Second, we obtain an ear-pendant decomposition of any 2-bicritical graph. We can restrict the ear-pendant decomposition further by requiring it to start with an odd homeomorph of K_3 or K_4, rather than an arbitrary blossom pair. This can be deduced using Lovász's characterization of KEGs (Theorem 2.3) but we present here a direct proof that every blossom pair contains an insulated odd homeomorph of one of Lovász's forbidden graphs. This also enables Theorem 2.3 to be deduced from Theorem 2.2.

Lemma 5.2. *Let M be a perfect matching of G and let B be a blossom pair in G with respect to M. Then B contains a subgraph H which is an odd homeomorph of one of the graphs of Fig. 2, and which is insulated with respect to a perfect matching of G.*

Proof. B consists of odd cycles C_1, C_2 and an odd path P joining a node w_1 of C_1 to a node w_2 of C_2. The only nodes of P belonging to C_1 and C_2 are w_1 and w_2. Moreover, w_1 is not in C_2 and w_2 is not in C_1. The edges of P, C_1, C_2 alternate with respect to M.

If C_1 and C_2 are node disjoint, then B is an odd subdivision of the graph of Fig. 2(b) and we are done. Otherwise, the alternating structure of the cycles implies that whenever they have a node in common, they will have an odd-length subpath in common, whose edges alternately are in M and not in M. Choose a direction of traversal of C_2 starting with w_2. Let $\pi_1, \pi_2, \ldots, \pi_k$ be the sequence of common subpaths encountered in this traversal, where we will assume that each path is oriented in the direction of traversal. We call these *intersection paths.* We proceed by induction on k.

If $k=1$, then B is an odd subdivision of K_4 and we are done. So assume $k\geq 2$. Choose the direction of traversal for C_1 which will cause π_1 to be traversed in a forward direction. We say that π_i is a *forward* path if its orientation agrees with that of C_1, and otherwise a *reverse* path. The following notation will be useful. For any π_i, let $s(\pi_i)$ and $e(\pi_i)$ be the start and end nodes respectively. For nodes u, v of C_i, where u occurs before v, we let $C_i(u, v)$ denote the path equal to the portion between u and v. Finally if paths π and σ have an identical end node, we may write $\pi \| \sigma$ to denote the concatenation of π and σ.

We will consider several cases, but the following is a useful preliminary fact:

> Suppose that there exist i, j such that $1\leq i<j\leq k$, π_i is a forward path and π_j occurs after π_i on C_1 and no node of C_2 occurs on C_1 between the end of π_i and π_j. Then we can assume that π_j is a reverse path. (5.4)

For if π_j were also a forward path we could replace $C_2(e(\pi_i), s(\pi_j))$ with $C_1(e(\pi_i), s(\pi_j))$ and get a new blossom pair such that the odd cycles have fewer common paths, then apply induction.

Case 1. *Some intersection path occurs on C_1 after π_1 but before w_1.* Let π_i be the first such path encountered (as C_1 is traversed in the forward direction). By (5.4) we can assume that π_i is a reverse path. Let

$$C_1' = P \| C_1(w_1, s(\pi_1)) \| C_2(w_2, s(\pi_1)),$$

$$C_2' = C_2(e(\pi_1), e(\pi_i)) \| C_1(e(\pi_1), e(\pi_i)),$$

$$P' = \pi_1.$$

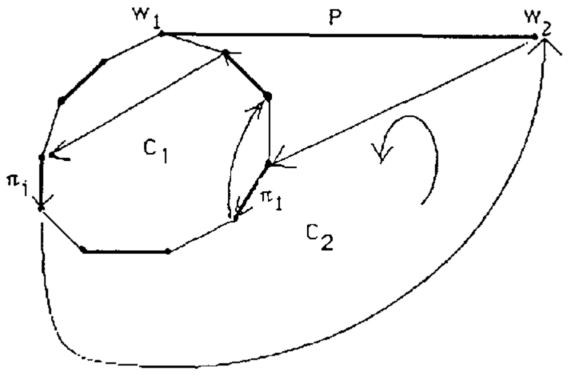

Fig. 8.

(See Fig. 8. In this and subsequent figures, edges of C_2 are drawn as arrows indicating the direction of traversal.) Then C_1', C_2' and P are an insulated blossom pair with respect to M and C_1' and C_2' have fewer intersection paths so the result follows by induction.

Case 2. *No node of C_2 occurs on C_1 between the end of π_1 and w_1.* Since $k \geq 2$, the path π_2 exists, and so precedes π_1 on C_1.

(a) *π_2 is a reverse path.* Suppose some other intersection path occurs on C_1 before π_2. Let π_i be the last such path encountered, as C_1 is traversed from w_1 to $e(\pi_2)$. If π_i were a reverse path, we could reverse the direction of traversal of C_2, then apply (5.4). So assume π_i is a forward path. (See Fig. 9.) Let

$$C_1' = C_2(s(\pi_1), s(\pi_2)) \,\|\, C_1(s(\pi_2), s(\pi_1)),$$

$$C_2' = C_2(e(\pi_2), e(\pi_i)) \,\|\, C_1(e(\pi_i), e(\pi_2)),$$

$$P' = \pi_2.$$

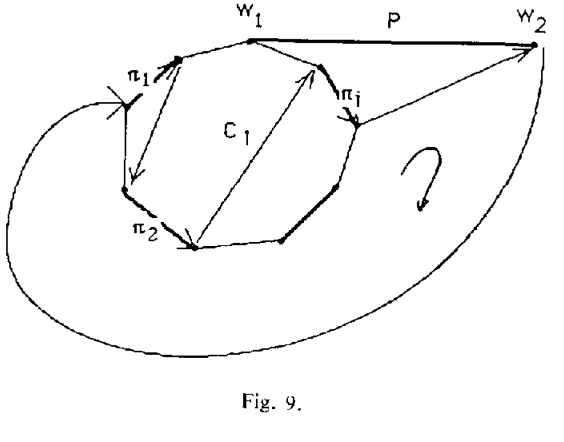

Fig. 9.

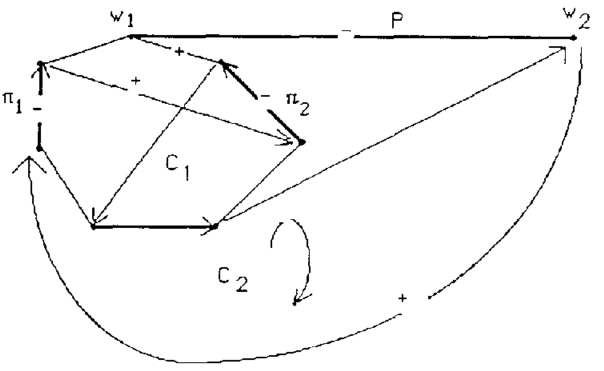

Fig. 10.

Then again the result follows by applying induction to the blossom pair comprised of C_1', C_2', P'.

Therefore, assume that no node of C_2 lies on C_1 between w_1 and the end of π_2. (See Fig. 10.)

Let $\hat{C} = P \| C_1(w_1, e(\pi_2)) \| C_2(w_2, e(\pi_2))$. Then $\hat{C}$ is an alternating cycle with respect to M. Let M' be the symmetric difference of M and $\hat{C}$. Let

$$\begin{aligned} C_1' &= P \| C_2(e(\pi_2), w_2) \| C_1(w_1, e(\pi_2)), \\ C_2' &= C_2(s(\pi_1), s(\pi_2)) \| C_1(s(\pi_2), s(\pi_1)), \\ P' &= C_2(w_2, s(\pi_1)). \end{aligned}$$

Then C_1', C_2' and P' are a blossom pair insulated with respect to M' having fewer intersection paths (for π_2 is no longer present). The result follows by induction.

(b) π_2 *is a forward path*. Suppose no edges of C_2 lie on C_1 between the end of π_2 and the start of π_1. (See Fig. 11.)

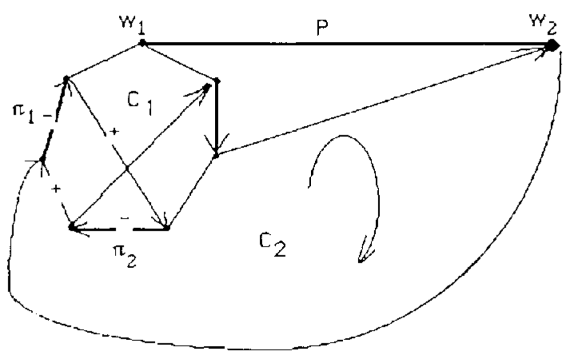

Fig. 11.

Let $\hat{C}=C_2(s(\pi_1),e(\pi_2))\,\|\,C_1(e(\pi_2),s(\pi_1))$. Then $\hat{C}$ is an alternating cycle with respect to M. Let M' be the symmetric difference of M and $\hat{C}$. Let

$$C_1' = C_1(e(\pi_1), w_1)\,\|\,C_1(w_1, s(\pi_2))\,\|\,C_2(e(\pi_1), s(\pi_2)),$$

$$C_2' = C_2(e(\pi_2), w_2)\,\|\,C_2(w_2, s(\pi_1))\,\|\,C_1(e(\pi_2), s(\pi_1)).$$

Then C_1', C_2' and P give an insulated blossom pair with respect to M', and again we apply induction.

So suppose there is an intersection path in the portion of C_1 between $e(\pi_2)$ and $s(\pi_1)$. Let π_i be the first such path. By (5.4), we can assume π_i is a reverse path on C_1. See Fig. 12.

Let $\hat{C}=C_2(w_2,e(\pi_1))\,\|\,C_1(e(\pi_1),w_1)\,\|\,P$. Then $\hat{C}$ is an alternating cycle. Let M' be the symmetric difference of $\hat{C}$ and M. Let

$$C_1' = C_2(e(\pi_2), e(\pi_i))\,\|\,C_1(e(\pi_2), e(\pi_i)),$$

$$C_2' = C_1(e(\pi_1), w_1)\,\|\,C_1(w_1, s(\pi_2))\,\|\,C_2(e(\pi_1), s(\pi_2)),$$

$$P' = \pi_2.$$

Then C_1', C_2', P' comprises an insulated blossom pair with respect to M', with fewer than k intersection paths, and the result follows by induction. □

Note that the proof of the preceding lemma can easily be adapted to give a polynomially bounded algorithm for finding an insulated odd homeomorph of one of the graphs of Fig. 2. Since an odd homeomorph of the graph of Fig. 2(b) can be constructed by adding a single pendant to an odd cycle, we could modify Step 4 of the decomposition algorithm so that the starting point for the construction of a 2-bicritical component would be an odd homeomorph of K_3 or K_4. Thus we have the following:

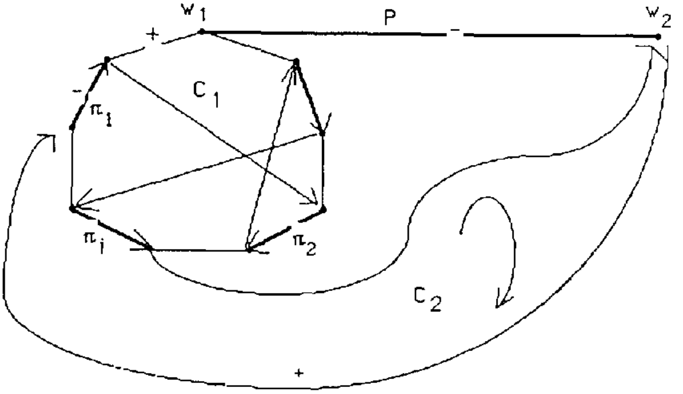

Fig. 12.

Theorem 5.3. *A connected graph $G=(V,E,\psi)$ is 2-bicritical if and only if it has an ear-pendant decomposition $G_0, G_1, \ldots, G_p = G$ where G_0 is an odd homeomorph of K_3 or K_4.*

Lemma 5.2 also provides a direct derivation of Theorem 2.3 from Theorem 2.2.

Finally, we note that the decomposition algorithm has complexity $O(|V| \cdot |E|)$, assuming $|E| > |V|$. The labelling procedure of the Fractional Matching Algorithm can be carried out in $O(|E| + |V|)$. We only have to perform this at most twice before a node is either made part of a shrunk subgraph, i.e., assigned to F, or else assigned to W_0 or W_1. Since the algorithm terminates as soon as all nodes are classified, we have our bound.

References

[1] J.-M. Bourjolly, Integral and fractional node-packings and pseudo-Boolean programming, Doctoral Dissertation, University of Waterloo, Ontario (1986).

[2] J.-M. Bourjolly, P.L. Hammer and B. Simeone, Node-weighted graphs having the König–Egerváry property, Math. Programming Study 22 (1984) 44–63.

[3] R.W. Deming, Independence numbers of graphs: An extension of the König–Egerváry theorem, Discrete Math. 27 (1979) 23–34.

[4] J. Edmonds, Paths, trees and flowers, Canad. J. Math. 17 (1965) 449–467.

[5] P.L. Hammer, P. Hansen and B. Simeone, Vertices belonging to all or to no maximum stable sets of a graph, SIAM J. Algebraic Discrete Methods 3 (1982) 511–522.

[6] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller and J.W. Thatcher, eds., Complexity of Computer Computations (Plenum, New York, 1972) 85–104.

[7] L. Lovász, Minimax theorems for hypergraphs, in: C. Berge and D.K. Ray-Chaudhuri, eds., Hypergraph Seminar 1972, Lecture Notes 411 (Springer, Berlin, 1972).

[8] L. Lovász, A note on factor-critical graphs, Studia Sci. Math. Hungar. 7 (1972) 279–280.

[9] L. Lovász, Ear-decompositions of matching-covered graphs, Combinatorica 3 (1983) 105–118.

[10] L. Lovász and M.D. Plummer, Matching Theory, Annals of Discrete Mathematics 29 (North-Holland, Amsterdam, 1986).

[11] E. Korach, On dual integrality, min max equalities and algorithms in combinatorial linear programs, Doctoral Dissertation, University of Waterloo, Ontario (1982).

[12] G.L. Nemhauser and L.E. Trotter, Vertex packings: Structural properties and algorithms, Math. Programming 8 (1975) 232–248.

[13] J.-C. Picard and M. Queyranne, On the integer valued variables in the linear vertex packing problem, Math. Programming 12 (1977) 97–101.

[14] J.-C. Picard and M. Queyranne, Vertex packings (VLP): reductions through alternate labelling, Tech. Rept. EP75-R-47, Ecole Polytechnique, Université de Montréal, Québec (1975).

[15] W.R. Pulleyblank, Minimum node-covers and 2-bicritical graphs, Math. Programming 17 (1979) 91–103.

[16] W.R. Pulleyblank and J. Edmonds, Facets of 0 1 matching polyhedra, in: C. Berge and D.K. Ray-Chaudhuri, eds., Hypergraph Seminar (Springer, Berlin, 1974) 214–242.

[17] F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, J. Combin. Theory Ser. B 27 (1979) 228–229.