ECE 470 LECTURE 5

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PLAN

- rotation matrices rotate things
- axis/angle representation
- angular velocity (and linear velocity)
- exponential coordinates

DOT PRODUCT

$$v \cdot w = \|v\| \, \|w\| \cos \theta_{vw} = (v^o)^T w^o$$

CROSS PRODUCT

$$v \times w = \left(\|v\| \, \|w\| \sin \theta_{vw} \right) n_{vw}$$

$$(v \times w)^o = [v^o] w^o$$

Suppose a and v are vectors and that $\|a\| = 1$.

What is u?

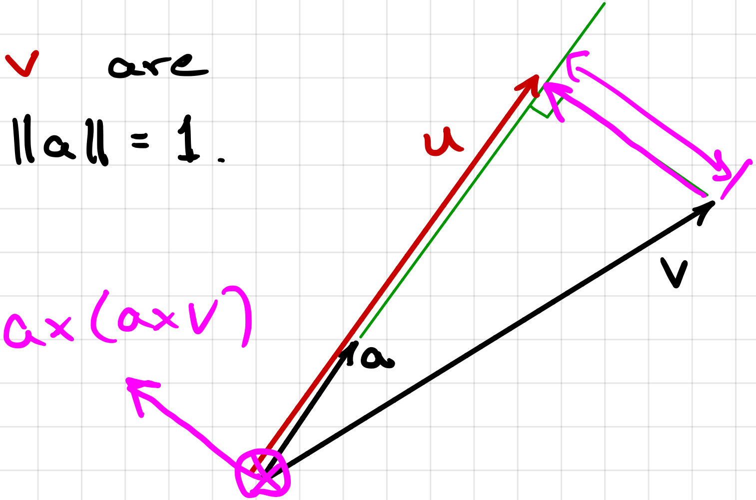
A $a \cdot v$

B $(a \cdot v)a$

C $v + a \times (a \times v)$

D A and C

E B and C



REMEMBER: ORIENTATION OF FRAME 1 $\longleftrightarrow$ x_1, y_1, z_1

"axes are unit length"

$$x_1 \cdot x_1 = 1 \longrightarrow (x_1^0)^T x_1^0 = 1$$

$$y_1 \cdot y_1 = 1 \qquad (y_1^0)^T y_1^0 = 1$$

$$z_1 \cdot z_1 = 1 \qquad (z_1^0)^T z_1^0 = 1$$

$$R_1^0 = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$a^2 + b^2 + c^2 = 1$$

$$\vdots$$

"axes are orthogonal"

$$x_1 \cdot y_1 = 0 \qquad (x_1^0)^T y_1^0 = 0$$

$$y_1 \cdot z_1 = 0 \qquad (y_1^0)^T z_1^0 = 0$$

$$z_1 \cdot x_1 = 0 \qquad (z_1^0)^T x_1^0 = 0$$

"axes satisfy the right-hand rule"

$$(x_1 \times y_1) \cdot z_1 = 1 \qquad ([x_1^0]\, y_1^0)^T z_1^0 = 1$$

How many numbers are necessary to completely specify the orientation of a frame?

A 1

B 2

C 3

D 4

E 9

WHAT IS ... x_0^0

... in the coordinates of frame 0.

... of frame 0 ...

the x axis ...

A undefined (it makes no sense)

B $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

E $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

REMINDER ABOUT MATRIX MULTIPLICATION

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \leftarrow m \times n$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \leftarrow n \times 1$$

$$Av = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

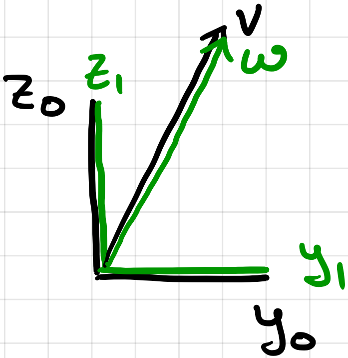
$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix}$$

$$x_0^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

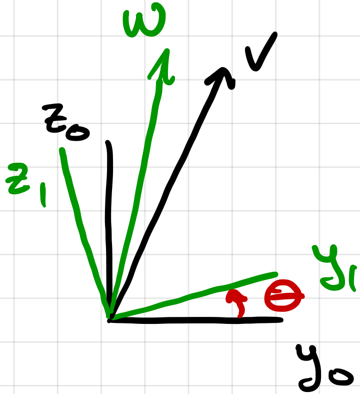
$$\boxed{R_1^0}\boxed{x_0^0} = (x_1^0)1 + (y_1^0)(0) + (z_1^0)(0)$$

$$= \boxed{x_1^0}$$

BEFORE

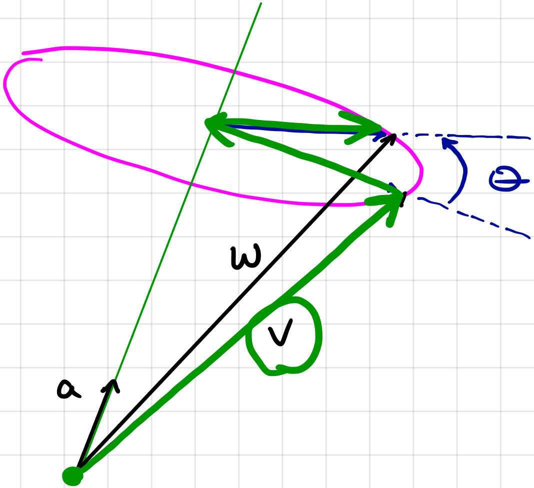


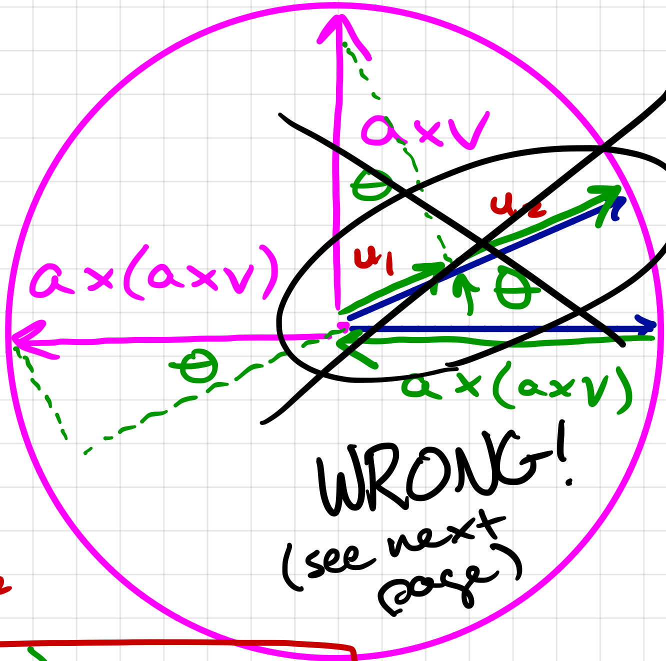
AFTER



$$w^0 = R_1^0 v^0$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$





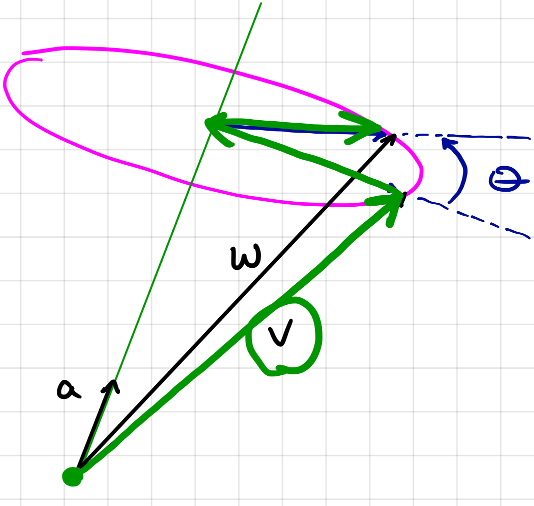
$$w = v + a\times(a\times v)$$

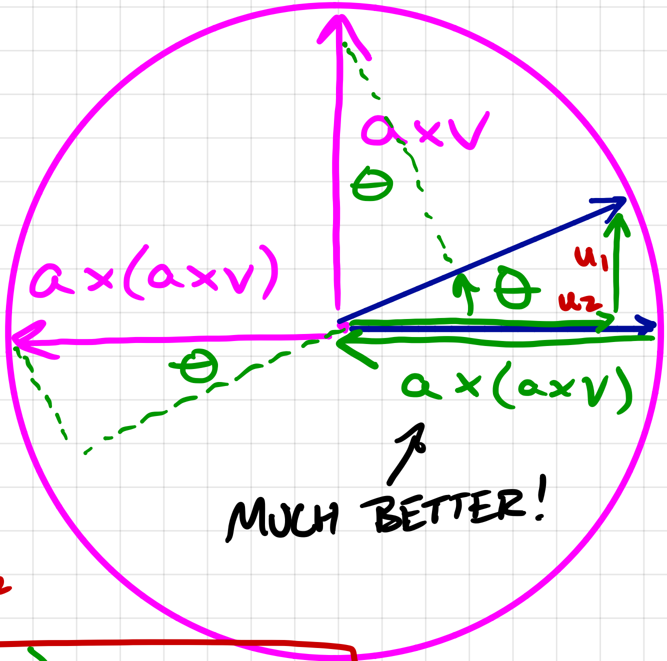
$$+ \overbrace{(\sin\theta)\, a\times v}^{u_1} \overbrace{- (\cos\theta)\, a\times(a\times v)}^{u_2}$$

$$= v + (\sin\theta)\, a\times v + (1-\cos\theta)\, a\times(a\times v)$$

$$w^0 = v^0 + (\sin\theta)[a^0]v^0 + (1-\cos\theta)[a^0]^2 v^0$$

$$= \boxed{\left(I + (\sin\theta)[a^0] + (1-\cos\theta)[a^0]^2\right)} v^0$$





$$w = v + a\times(a\times v) + \overbrace{(\sin\theta)\, a\times v}^{u_1} - \overbrace{(\cos\theta)\, a\times(a\times v)}^{u_2}$$

$$= v + (\sin\theta)\, a\times v + (1-\cos\theta)\, a\times(a\times v)$$

$$w^o = v^o + (\sin\theta)[a^o]v^o + (1-\cos\theta)[a^o]^2 v^o$$

$$= \boxed{\left(I + (\sin\theta)[a^o] + (1-\cos\theta)[a^o]^2\right)} v^o$$

AXIS/ANGLE REPRESENTATION

$$(a^0, \theta) \rightarrow R_1^0$$

$$R_1^0 = I + (\sin\theta)[a^0] + (1-\cos\theta)[a^0]^2$$

$$R_1^0 \rightarrow (a^0, \theta)$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\left(\mathrm{Trace}(R_1^0) - 1\right)\right)$$

$$[a^0] = \frac{1}{2\sin\theta}\left(R_1^0 - (R_1^0)^T\right)$$

this works when $\theta \in (0, \pi)$ ...

see text for how to handle $\theta = 0$ and $\theta = \pi$ ...

note that (a, θ) and $(-a, -\theta)$ produce the same rotation