Operations Research II: Algorithms

Branch & Bound and Heuristic Algorithms

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Introduction

- In some cases, variables must only take **integer values**.
- The subject of formulating and solving models with integer variables is **Integer Programming** (IP).
  - An IP is typically a linear IP (LIP).
  - If the objective function or any functional constraint is nonlinear, it is a nonlinear IP (NLIP).
  - We will focus on linear IP in this course.

Introduction

- The **branch-and-bound algorithm** finds an **optimal** solution for any IP.
  - It "decomposes" an IP to multiple LPs, solve all the LPs, and compares those outcomes to reach a conclusion.
  - Each LP is solved separately (with the simplex method or other ways).
  - In general, the process may take a very long time.
- As finding an optimal solution for an IP may be too time-consuming, we often look for a **near-optimal** feasible solution instead in practice.
  - An algorithm that generates a **feasible** solution in a **short time** is called a **heuristic algorithm**.
  - **Hopefully** it is **near-optimal**.
  - A good heuristic algorithm does not always work, but it works for most of the time.

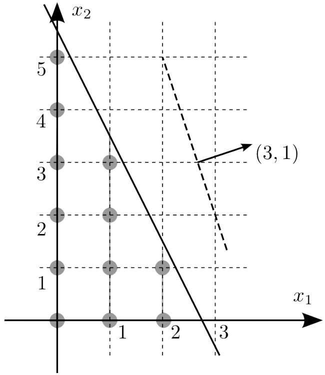
Road map

- **Linear relaxation**.
- The branch-and-bound algorithm.
- Solving the knapsack problem.
- Heuristic algorithms.

Solving an IP

- Suppose we are given an IP, how may we solve it?
- The simplex method does not work!
  - The feasible region is not "a region".
  - It is **discrete**.
  - There is no way to "move along edges".
- But all we know is how to solve LPs. How about solving a **linear relaxation** first?

$$\begin{array}{rrcrcr} \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 11 \\ & & x_i \in \mathbb{Z}_+ & \forall i = 1, 2. & & \end{array}$$



Definition 1 (Linear relaxation)

For a given IP, its linear relaxation is the resulting LP after removing all the integer constraints.

Linear relaxation

- What is the linear relaxation of

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & 3x_2 & \leq & 10 \\ & 2x_1 & - & x_2 & \geq & 5 \\ & & & \multicolumn{3}{l}{x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2?} \end{array}$$

  - $\mathbb{Z}$ is the set of all integers. $\mathbb{Z}_+$ is the set of all nonnegative integers.
- The linear relaxation is

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & 3x_2 & \leq & 10 \\ & 2x_1 & - & x_2 & \geq & 5 \\ & & & \multicolumn{3}{l}{x_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

Linear relaxation

- For the knapsack problem

$$\begin{array}{rcccccccccc} \max & 16x_1 & + & 22x_2 & + & 12x_3 & + & 8x_4 & & \\ \text{s.t.} & 5x_1 & + & 7x_2 & + & 4x_3 & + & 3x_4 & \leq & 10 \\ & & & \multicolumn{5}{c}{x_i \in \{0,1\} \quad \forall i = 1, ..., 4,} & & \end{array}$$

  the linear relaxation is

$$\begin{array}{rcccccccccc} \max & 16x_1 & + & 22x_2 & + & 12x_3 & + & 8x_4 & & \\ \text{s.t.} & 5x_1 & + & 7x_2 & + & 4x_3 & + & 3x_4 & \leq & 10 \\ & & & \multicolumn{5}{c}{x_i \in [0,1] \quad \forall i = 1, ..., 4,} & & \end{array}$$

  - $x_i \in [0,1]$ is equivalent to $x_i \geq 0$ and $x_i \leq 1$.

Linear relaxation provides a bound

- For a **minimization** IP, its linear relaxation provides a **lower bound**.

Proposition 1

Let z^* and z' be the objective values associated to optimal solutions of a minimization IP and its linear relaxation, respectively, then $z' \leq z^*$.

Proof. They have the same objective function. However, the linear relaxation's feasible region is (weakly) larger than that of the IP. □

- For a **maximization** IP, linear relaxation provides an **upper bound**.

Linear relaxation may solve the IP

- If we are lucky, the linear relaxation may be infeasible or unbounded.
  - The IP is then infeasible or unbounded.
- If we are lucky, an optimal solution to the linear relaxation may be **feasible** to the original IP. When this happens, the IP is solved:

  Proposition 2

  Let x' be an optimal solutions to the linear relaxation of an IP. If x' is feasible to the IP, it is optimal to the IP.

  Proof. Suppose x' is not optimal to the IP, there must be another feasible solution x'' that is better. However, as x'' is feasible to the IP, it is also feasible to the linear relaxation, which implies that x' cannot be optimal to the linear relaxation. □
- What if we are **unlucky**?

Rounding a fractional solution

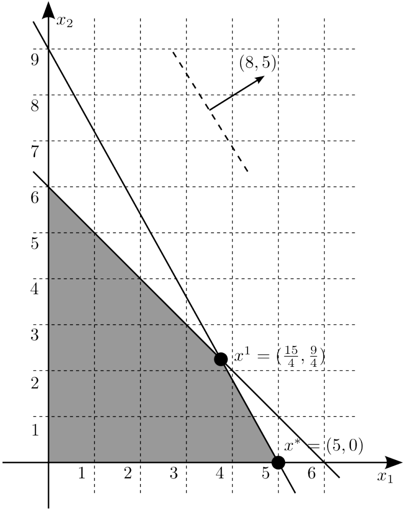
- Suppose we solve a linear relaxation with an LR-optimal solution x'.
  - "LR-optimal" means x' is optimal to the linear relaxation.
- x', however, has at least one variable violating the integer constraint in the original IP.
- We may choose to **round** the variable.
  - Round up or down?
  - Is the resulting solution always feasible?
  - Will the resulting solution be close to an IP-optimal solution x^*?

Rounding a fractional solution

- Consider the following IP

$$\begin{array}{rrcrcl} \max & 8x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & 9x_1 & + & 5x_2 & \leq & 45 \\ & \multicolumn{5}{c}{x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2.} \end{array}$$

- $x^* = (5, 0)$ is IP-optimal.
- But $x^1 = (\frac{15}{4}, \frac{9}{4})$ is LR-optimal!
  - Rounding up any variable results in infeasible solutions.
  - None of the four grid points around x^1 is optimal.
- We need a way that guarantees to find an optimal solution.

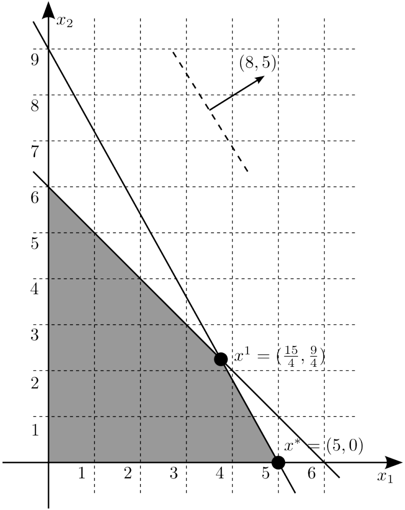


Road map

- Linear relaxation.
- **The branch-and-bound algorithm**.
- Solving the knapsack problem.
- Heuristic algorithms.

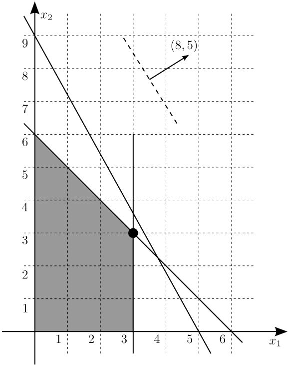
Rounding a fractional solution

- $x^1 = (\frac{15}{4}, \frac{9}{4})$ is LR-optimal.
  - Rounding up or down x_1 (i.e., adding $x_1 = 4$ or $x_1 = 3$ into the program) both **fail** to find the optimal solution.
  - Because we eliminate too many feasible points!
  - Instead of adding equalities, we should add **inequalities**.
- What will happen if we add $x_1 \geq 4$ or $x_1 \leq 3$ into the program?
- We will **branch** this problem into two problems, one with an additional constraint.

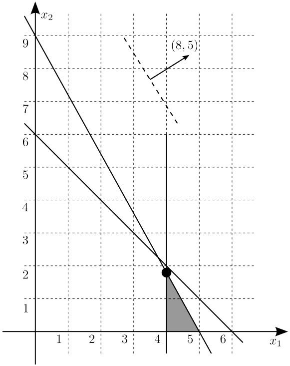


Rounding a fractional solution

If we add $x_1 \leq 3$:



If we add $x_1 \geq 4$:



- The optimal solution to the IP must be contained in one of the above two feasible regions. Why?

Rounding a fractional solution

- So when we solve the linear relaxation and find any variable violating an integer constraint, we will **branch** this problem into two problems, one with an additional constraint.
- The two new programs are still linear programs.
- Once we solved them:
  - If their LR-optimal solutions are both IP-feasible, compare them and choose the better one.
  - If any of them results in a variable violating the integer constraint, **branch** on that variable **recursively**.
  - Eventually compare all the IP-feasible solutions we obtain.