Numerical Optimization

Unconstrained Optimization (I)

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NPTEL Course on Numerical Optimization

Global Minimum

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$
Consider the problem,

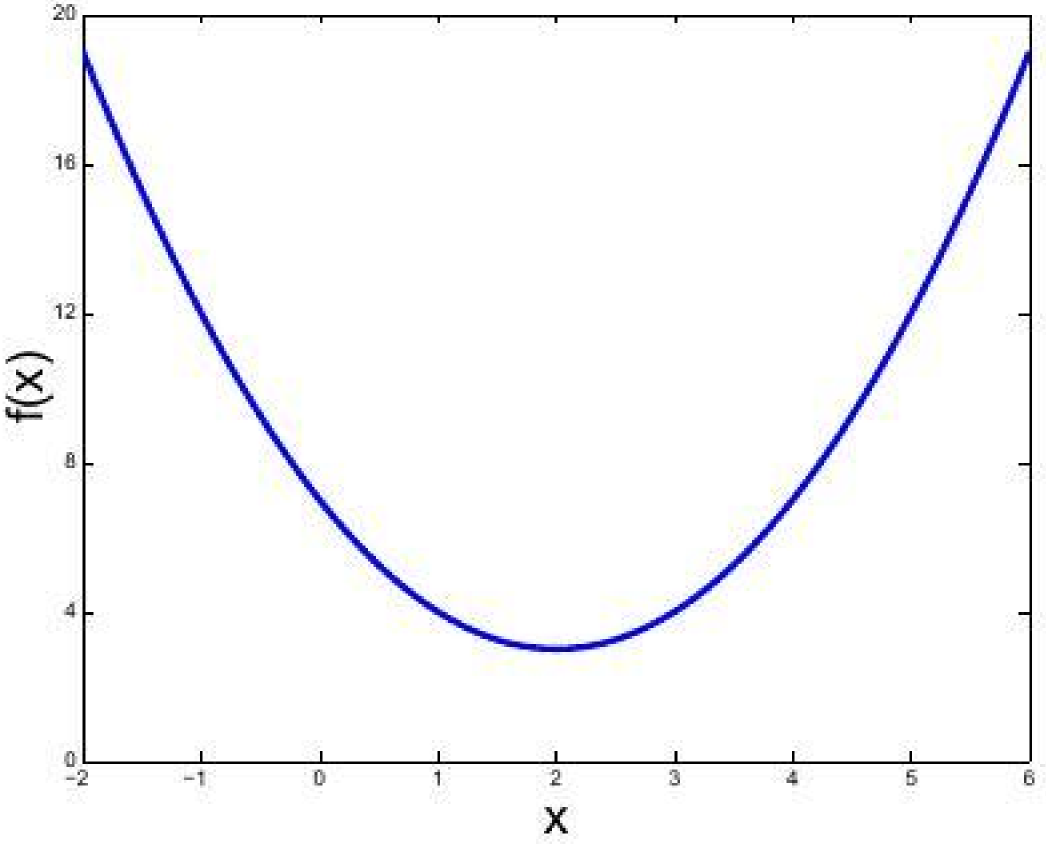
Constrained optimization problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{x} \in X \end{aligned}$$

Definition

$\boldsymbol{x}^* \in X$ is is said to be a *global minimum* of f over X if
$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X.$

Question: Under what conditions on f and X does the function f attain its maximum and/or minimum in the set X?


20
16
12
8
4
0
−2
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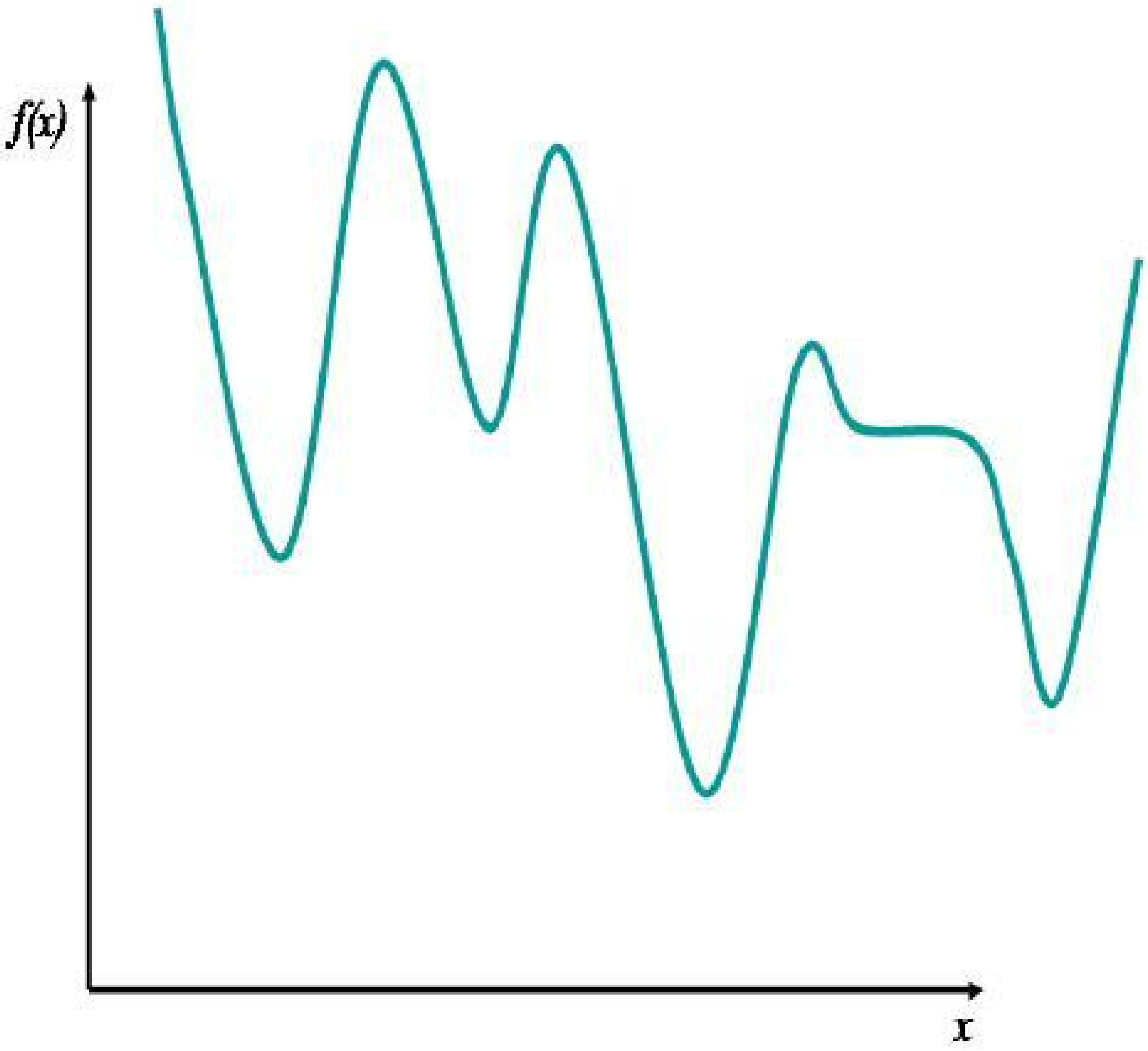
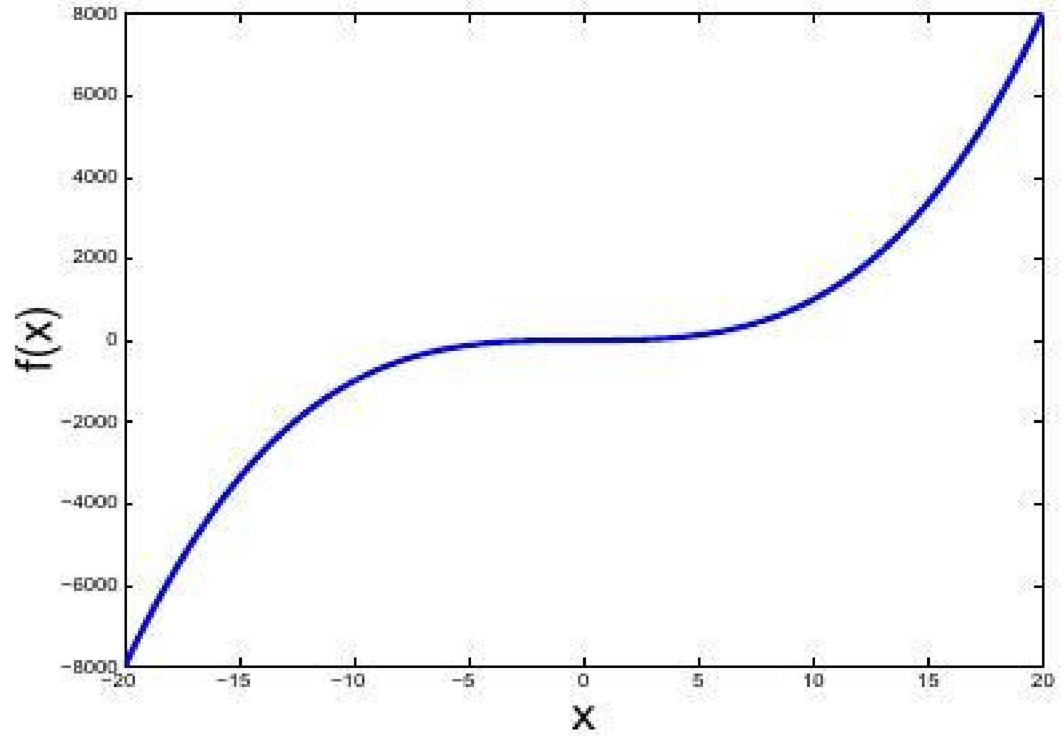
0
1
2
3
4
5
6
f(x)
x


f(x)
x

Global Minimum

- $X = \mathbb{R}, f : X \to \mathbb{R}$ defined as $f(x) = x^3$.



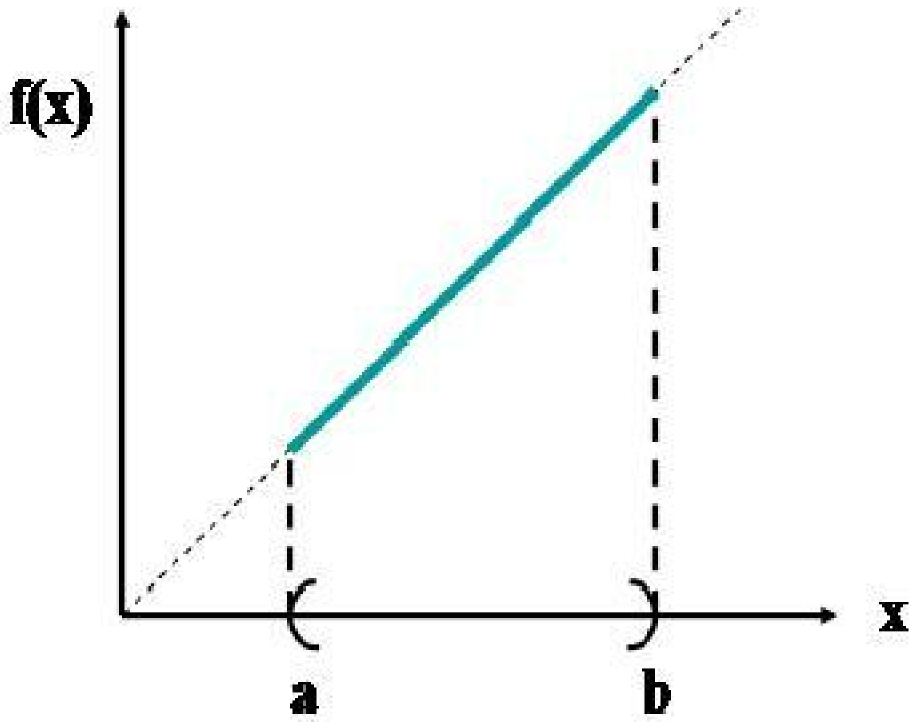
f attains neither a minimum nor a maximum on X

Note: X is closed, but not bounded; that is, X is not a compact set

closed by [-inf,inf]

Constrained Optimization

- $X = (a, b), f : X \to \mathbb{R}$ defined as $f(x) = x$.



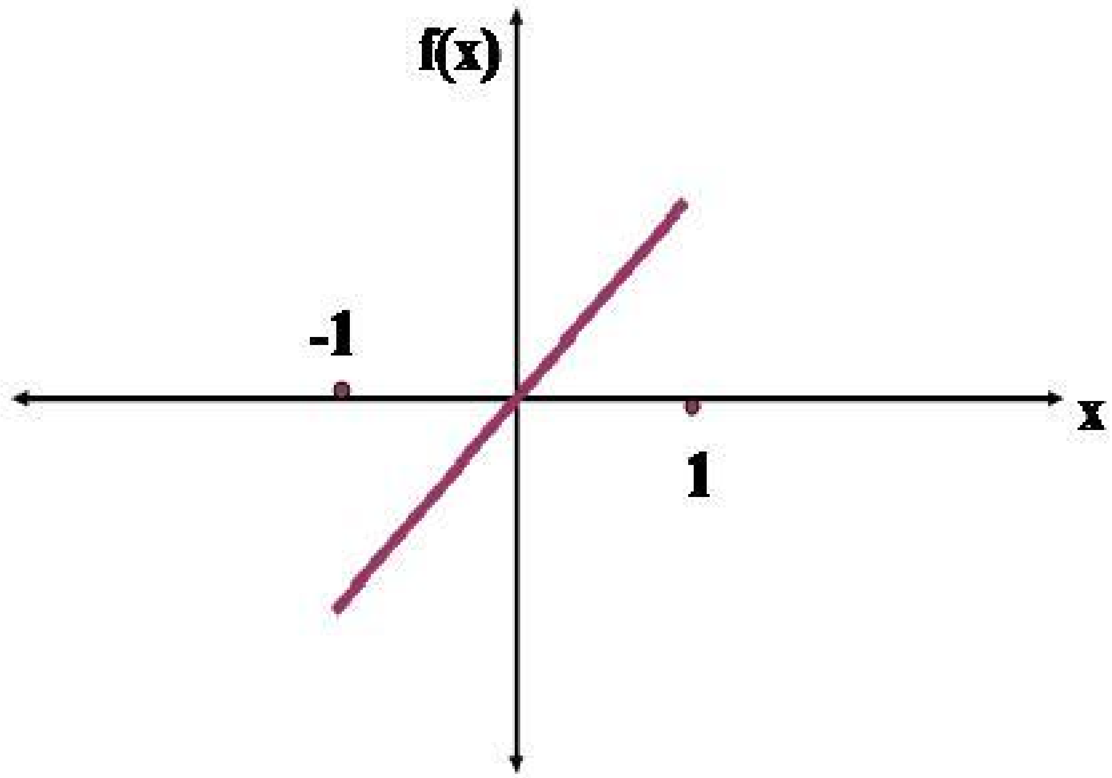
f attains neither a minimum nor a maximum on X

Note:

- X is bounded, but not closed; that is, X is not a compact set
- f does attain infimum at a and supremum at b

Constrained Optimization

- $X = [-1, 1], f : X \to \mathbb{R}$ defined as $f(x) = x$ if $-1 < x < 1$ and 0 otherwise.



f attains neither a minimum nor a maximum on X

Note:

- X is closed and bounded; X is compact
- f is not continuous on X

Weierstrass' Theorem

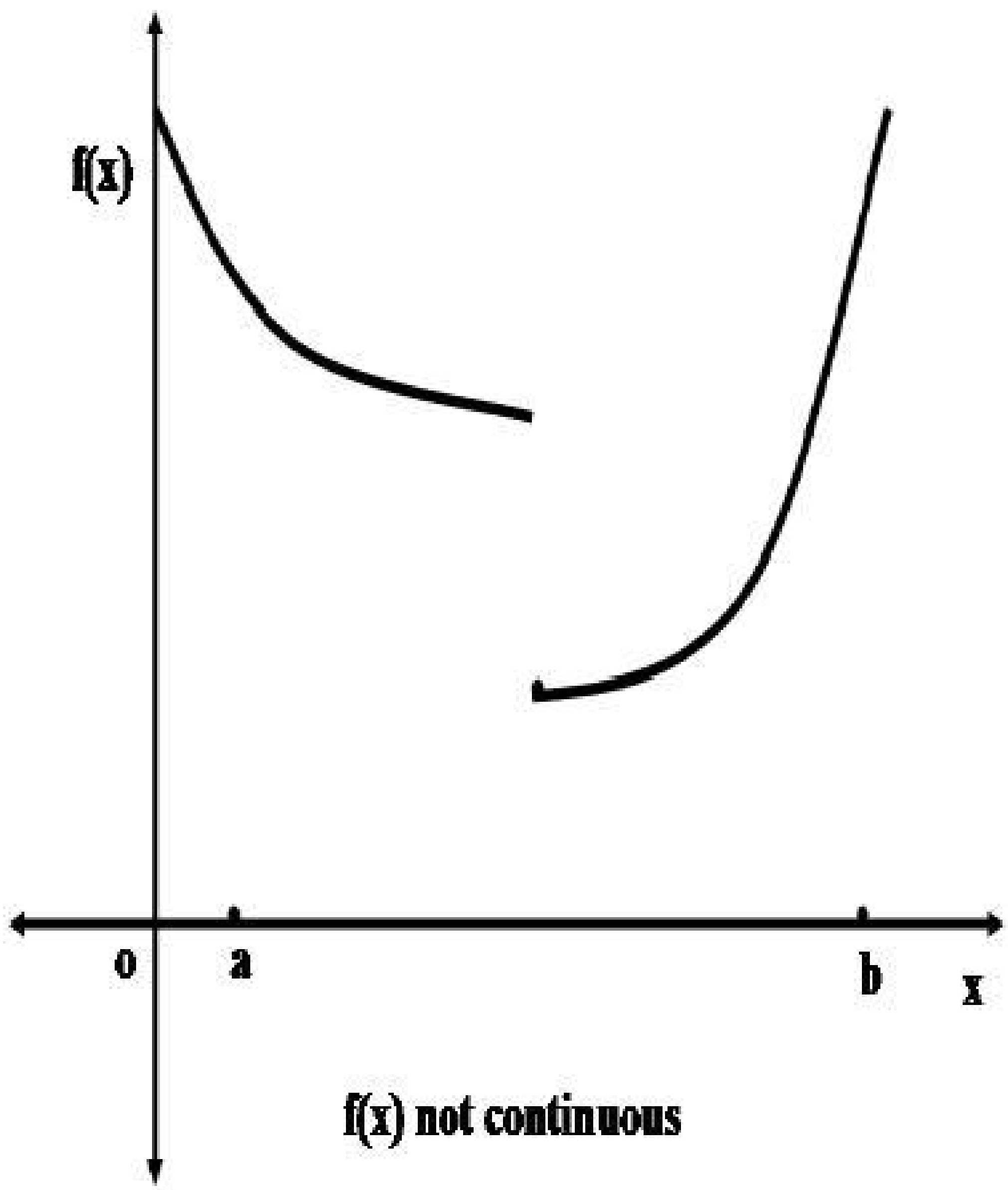
Theorem

Let $X \subset \mathbb{R}^n$ be a nonempty compact set and $f : X \to \mathbb{R}$ be a continuous function on X. Then, f attains a maximum and a minimum on X; that is, there exist $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in X such that

$$f(\boldsymbol{x}_1) \geq f(\boldsymbol{x}) \geq f(\boldsymbol{x}_2) \quad \forall \boldsymbol{x} \in X.$$

Note: Weierstrass' Theorem provides only *sufficient* conditions for the existence of optima.

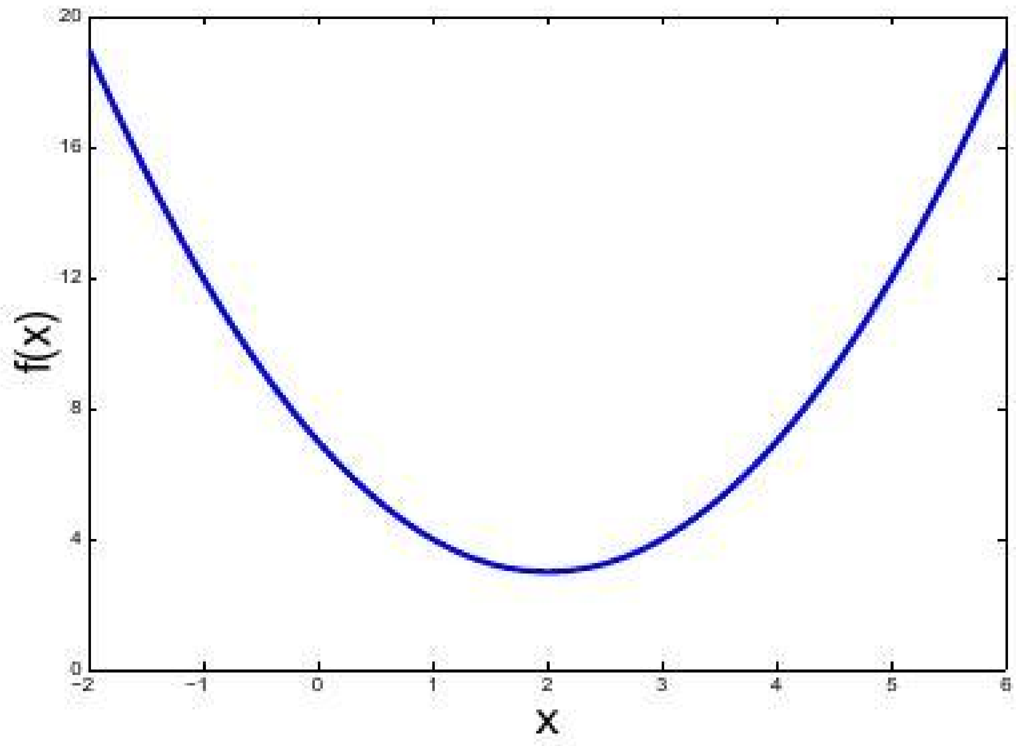
- $X = [a, b], f : X \to \mathbb{R}$



- $f(x)$ not continuous; but f attains a minimum

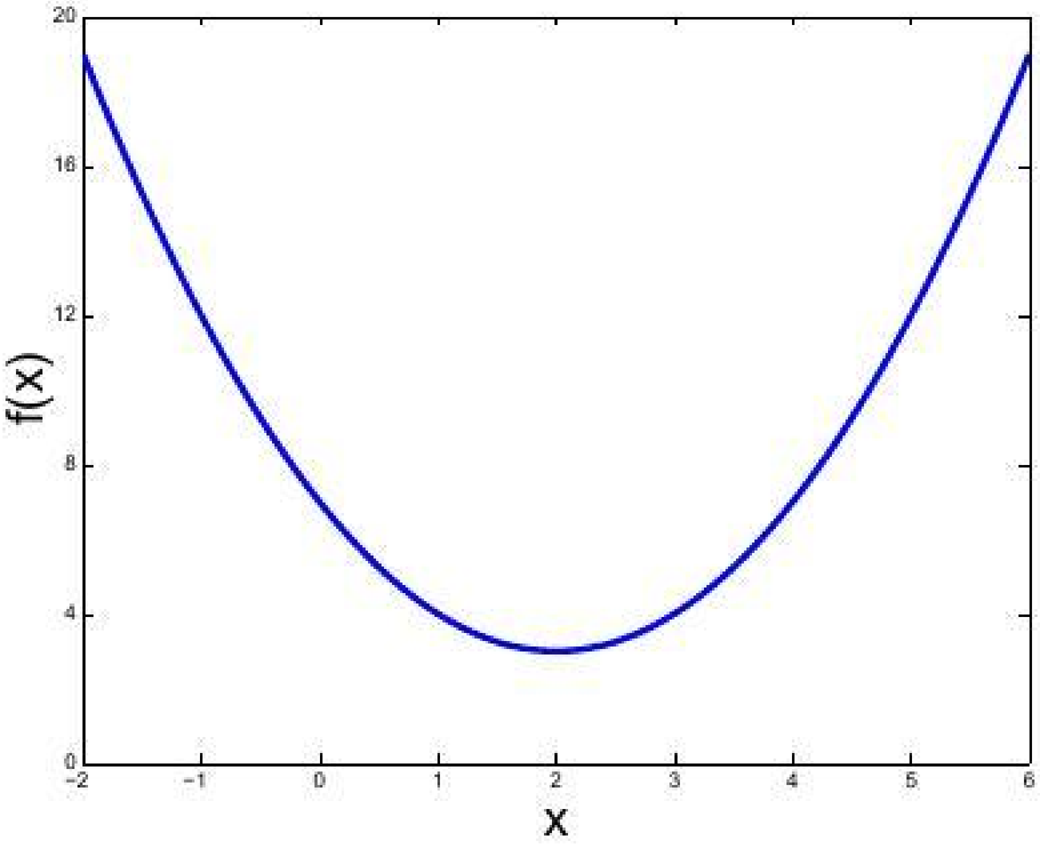
Constrained Optimization

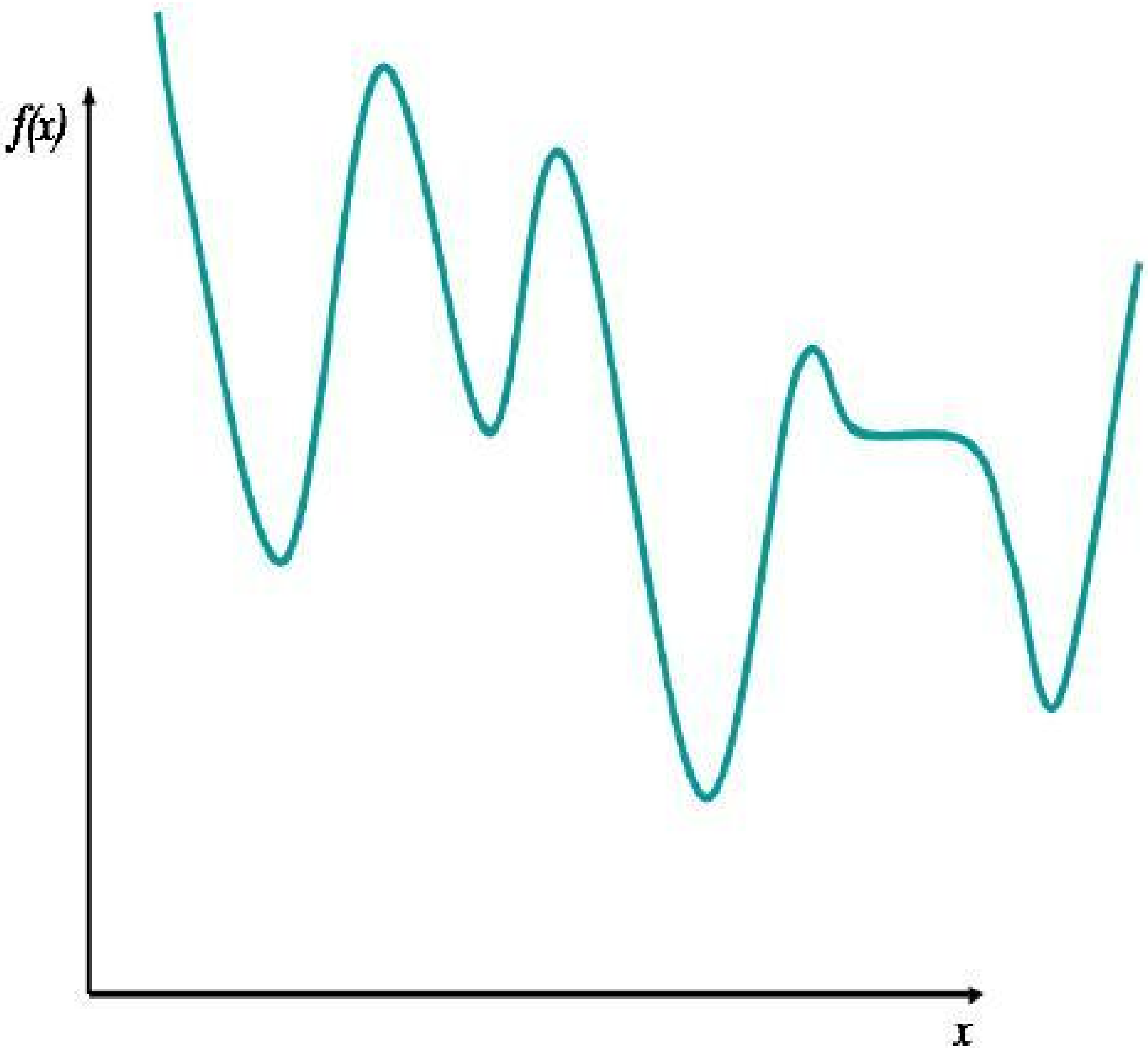
- $X = \mathbb{R}, f : X \to \mathbb{R}$ defined as $f(x) = (x-2)^2$.



- $f(x)$ continuous, X not compact; but f attains a minimum

Unconstrained Optimization





Global Minimum

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$
Consider the problem,

Constrained optimization problem

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

constrained

Definition

$\boldsymbol{x}^* \in X$ is is said to be a *global minimum* of f over X if
$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X$.

- Global minimum is difficult to find or characterize for a general nonlinear function

Local Minimum

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$
Consider the problem,

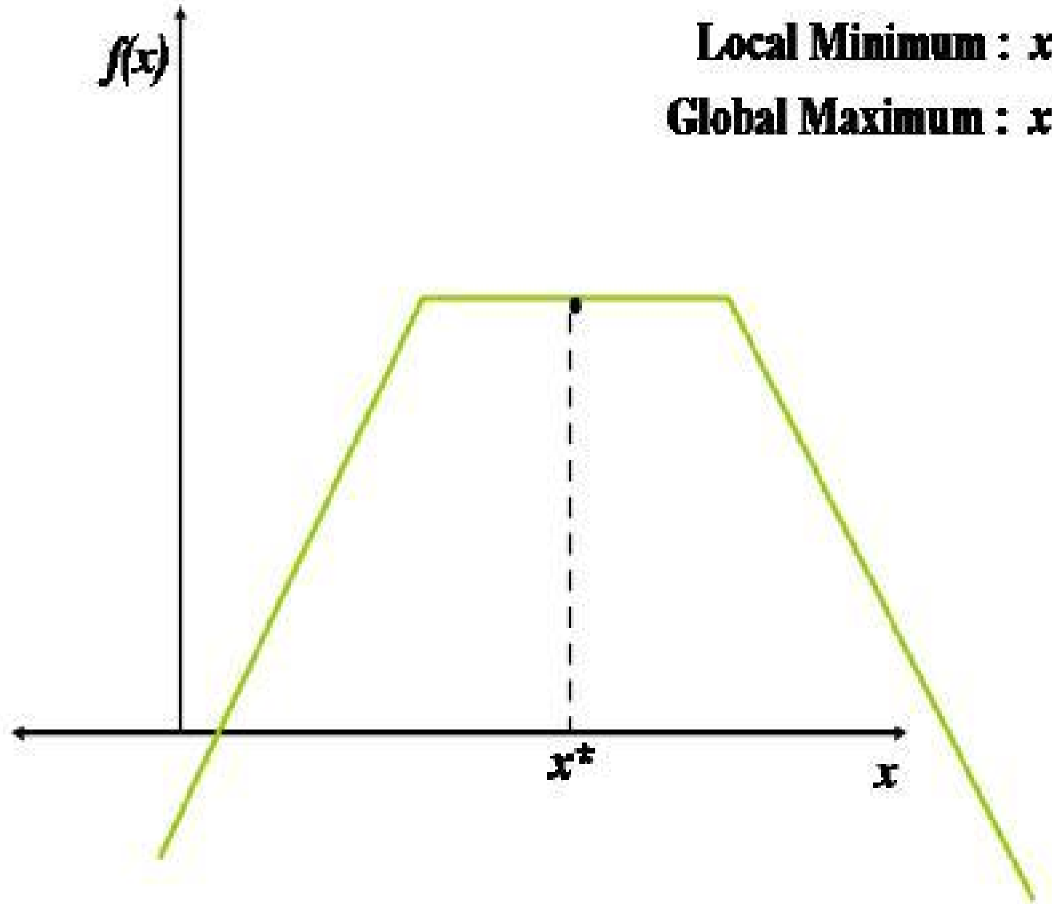
Constrained optimization problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & x \in X \end{array}$$

Definition

$\boldsymbol{x}^* \in X$ is is said to be a *local minimum* of f if there is a $\delta > 0$ such that $f(\boldsymbol{x}^*) \le f(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X \cap B(\boldsymbol{x}^*, \delta)$.

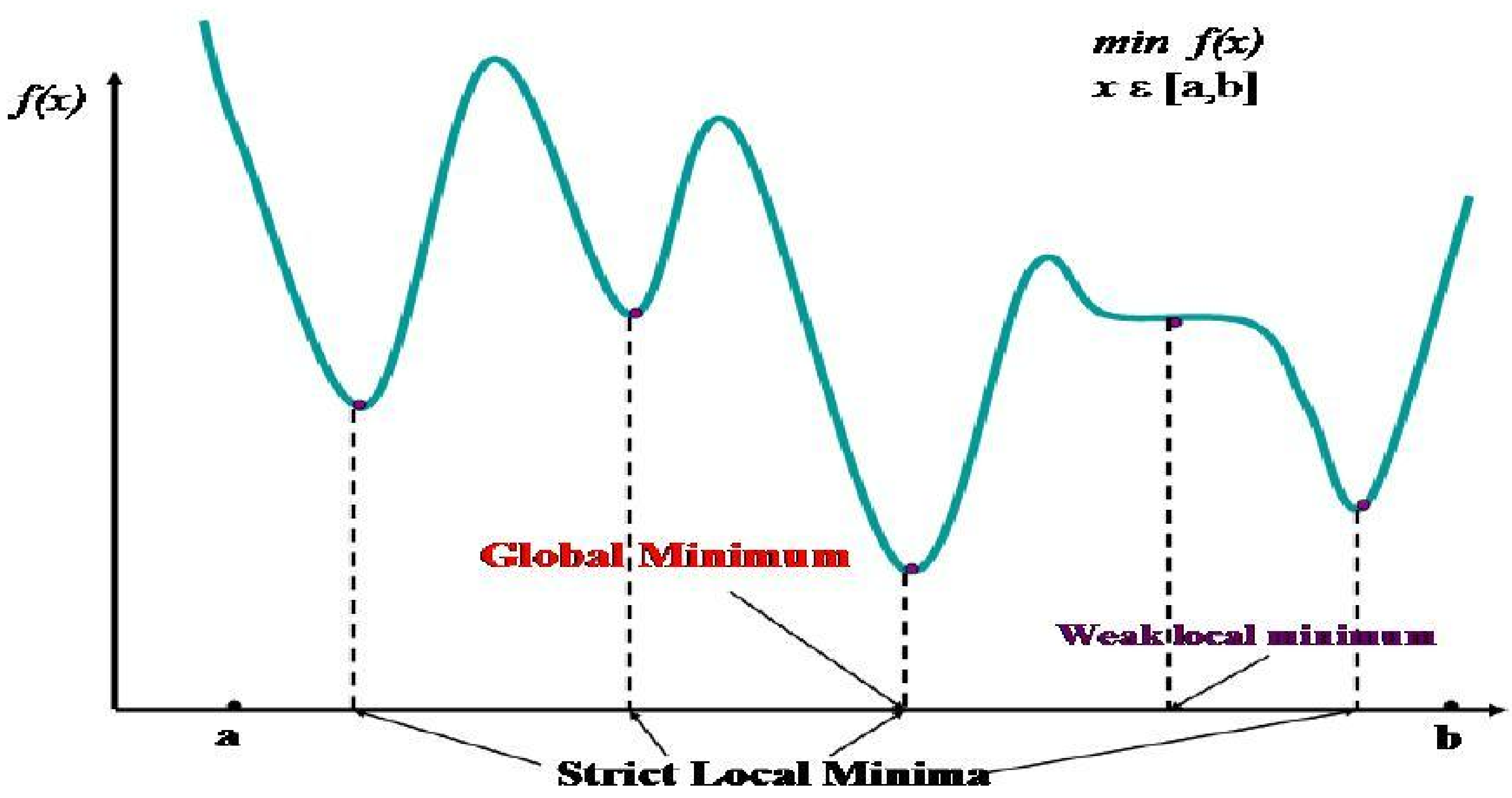
Strict Local Minimum



Definition

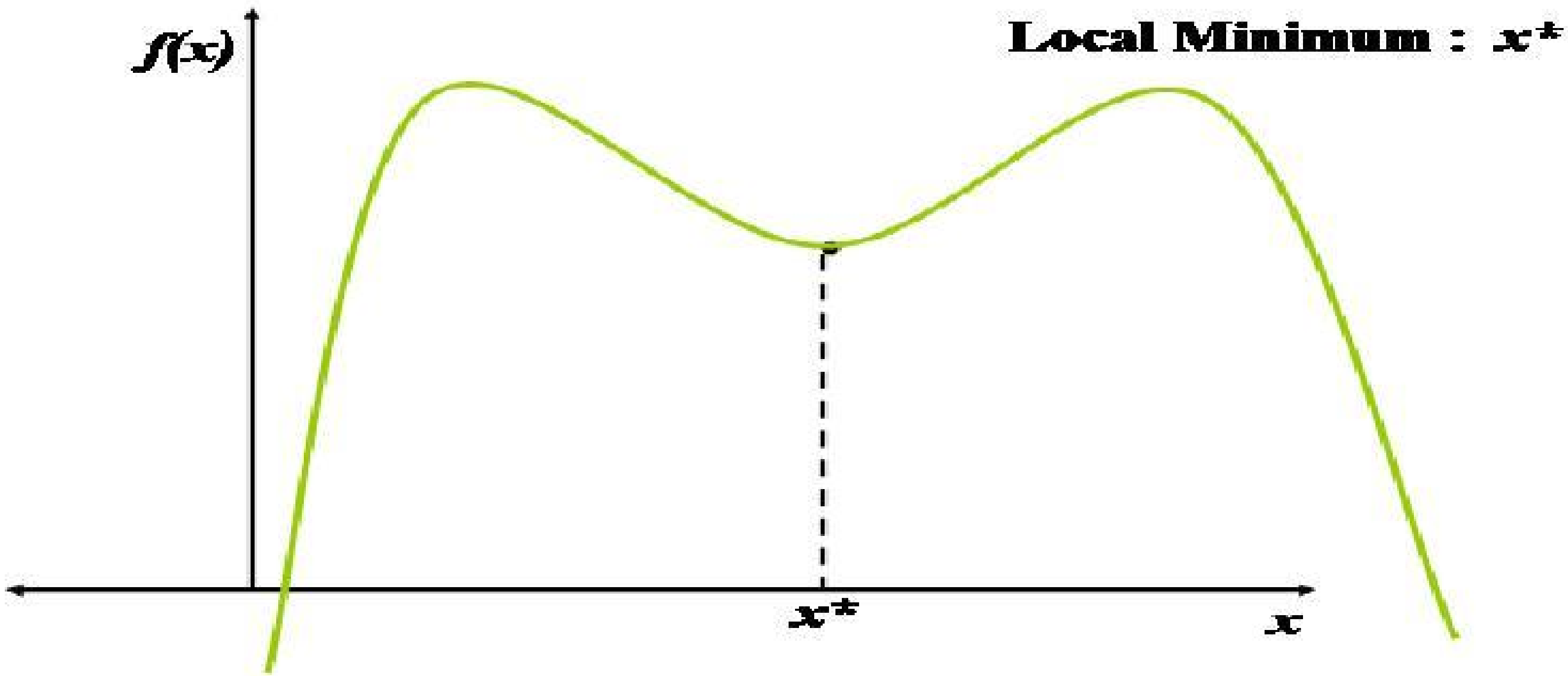
$\boldsymbol{x}^* \in X$ is is said to be a *strict local minimum* of f if
$f(\boldsymbol{x}^*) < f(\boldsymbol{x}) \quad \forall\ \boldsymbol{x} \in X \cap B(\boldsymbol{x}^*, \delta), \boldsymbol{x} \neq \boldsymbol{x}^*.$

Different Types of Minima



Global Minimum and Local Minimum

- Every global minimum is also a local minimum.
- It may not be possible to identify a global min by finding all local minima



- f does not have a global minimum

Optimization Problems

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$

- Constrained optimization problem:

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

- Unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} \quad f(x)$$

Now, consider $f : \mathbb{R} \to \mathbb{R}$

- Unconstrained one-dimensional optimization problem:

$$\min_{x \in \mathbb{R}} \quad f(x)$$

Unconstrained Optimization

Let $f : \mathbb{R} \to \mathbb{R}$

Unconstrained problem

$$\min_{x \in \mathbb{R}} \quad f(x)$$

- What are *necessary and sufficient conditions* for a local minimum?
  - Necessary conditions: Conditions satisfied by every local minimum
  - Sufficient conditions: Conditions which guarantee a local minimum
- Easy to characterize a local minimum if f is *sufficiently* smooth

First Order Necessary Condition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in \mathcal{C}^1$.
Consider the problem, $\min_{x \in \mathbb{R}} \quad f(x)$

Result (First Order Necessary Condition)

If x^* is a local minimum of f, then $f'(x^*) = 0$.

Proof.

Suppose $f'(x^*) > 0$. $f \in \mathcal{C}^1 \Rightarrow f' \in \mathcal{C}^0$.
Let $D = (x^* - \delta, x^* + \delta)$ be chosen such that $f'(x) > 0 \quad \forall\, x \in D$.
Therefore, for any $x \in D$, using first order truncated Taylor series,

$$f(x) = f(x^*) + f'(\bar{x})(x - x^*) \quad \text{where } \bar{x} \in (x^*, x).$$

Choosing $x \in (x^* - \delta, x^*)$ we get,

$$f(x) < f(x^*), \quad \text{a contradiction.}$$

Similarly, one can show, $f(x) < f(x^*)$ if $f'(x^*) < 0$. $\square$