Model structures

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Outline

Model structures

Choice of a structure = caracterisation.

Example:

$$\frac{dy_m}{dt} = -p_1 y_m + p_2 u,\ y_m(0) = 0,$$

with positive p_1 and p_2 defines

- a class of possible behaviors (no oscillations, stable, linear).
- an *a priori* set of admissible parameters, to which p has to belong.

Model structures

In what follows

- $M(\cdot)$ is the model structure
- $M(\mathrm{p})$ is the model of structure $M(\cdot)$ with a value p of the parameter vector.
- $\mathbb{P}$ set of *a priori* admissible parameter vectors.

The choice of $M(\cdot)$ and $\mathbb{P}$ is

- difficult, since intuition (and expertise) play an important role,
- fondamental, since it limits the possible model outputs.

Model structures

We will only consider model with localized parameters, described by

- algebraic
- differential (ODE)
- finite difference equations.

The methods described here may be adapted to PDEs.

Model structures

The choice of the model structure (characterization) will have to account for

- the goal pursued,
- the conditions under which the model will be used (operating ranges, nature of inputs, communication with other elements of a control system, etc.),
- the cost of building the model (time and resources),
- the possibilities of investigation (there is no point in developing a very complex model with many parameters if detailed data cannot be obtained).

Remark

Most of the models considered assume that if the initially idle system is subjected to an identically null input u, its output y will be identically null in the absence of any perturbation. It will therefore be necessary to ensure that u and y are expressed in appropriate references.

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Knowledge-based models

Built using the main principles of physics

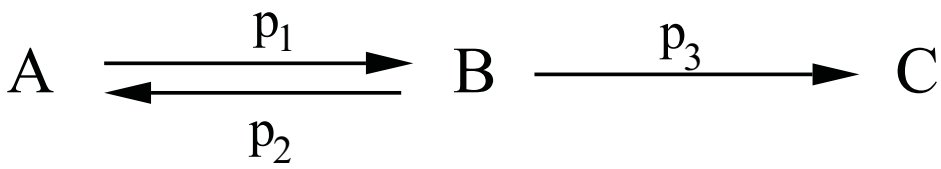
- conservation equations (mass, moment, energy...)
- balance equations.

Suitable for

- taking into account the *a priori* available information
- the *a posteriori* control of the orders of magnitude of the parameters obtained.

Knowledge-based models

Example: chemical reaction



Hypotheses :

- all kinetics are of the first order
- isothermal and perfectly agitated reactor

$$\frac{d[A]}{dt} = -p_1[A] + p_2[B],$$
$$\frac{d[B]}{dt} = p_1[A] - (p_2 + p_3)[B],$$
$$\frac{d[C]}{dt} = p_3[B].$$

Knowledge-based models

Model structure imposed by the *a priori* knowledge of the system under study.

Parameters and state variables have a concrete meaning:

- the parameters p_i are kinetic constants of the elementary reactions,
- the state variables $[A]$, $[B]$,and $[C]$ are the concentrations of the reactants.

Knowledge-based models

For complex processes, one may obtains models

- of large dimension (discretization of PDEs),
- consisting of many equations,
- often non-linear,
- Sometimes difficult to simulate,
- rarely usable as is to design a control law,
- but well-suited for an accurate long-term simulation.

Behavioral models

A behavioral model simply reproduce an observed behavior:

- You don't even need to know what the output is.
- Its structure does not correspond to the structure of the process.
- Its parameters are therefore not clearly related to the process being studied.

Behavioral models

Example: polynomial model

$$y_{\mathrm{m}}(t, \mathrm{p}) = p_1 + p_2 t + p_3 t^2 + p_4 t^3 + \ldots$$

will be able to reproduce any experimental data set with arbitrary precision.

Behavioral models

More sophisticated behavioral models

- (deep) neural networks,
- splines
- Support Vector Machines

To wrap up

	Knowledge-based models	Behavioral models
Parameters	concrete meaning	no concrete meaning
Simulation	difficult	easy
A priori information	can be taken into account	neglected
Validity domain	large (if correct!)	restricted

Linear and nonlinear models

Two types of linearity :

- Linearity with respect to the input (LI) : superposition principle applies to u

$$\forall \lambda, \mu \in \mathbb{R}^2, \forall t \in \mathbb{R}^+, \ y_m(t, p, \lambda u_1 + \mu u_2) = \lambda y_m(t, p, u_1) + \mu y_m(t, p, u_2)$$

  = linearity in a control-theoretic sense
  (assumes also stationnarity = invariance with respect to any translation of the time origin).

- Linearity with respect to the parameters (LP) : superposition principle applies to p

$$\forall \lambda, \mu \in \mathbb{R}^2, \forall t \in \mathbb{R}^+, \ y_m(t, \lambda p_1 + \mu p_2, u) = \lambda y_m(t, p_1, u) + \mu y_m(t, p_2, u)$$

  = linearity in a statistic sense.

Linear and nonlinear models

One has to determine whether $M(.)$ is LP or non LP, LI or non LE.

(Important consequences for estimation algorithms)

Examples:

$$y_m(t,p) = pu(t) \rightarrow \text{LI, LP}$$
$$y_m(t,\mathrm{p}) = p_1 y_m(t-1,\mathrm{p}) + p_2 u(t) \rightarrow \text{NLP, LI}$$
$$y_m(t,p) = pu^2(t)$$
$$y_m(t,\mathrm{p}) = p_1 y_m^2(t-1,\mathrm{p}) + p_2 u(t)$$
$$y_m(t,\mathrm{p}) = p_1 y(t,\mathrm{p}) + p_2 u(t)$$

$u(0) = 1, \quad u(t) = 0, \forall t \neq 0.$

$y_m(0, p) = p_2, \quad y_m(1, p) = p_1 p_2, \quad y_m(2, p) = p_1^2 p_2.$

$y_m(t, p) = \underline{r}^T(t)\, p.$ ← $\underline{r}^T(t)$: vector of regressors does not depend on p!

$y_m(t, p) = (y(t), u(t))^T \cdot p$

Linear and nonlinear models

One has to determine whether $M(.)$ is LP or non LP, LI or non LE.

(Important consequences for estimation algorithms)

Examples:

$y_m(t,p) = pu(t)$	is LP and LI,
$y_m(t,p) = p_1 y_m(t-1,p) + p_2 u(t)$	is non LP and LI,
$y_m(t,p) = pu^2(t)$	is LP and non LI,
$y_m(t,p) = p_1 y_m^2(t-1,p) + p_2 u(t)$	is non LP and non LI
$y_m(t,p) = p_1 y(t,p) + p_2 u(t)$	is LP (for LI one does not know)

Linear and nonlinear models

When possible, LP and LI are preferred.

LI models

- powerful mathematical results (stability, optimal control, behavior in the presence of disturbances, *etc.*),
- limited validity domain (all systems are non LE for large inputs).

LP models

Linearity with respect to the parameters facilitates

- their estimation,
- determination of the uncertainty on these parameters
- experiment design.

But the parameters of LP models often have no physical meaning.

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Linear and nonlinear models

Sometimes, it is possible to transform a non-LP into an LP model by a change of variables.

For example, the non-LP model

$$y_m(t_k, \mathrm{p}) = p_1 \exp(-p_2 t_k)$$

becomes LP considering

$$\ln y_m(t_k, \mathrm{q}) = q_1 - q_2 t_k$$

where $q_1 = \ln p_1$ and $q_2 = p_2$.

Not without consequences on the value of the estimates obtained...
In the presence of noise, the same results will not be obtained by estimating p directly or from an estimate of q.

Can be used to initialize an iterative search of the parameters of a non-LP model.

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Continuous-time models

Natural system usually considers continuous time.

Are often described by (ODE or PDE).

State equation

$$\frac{dx(t)}{dt} = f(x, p, u, t),\ x(0) = x_0(p),$$
$$y_m(t) = h(x, p, u, t)$$

$$m\ddot{x} = -k\dot{x} \qquad x(0) = 0, \quad \dot{x}(0) = 1$$

Consider

$$x_1 = x$$

$$x_2 = \dot{x}$$

Then

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = 0 \\ \dot{x}_2 = -\frac{k}{m} x_2 & x_2(0) = 1 \end{cases}$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad \underline{x}(0) = \underline{x}_0 .$$

$$\frac{\underline{x}(t+\delta t) - \underline{x}(t)}{\delta t} = \underline{f}(\underline{x}, t)$$

$$\boxed{\underline{x}(t+\delta t) = \underline{x}(t) + \delta t\, \underline{f}(\underline{x}(t), t) .}$$

Continuous-time models

Stationary LI model

$$\frac{dx(t)}{dt} = A(p)x(t) + B(p)u(t),\ x(0) = x_0(p),$$
$$y_m(t) = C(p)x(t) + D(p)u(t)$$

If $x(0) = 0$, one may consider a transfer representation

$$y_m(s,p) = H(s,p)u(s)$$

where s is the Laplace variable, thanks to

$$H(s,p) = C(p)[sI - A(p)]^{-1}B(p) + D(p).$$

In time domain, one gets the system of EDOs linking the input to the output

$$\sum_{i=0}^{n} P_i(p)\frac{d^i}{dt^i}y_m = \sum_{i=0}^{m} Q_i(p)\frac{d^i}{dt^i}u.$$

Easy to switch between representations (difficult for non-LI models).

Continuous-time models

Stationary LI model

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Discrete-time models

Numerical simulation of discrete-time models

- simpler
- faster.

This is why they are widely used for real-time process control.

(But loss of information on the behavior of the underlying continuous-time model).

Discrete-time models

As in the continuous-time case, one may consider the state equations

$$x(t+1) = f(x, p, u, t),\ x(0) = x_0(p),$$
$$y_m(t) = h(x, p, u, t)$$

In the LI case

$$x(t+1) = A(p)x(t) + B(p)u(t),\ x(0) = x_0(p),$$
$$y_m(t) = C(p)x(t) + D(p)u(t)$$

If $x(0) = 0$, one may consider the following transfer representation

$$y_m(z, p) = H(z, p)u(z)$$

where

$$H(z, p) = C(p)[zI - A(p)]^{-1}B(p) + D(p).$$

Discrete-time models

In the time domain, one gets the system of difference equations

$$\mathrm{y_m}(t+1) = \sum_{i=0}^{n} \mathrm{P}_i(\mathrm{p})\mathrm{y_m}(t-i) = \sum_{i=0}^{m} \mathrm{Q}_i(\mathrm{p})\mathrm{u}(t-i).$$

Introducing the unit-delay operator q^{-1}, such that

$$q^{-1}x(t) = x(t–1).$$

Then, for example,

$$y_\mathrm{m}(t+1) = -a_1 y_\mathrm{m}(t) - a_2 y_\mathrm{m}(t–1) + b_1 u(t) + b_2 u(t–1),$$

becomes

$$A(q^{-1})y_\mathrm{m}(t+1) = B(q^{-1})u(t+1),$$

with $A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2}$ et $B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2}$.

Discrete-time models

These models

- imposes constraints on the measurement time instants,
- can hide oscillations of the associated continuous system,
- have parameters without a clear physical meaning and whose value depends on the discretization step.

Properties of discrete and continuous models not always analogous.

Example: Discrete-time oscillating first-order linear model

$$y_m(t+1) = -p_1 y_m(t) + p_2 u(t), \text{ avec } 0 < p_1 \leqslant 1.$$

(Impossible for a continuous time first-order linear model).

Complicates the determination of a continuous time model via a discrete time model.

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Complicates the determination of a continuous time model via a discrete time model.

$$\dot{x} = px + u \quad , \quad x(0) = 0.$$

Discretise with a period T:

$$\frac{x(kT+T) - x(kT)}{T} = p\,x(kT) + u.$$

$$x(kT+T) = x(kT) + \underbrace{T}_{1}\,p\,x(kT) + Tu$$

$$x_{k+1} = x_k + q\,x_k + Tu$$

$T = 0{,}1\,s$
$\hookrightarrow q = 0{,}1$

$T = 0{,}01\,s$
$\hookrightarrow q = 0{,}01$

Discrete-time models

	Continuous time	Discrete time
Parameters	independent of sampling time	dependent of sampling time
Simulation	complicated	easy
A priori information	usable	unusable
Measurements time instants	any	imposed

Accounting for perturbations

Perturbations are usually described by realizations of stochastic processes (sequence of r.v.).

One considers here systems with

- a scalar input $u(.)$,
- a sclara output $y(.)$
- a scalar perturbation $b(.)$

described by a difference equation.

Accounting for perturbations

Simplest way is to add a noise to the recurrence equation

$$y(t)+a_1 y(t-1)+\cdots+a_{n_\mathrm{a}} y(t-n_\mathrm{a}) = b_1 u(t-n_\mathrm{r})+\cdots+b_{n_\mathrm{b}} u(t-n_\mathrm{r}-n_\mathrm{b}+1) + b(t),$$

$$\text{auto-regressive} = \text{exogenous} + \text{noise}$$

where $b(t)$ $(t = 1, 2, \ldots)$ are realizations of a sequence of iid r.v. $\varepsilon(t)$.

(For a discretized continuous-time system, the input-output delay is such that $n_\mathrm{r} \geqslant 1$.)

Once n_a, n_b, and n_r have been chosen,

$$\mathrm{p} = \left(a_1, \ldots, a_{n_\mathrm{a}}, b_1, \ldots, b_{n_\mathrm{b}}\right)^\mathsf{T}.$$

Usually $n_a = n_b$, which simplifies characterization.
ARX structure (Auto-Regressive with eXogeneous variable).

Accounting for perturbations

ARX lacks of flexibility in the description of the noise (some noise in the recurrence equation is not necessarily described by iid r.v.).

One may replace $b(t)$ by a linear combination of successive $\varepsilon(t)$, called moving average

$$\begin{aligned} y(t)+a_1 y(t-1)+\cdots+a_{n_\mathrm{a}} y(t-n_\mathrm{a}) &= b_1 u(t-n_\mathrm{r})+\cdots+b_{n_\mathrm{b}} u(t-n_\mathrm{r}-n_\mathrm{b}+1) \\ &\quad + c_1 \varepsilon(t-1)+\cdots+c_{n_c} \varepsilon(t-n_c), \\ \text{auto-regressive} &= \text{exogenous} + \text{moving average} \end{aligned}$$

Once n_a, n_b, n_c, and n_r have been chosen,

$$\mathrm{p} = \left(a_1, \ldots, a_{n_\mathrm{a}}, b_1, \ldots, b_{n_\mathrm{b}}, c_1, \ldots, c_{n_\mathrm{c}}\right)^\mathsf{T}.$$

ARMAX structure (Auto-Regressive with Moving Average and eXogeneous variable).

Accounting for perturbations

Alternatively $b(t)$ in the ARX structure can be replaced by an autoregressive part $\eta(t)$

$$y(t)+a_1 y(t-1)+\cdots+a_{n_\mathsf{a}} y(t-n_\mathsf{a}) = b_1 u(t-n_\mathsf{r})+\cdots+b_{n_\mathsf{b}} u(t-n_\mathsf{r}-n_\mathsf{b}+1) + \eta(t),$$

where

$$\eta(t)+d_1\eta(t-1)+\cdots+d_{n_\mathsf{d}}\eta(t-n_\mathsf{a}) = \varepsilon(t)$$

Once n_a, n_b, n_d, and n_r are chosen,

$$\mathsf{p} = \left(a_1,\ldots,a_{n_\mathsf{a}},b_1,\ldots,b_{n_\mathsf{b}},d_1,\ldots,d_{n_\mathsf{d}}\right)^\mathsf{T}.$$

ARARX structure.

Accounting for perturbations

Finally, $b(t)$ may be replaced by an auto-regressive with moving average (ARMA) term:

$$y(t)+a_1 y(t-1)+\cdots+a_{n_\mathsf{a}} y(t-n_\mathsf{a}) = b_1 u(t-n_\mathsf{r})+\cdots+b_{n_\mathsf{b}} u(t-n_\mathsf{r}-n_\mathsf{b}+1) + \eta(t),$$

where

$$\eta(t)+d_1\eta(t-1)+\cdots+d_{n_\mathsf{d}}\eta(t-n_\mathsf{a}) = c_1\varepsilon(t-1)+\cdots+c_{n_c}\varepsilon(t-n_c)$$

Once n_a, n_b, n_d, and n_r have been chosen,

$$\mathsf{p} = \left(a_1,\ldots,a_{n_\mathsf{a}},b_1,\ldots,b_{n_\mathsf{b}},c_1,\ldots,c_{n_\mathsf{c}},d_1,\ldots,d_{n_\mathsf{d}}\right)^\mathsf{T}.$$

ARARMAX structure.

Accounting for perturbations

Consider

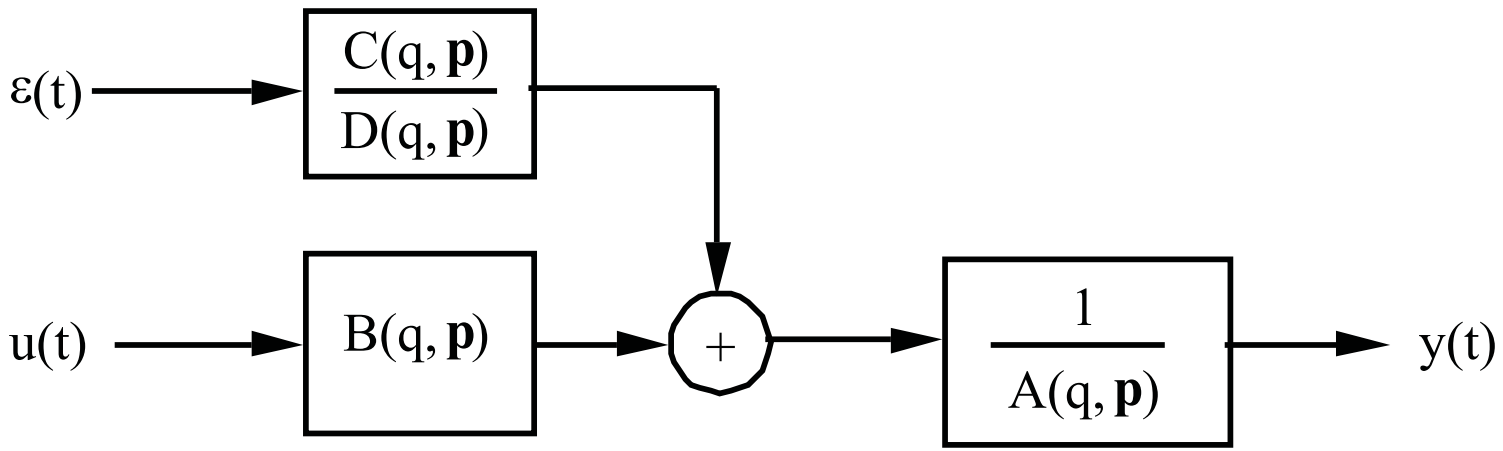
$$
\begin{aligned}
A(q,\mathrm{p}) &= 1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a}, \\
B(q,\mathrm{p}) &= q^{1-n_\mathrm{r}} \left(b_1 q^{-1} + \ldots + b_{n_b} q^{-n_b} \right), \\
C(q,\mathrm{p}) &= 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c}, \\
D(q,\mathrm{p}) &= 1 + d_1 q^{-1} + \ldots + d_{n_d} q^{-n_d}.
\end{aligned}
$$

Accounting for perturbations

ARARMAX structure becomes

$$A(q,\mathrm{p})y(t) = B(q,\mathrm{p})u(t) + \frac{C(q,\mathrm{p})}{D(q,\mathrm{p})}\varepsilon(t),$$

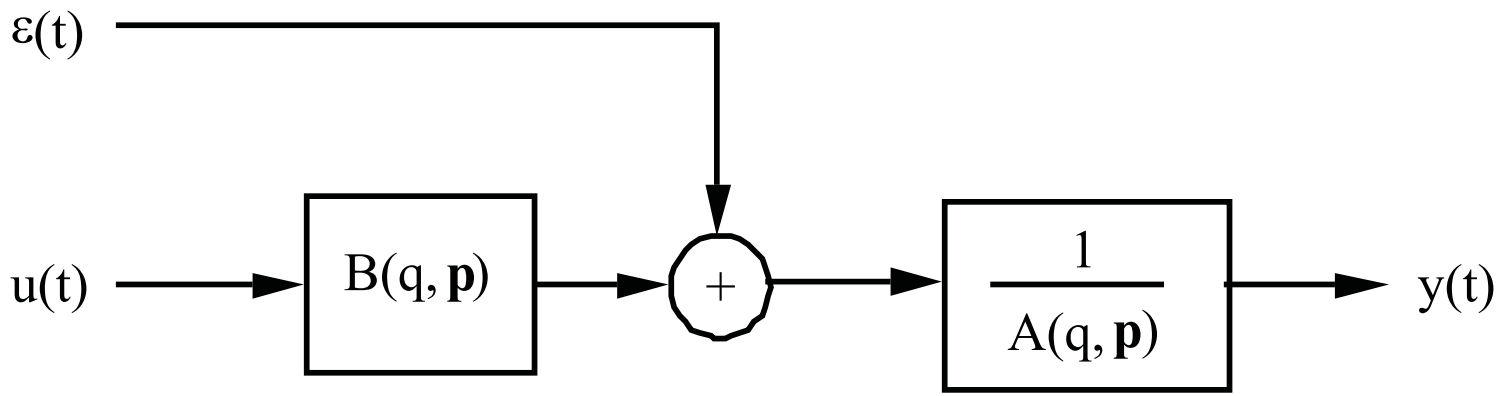
where p^* is the true value of the parameter vector.



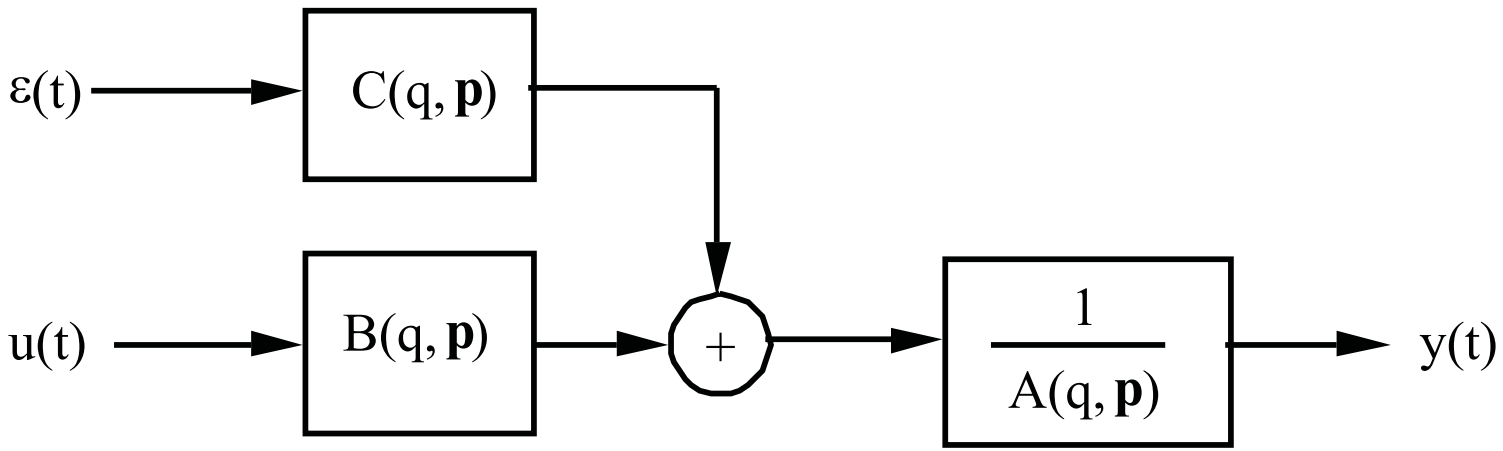
Accounting for perturbations

Contains the ARX, ARARX, and ARMAX structures as special cases:

ARX

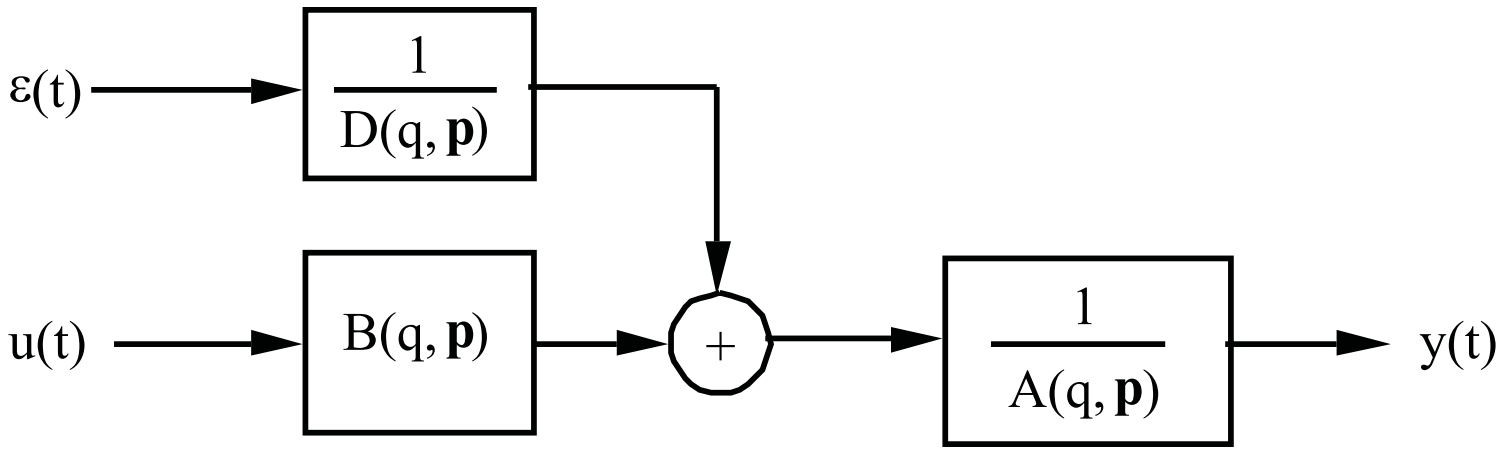


ARMAX



Accounting for perturbations

ARARX

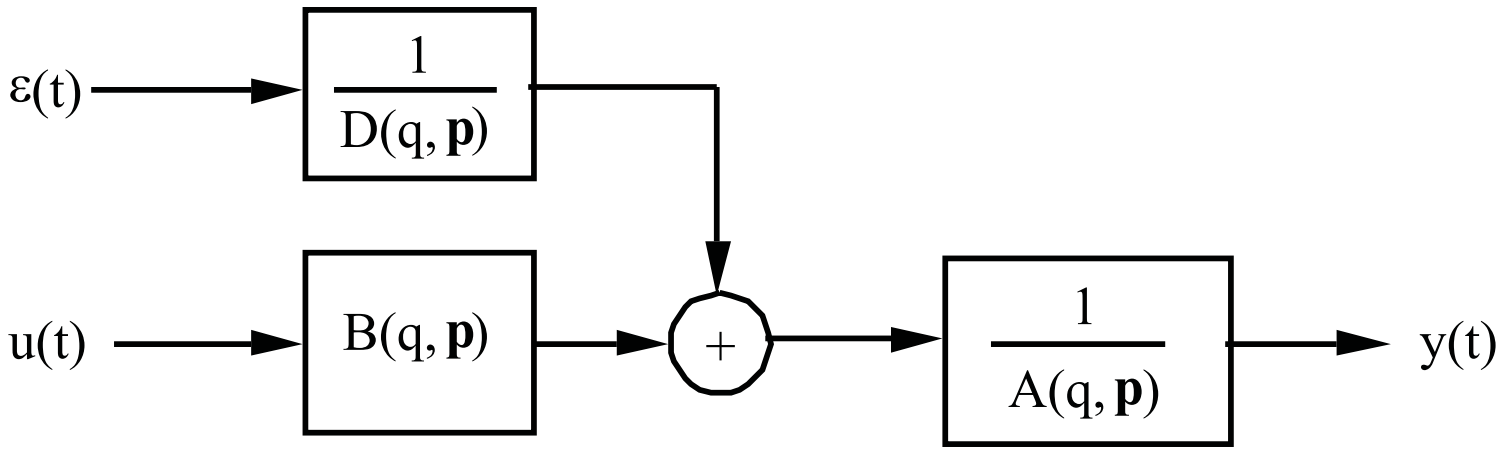


All these structures impose to the transfer functions linking noise and input to output to have a part of their denominators in common since

$$y(t) = \frac{B(q,\mathrm{p}^*)}{A(q,\mathrm{p}^*)}u(t) + \frac{C(q,\mathrm{p}^*)}{A(q,\mathrm{p}^*)D(q,\mathrm{p}^*)}\varepsilon(t).$$

Accounting for perturbations

ARARX



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Choosing the complexity

Consider two model structures $M_1(.)$ and $M_2(.)$, where $M_2(.)$ is a super-set of $M_1(.)$.
$\hookrightarrow$ $M_2(.)$ has thus more degrees of freedom than $M_1(.)$.

Example 1:

$$H_1(s,\mathrm{p}) = \frac{p_1}{1+p_2 s},\ \mathrm{p} \in \mathbb{R}^2,$$

$$H_2(s,\mathrm{p}) = \frac{p_1}{1+p_2 s+p_3 s^2},\ \mathrm{p} \in \mathbb{R}^3,$$

Example 2:

$M_1(.)$ and $M_2(.)$ correspond to ARMAX structures with increasing complexity.

Example 3:

Polynomials of increasing degrees.

Choosing the complexity

Because of the extra degrees of freedom it has, $M_2(.)$ can describe more finement of the given experimental results than $M_1(.)$.

One might think that the only problem to be solved is a complexity/performance trade-off.

In fact, the problem is not so simple...

Choosing the complexity

Consider a process of structure $M_1(.)$ with very noisy experimental data.

- The behavior of the best model of structure $M_1(.)$ only very approximately behaves like the process.
- The $M_2(.)$ structure may then seem to lead to better results.
- Nevertheless, the additional degrees of freedom of $M_2(.)$ only serve here to model a particular realization of the noise.

If we repeat the experiment, we will obtain a different realization of this noise.

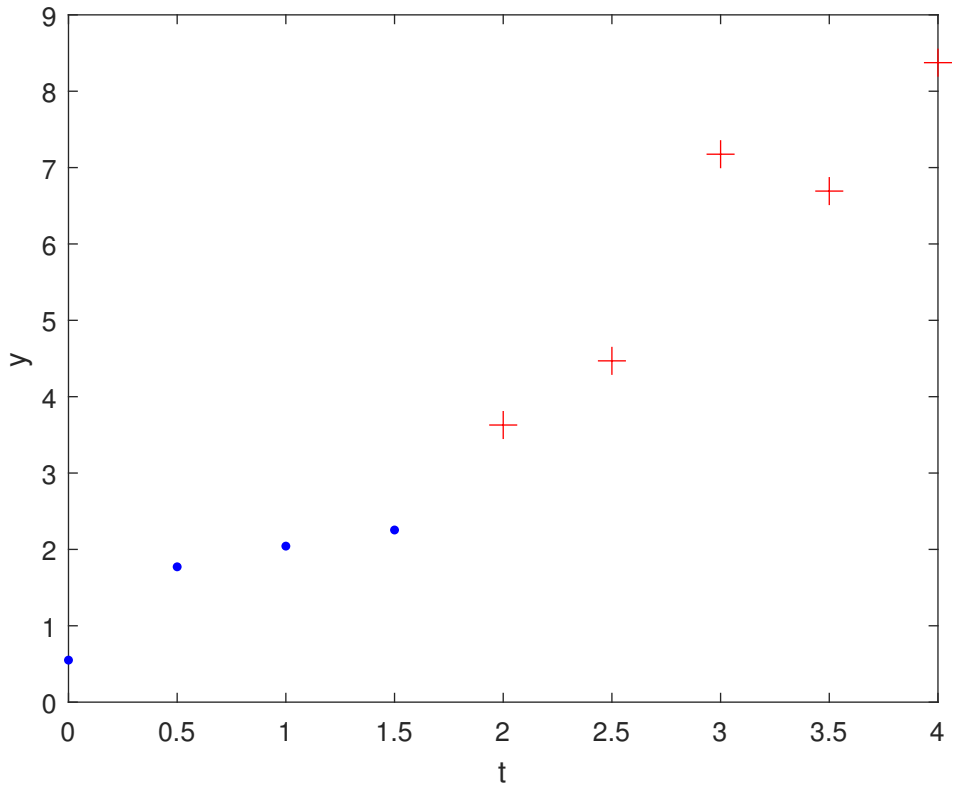
The best model of structure $M_2(.)$ will provide a worse prediction than the best model of structure $M_1(.)$.

Choosing the complexity - example

Consider data generated by the system

$$y(t_i) = at_i + b(t_i), \text{ avec } t_i = i\Delta.$$

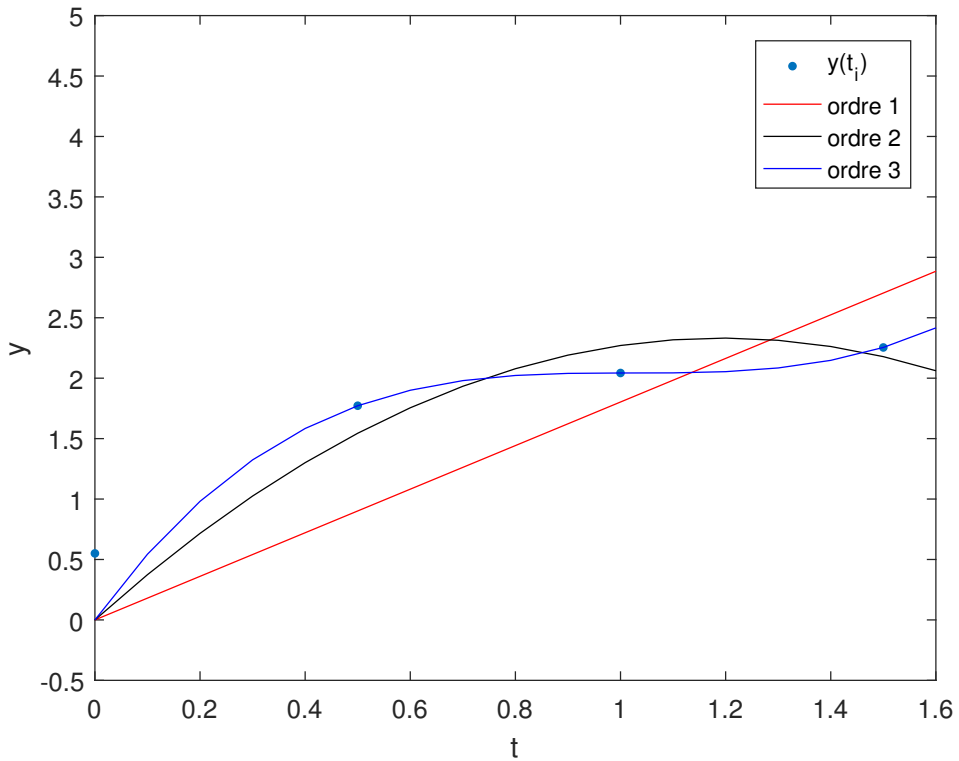
One takes $\Delta = 0.5$ s and $i = 0, \ldots, 3$ as training data and $i = 4, \ldots, 8$ as testing data



Choix de la complexitï¿œ - exemple

A least-squares parameter estimation is performed considering the three models

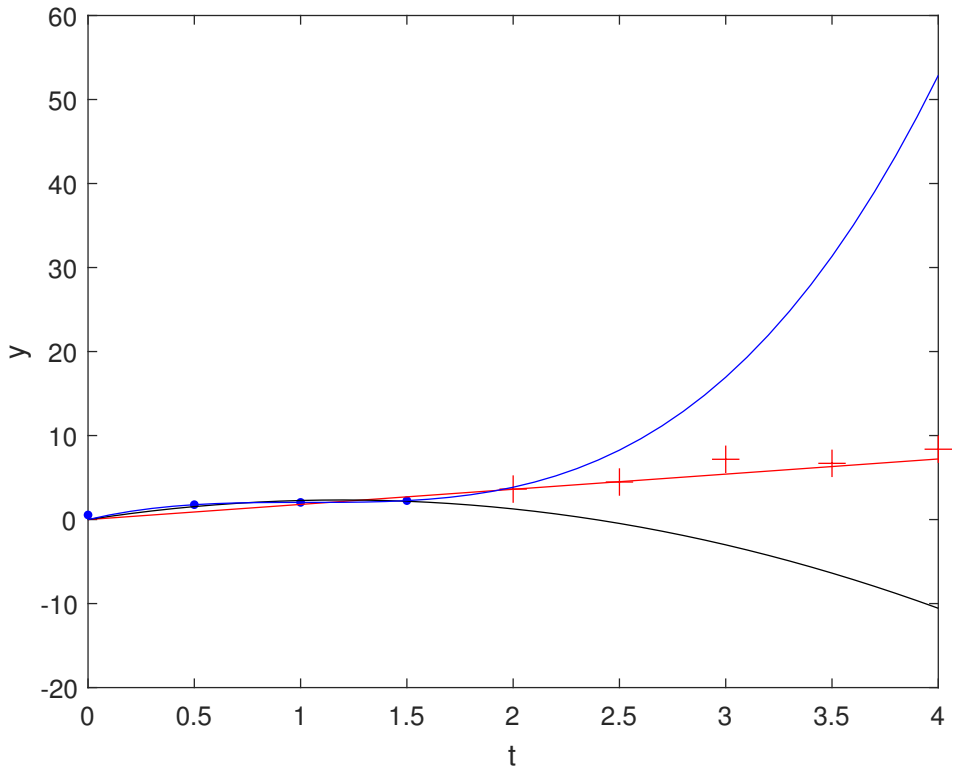
$$y_m^1(t_i) = a t_i$$
$$y_m^2(t_i) = b_1 t_i + b_2 t_i^2$$
$$y_m^3(t_i) = c_1 t_i + c_2 t_i^2 + + c_3 t_i^2$$



The most complex model minimizes the output error

Choosing the complexity - example

One compares the prediction capability of three models



The simplest model has the best prediction capability

Choosing the complexity

Statistical creteria used to decide when it is not worth further increasing the complexity.

But

- *a priori* knowledge (physics) may lead to a more complex model,
- application constraints may impose a simpler model (real-time control).

Choosing the complexity

One can, for example, look for a very simple model to describe the behavior of the process only in a given frequency band (near a critical frequency). This could lead to filter data with a bandpass filter to

- eliminate irrelevant information
- reduce errors induced by offsets and high-frequency noise.

Experiment design tools
for the choice of the model structure.

Structural properties of models

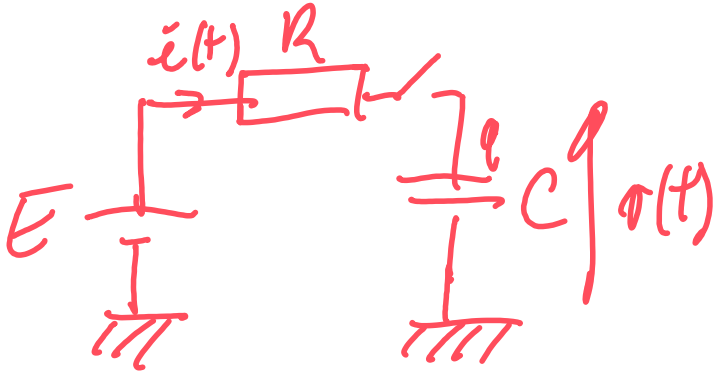
Suppose we choose a model structure (possibly several, between which we will have to decide).

It is important to study its properties as independently as possible from the values taken by the parameters in order to detect possible problems before collecting data.

A property is said to be structural if it is true for almost any value of the parameters, and possibly false on a sub-space of the parametric space of null measure.

We therefore have a null probability of finding an atypical value of the parameters by drawing p at random.

Two structural properties are of particular importance in our context, those of identifiability and distinguishability.



At $t = 0$, $q(t) = 0$.

E is known

R and C have to be determined.

$$v(t) = E\left(1 - e^{-t/RC}\right) u(t).$$

$u(t)$: Heaviside function

$$R \to \alpha R = R'$$

$$C \to \frac{C}{\alpha} = C'$$

$$R'C' = RC.$$

$$i(t) = \frac{E}{R} e^{-t/RC} \cdot u(t)$$

Identifiability

It is natural to question whether there is any chance of success for the estimation of p.

In such vague terms, the question has no answer.

We consider an idealized framework where

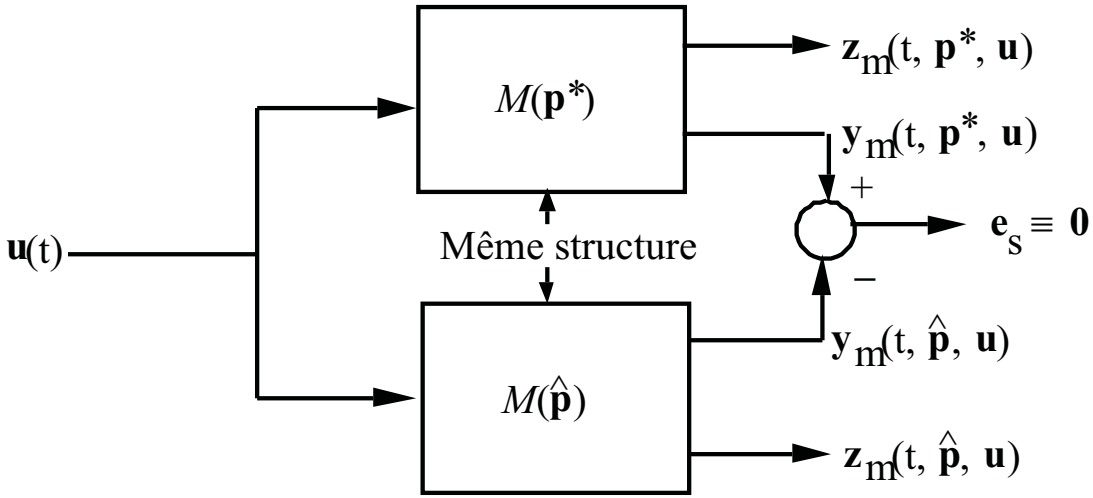
- process and model have identical structures (no characterization error),
- the data is noiseless,
- the applied input and the measuring times can be freely selected.

It is then always possible to set the model parameters $\widehat{\mathrm{p}}$ so that the model has the same input-output behavior as the process.

$$M(\mathrm{p}^*) = M(\widehat{\mathrm{p}}).$$

↳ true value of the parameter vector

Structural Identifiability



Idealized framework of the structural identifiability study.

$M(\hat{p}) = M(p^*)$ has always the solution $\hat{p} = p^*$. We hope that there is no other solution.

More precisely, the parameter p_i is
structurally globally identifiable (s.g.i.)
if for almost all $p^* \in \mathbb{P}$,

$$M(\hat{p}) = M(p^*) \Rightarrow \hat{p}_i = p_i^*$$

$M(.)$ is s.g.i. if all its parameters are s.g.i.

Structural Identifiability

Unfortunately,

- $M(.)$ is not always s.g.i.
- even if it is, it may be difficult to prove this.

A weaker notion:

The parameter p_i is structurally locally identifiable (s.l.i.)
if for almost all $\mathrm{p}^* \in \mathbb{P}$, there exists some neighborhood $\mathbb{N}(\mathrm{p}^*)$ such that

$$\widehat{\mathrm{p}} \in \mathbb{N}(\mathrm{p}^*) \text{ et } M(\widehat{\mathrm{p}}) = M(\mathrm{p}^*) \Rightarrow \widehat{p}_i = p_i^*$$

Local identifiability is a necessary condition for global identifiability.

$M(.)$ is s.l.i. when all its parameters are s.l.i.

Structural Identifiability - examples

Example simpliste : Ohm's law (one measures U and I)

$$U = RI,\ p = R \text{ is s.g.i.}$$

If one would have chosen

$$U = \cos(p)\, I,\ p \text{ would be s.l.i.}$$

If one would have chosen (U is measured)

$$U = p_1 p_2,\ (p_1, p_2)^T \text{ would be s.n.i.}$$

Test methods are now required.

Structural Identifiability

The parameter p_i is structurelly non-identifiable (s.n.i.) if for almost all $\mathrm{p}^* \in \mathbb{P}$, there exists a non-countable infinity of values $\widehat{p}_i$ such that $M(\widehat{\mathrm{p}}) = M(\mathrm{p}^*)$.

$M(.)$ is s.n.i. if at least one of its parameters is s.n.i.

Structural Identifiability

Some parameters of a s.n.i. model may be s.l.i. or even s.g.i.

The identifiabilitï¿œ sometimes depends on the numerical value taken by the parameters without the atypical region being of null measurement. It is then impossible to give a structural conclusion.

Structural Identifiability - Test method

This method may be applied to stationnary LI models.

Consider the strcuture $M(.)$ described by

$$\frac{dx(t)}{dt} = A(p)x(t) + B(p)u(t),\ x(0) = 0,$$
$$y_m(t) = C(p)x(t) + D(p)u(t)$$

Structural Identifiability - Test method

1. Evaluate the associated transfer matrix

$$H(s,p) = C(p)[sI - A(p)]^{-1}B(p) + D(p).$$

2. Write $H(s,p)$ in canonical form $H_c(s,p)$, *i.e.*, in a form where there is a single way to write the transfer matrix. For example :
   1. simplify each element of $H(s,p)$ by the GCD of the numerator and denominator,
   2. develop numerators and denominators with increasing powers of s,
   3. normalizer to one the term of ~~highest~~ lowest degree in each denominator.
3. Perform a term-by-term identification of the coefficients of $H_c(s,\hat{p})$ and $H_c(s,p^*)$.
4. Solve the resulting equations in $\hat{p}$.

Structural Identifiability - Test method

If for almost all p^*

- there is a unique solution for $\widehat{p}$, equal to p^*, then $M(.)$ is s.g.i.
- the set of solutions is countable (usually finite), $M(.)$ is s.l.i.
- the set of solutions is uncountable, $M(.)$ is s.n.i.

Structural Identifiability - Test method

A parameter which takes always the same value in all solutions is s.g.i.

A parameter taking its value in a countable set is s.l.i.

For a given structure $M(.)$, some parameters may be s.g.i., while others are s.l.i. and even s.n.i.

The canonical form is mandatory.

Consider for example

$$H(s,p) = \frac{p_1}{p_2 + p_3 s}$$

one may too quickly conclude that

$$H(s,\hat{p}) = H(s,p^*) \Longrightarrow \hat{p} = p^*$$

$$H_d(s,p) = \frac{p_1/p_2}{1 + p_3/p_2 s}$$

$$\begin{cases} \hat{p}_1/\hat{p}_2 = p_1^*/p_2^* \\ \hat{p}_3/\hat{p}_2 = p_3^*/p_2^* \end{cases}$$

Structural Identifiability - Test method

This is not the case, since

$$\mathrm{H}(s,\mathrm{p}) = \frac{p_1}{p_2 + p_3 s}$$

corresponds to

$$\mathrm{H}_c(s,\mathrm{p}) = \frac{p_1/p_2}{1 + p_3/p_2 s}$$

entirely characterized by p_1/p_2 and p_3/p_2. Consequently $M(\widehat{\mathrm{p}}) = M(\mathrm{p}^*)$ is equivalent to

$$\frac{\widehat{p}_1}{\widehat{p}_2} = \frac{p_1^*}{p_2^*}$$
$$\frac{\widehat{p}_3}{\widehat{p}_2} = \frac{p_3^*}{p_2^*}$$

There are infinitely many solutions $\widehat{\mathrm{p}}$ such that $M(\widehat{\mathrm{p}}) = M(\mathrm{p}^*)$ and $M(.)$ is thus s.n.i.

Structural Identifiability - Test method

Consider the (LI, non-LP) model

$$\frac{dx_1}{dt} = -(p_1 + p_2)x_1 + p_3 x_2 + u,\ x_1(0) = 0,$$
$$\frac{dx_2}{dt} = p_1 x_1 - p_3 x_2,\ x_2(0) = 0,$$
$$y_m = x_2.$$

Using the Laplace transform

$$(s + p_1 + p_2)x_1(s) = p_3 x_2(s) + u(s),$$
$$(s + p_3)x_2(s) = p_1 x_1(s),$$
$$y_m(s) = x_2(s)$$

By elimination of x_1 and x_2, one gets

$$(s + p_1 + p_2)(s + p_3)y_m(s) = p_1 p_3 y_m(s) + p_1 u(s),$$

leading to the canonical transfer function

$$H_c(s, p) = \frac{p_1}{s^2 + s(p_1 + p_2 + p_3) + p_2 p_3}$$

Structural Identifiability - Test method

$$\mathrm{H}_c(s,\mathrm{p}) = \frac{p_1}{s^2 + s(p_1+p_2+p_3) + p_2 p_3}$$

The relation $\mathrm{H}_c(s,\widehat{\mathrm{p}}) = \mathrm{H}_c(s,\mathrm{p}^*)$ leads to

$$\begin{aligned} \widehat{p}_1 &= p_1^* \\ \widehat{p}_2 + \widehat{p}_3 &= p_2^* + p_3^* \\ \widehat{p}_2 \widehat{p}_3 &= p_2^* p_3^* \end{aligned}$$

which has two solutions in $\widehat{\mathrm{p}}$:

$$\widehat{\mathrm{p}}_1 = (p_1^*, p_2^*, p_3^*)^{\mathsf{T}} \text{ et } \widehat{\mathrm{p}}_2 = (p_1^*, p_3^*, p_2^*)^{\mathsf{T}}.$$

- p_1, is s.g.i.
- p_2 and p_3, which may take two different values, are only s.l.i.

Therefore, we can determine the true value of p_1 from noiseless data, but for p_2 and p_3 we will obtain two possible values between which we will not be able to decide.

Structural Identifiability - Remarks

- In the absence of noise, an iterative algorithm will converge towards one or the other solution.
- In the presence of noise, there is no reason for the problem to disappear. We have two vectors of parameters associated with the same behavior.
- Knowing one of these two solutions, we are now able to generate the other.
- This example illustrates the need for the statement "for almost any value of p^*" in dï¿œfinitions: If p^* belongs to the plane of the allowable parametric space defined by $p_1 = 0$, the output of the process will be identically null and p_2 and p_3 will become non-identifiable. Nevertheless, we will say that they are s.l.i., because $p_1 = 0$ is atypical.

Structural Identifiability - Remarks

- The fact that this model is not s.g.i. means that we will not be able to reconstruct the state x in a unique way from the sole knowledge of the input-output behavior. Depending on the model chosen, we will obtain two possible values for x_1.
- If the model considered is intended for the sole prediction of the process output, the notion of identifiability is not relevant.

Structural Identifiability - Remarks

- Testing identifiability often leads to cumbersome algebraic manipulations, which are made easier by computer algebra software like MAPLE.
- The calculations required are sometimes too complex to be performed on the most powerful computers. The following method can be used to test whether $M(.)$ is (at least locally) identifiable.
  - Choose a nominal value p_0 of the parameters (at random in $\mathbb{P}$).
  - Simulate $M(p_0)$ with high precision to obtain data fictives y_f.
  - Estimate p from y_f by minimizing a quadratic criterion on the output error using a Newton or Gauss-Newton method initialized at $\widehat{p}_0 = p_0$. If $\widehat{p}_k$ remains stable in p_0, $M(.)$ is s.l.i.

Distinguishability

One may hesitate between several model structures.

Natural, then, to ask whether the measures being considered will help decide which is the best one.

This is the question of the distinguishability of structures, which receives a partial answer in the same idealized framework as the identifiability study.

It is assumed that

- the "process" is a model of structure $M(.)$
- its "model" has a sructure $\widehat{M}(.)$ different from $M(.)$.

Distinguishability

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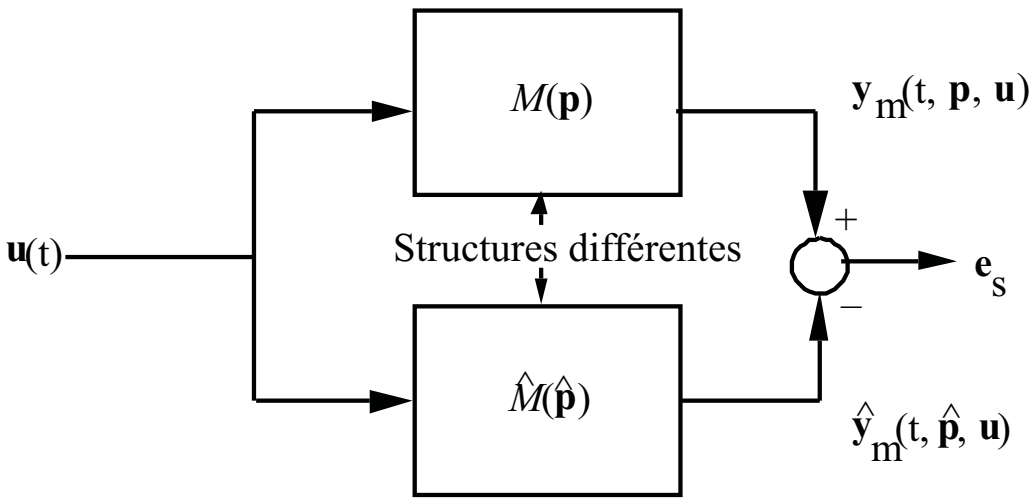
Distinguishability

The vectors of parameters (not necessarily of the same size)

- of $\widehat{M}(.)$ is denoted as $\widehat{\mathrm{p}}$,
- of $M(.)$ is denoted as p.



Idealized framework of a structural distinguishability study

We hope that there is no $\widehat{\mathrm{p}}$ such that $\widehat{M}(\widehat{\mathrm{p}}) = M(\mathrm{p})$.

This impossibility will help eliminating $\widehat{M}(.)$ in favor of $M(.)$.

Structural Distinguishability

More precisely, $\widehat{M}(.)$ is structurally distinguishable (s.d.) from $M(.)$ if, for almost all p, there is no $\widehat{\mathrm{p}}$ such that

$$\widehat{M}(\widehat{\mathrm{p}}) = M(\mathrm{p}).$$

Note that $\widehat{M}(.)$ s.d. from $M(.)$ does not imply $M(.)$ s.d. from $\widehat{M}(.)$.
$\hookrightarrow$ for example, the class of the first model may be a super set of the class of the other model.

When $\widehat{M}(.)$ is s.d. from $M(.)$ and $M(.)$ is s.d. from $\widehat{M}(.)$, $M(.)$ and $\widehat{M}(.)$ are s.d.

Structural Distinguishability

Structural distinguishability testing techniques are similar to those used for identifiability testing.

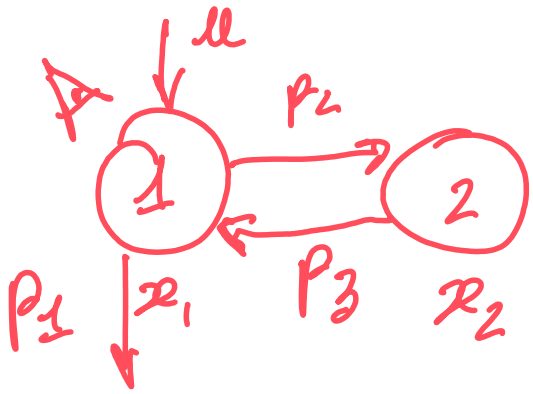
Two notable differences:

- we consider two models with different structures, instead of two models with the same structure,
- we hope to show the non-existence of a solution in $\widehat{p}$, rather than the uniqueness of this solution.

Structural Distinguishability - Example

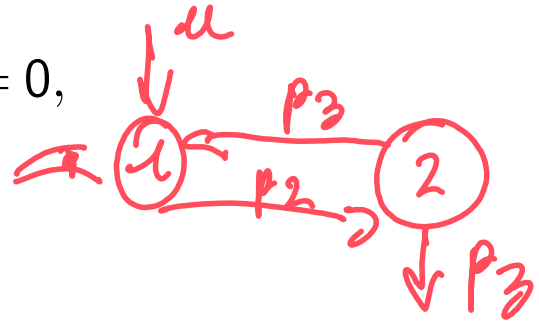
Consider $M(.)$ defined by

$$\frac{dx_1}{dt} = -(p_1 + p_2)x_1 + p_3 x_2 + u,\ x_1(0) = 0,$$
$$\frac{dx_2}{dt} = p_2 x_1 - p_3 x_2,\ x_2(0) = 0,$$
$$y_m = x_1$$



and $\widehat{M}(.)$ defined by

$$\frac{dx_1}{dt} = -\widehat{p}_2 x_1 + \widehat{p}_3 x_2 + u,\ x_1(0) = 0,$$
$$\frac{dx_2}{dt} = \widehat{p}_2 x_1 - (\widehat{p}_1 + \widehat{p}_3)x_2,\ x_2(0) = 0,$$
$$y_m = x_1$$



Structural Distinguishability - Example

Transfer function in canonical form:

$$\mathrm{H}_c(s,\mathrm{p}) = \frac{s+p_3}{s^2+(p_1+p_2+p_3)\,s+p_1p_3}$$

and

$$\widehat{\mathrm{H}}_c(s,\widehat{\mathrm{p}}) = \frac{s+\widehat{p}_1+\widehat{p}_3}{s^2+(\widehat{p}_1+\widehat{p}_2+\widehat{p}_3)\,s+\widehat{p}_1\widehat{p}_2}$$

One has

$$\widehat{M}(\widehat{\mathrm{p}}) = M(\mathrm{p}) \text{ iff } \begin{cases} \widehat{p}_1+\widehat{p}_3 = p_3 \\ \widehat{p}_1+\widehat{p}_2+\widehat{p}_3 = p_1+p_2+p_3 \\ \widehat{p}_1\widehat{p}_2 = p_1p_3 \end{cases}$$

For all p, there exists $\widehat{\mathrm{p}}$ such that both equations are satisfied and *vice-versa*.

$M(.)$ and $\widehat{M}(.)$ are thus structurally undistinguishable.

It is impossible to know if you have the wrong structure. One way to remove this uncertainty would be to make measurements of x_2.

Assume now that:

- only x_2 is observed, is $\hat{n}$ sd from n?
  is n sd from $\hat{n}$?

- both x_1 and x_2 are observed, same questions -

Structural Distinguishability - Remarks

As with identifiability, there are cases where no structural conclusion is possible.

Consider for example two models with transfer functions

$$\widehat{\mathrm{H}}_c(s,\widehat{\mathrm{p}}) = \frac{1}{s^2+\widehat{p}_1 s+\widehat{p}_2} \text{ and } \mathrm{H}_c(s,\mathrm{p}) = \frac{1}{(s+p_1)(s+p_2)}$$

with $\widehat{\mathrm{p}}$ and p belonging to $\mathbb{R}^2$.

- If $\widehat{\mathrm{p}}$ is such that $\widehat{\mathrm{H}}_c(s,\widehat{\mathrm{p}})$ has two real poles, $\mathrm{H}_c(s,\mathrm{p})$ is is undistinguishable from $\widehat{\mathrm{H}}_c(s,\widehat{\mathrm{p}})$;
- else $\mathrm{H}_c(s,\mathrm{p})$ is distinguishable from $\widehat{\mathrm{H}}_c(s,\widehat{\mathrm{p}})$.

Neither of these two situations can be considered atypical.

Relations between identifiability and distinguishability

- The identifiability of two structures is neither necessary nor sufficient for their distinguishability.
- The techniques to be implemented to test these two types of properties are quite similar.

Conclusions

- The choice of a model structure is not an exact science.
- It involves more or less arbitrary decisions but with important consequences (the best possible model has then to be found in the considered class).
- It must therefore be considered temporary, and questionable.
- A model structure may be challenged before taking any measurement, through the identifiability or distinguishability studies.

Conclusions

- The notions of identifiability and distinguishability are fundamental when
  - parameters or state variables have a physical meaning (knowledge-based model)
  - decisions are made on the basis of numerical values of parameters or state variables
- Their importance is marginal when we are only interested in reproducing an input/output behavior (behavioral model).