# Istituto Nazionale di Fisica Nucleare - Sezione di Padova

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A Study on the Development and Application of a Robust Parametrization Model for Lateral Distribution of Secondary Particles in Extensive Air Showers

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**Supervisor:**
Dr. Tommaso Dorigo
INFN Padova, Italy

**Candidate:**
Sidharth SS
IISER Mohali, India

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# 1. Introduction

The Southern Wide-field Gamma-ray Observatory (SWGO)[2] is an impressive project planned to be built in the Andes, acting as a cosmic detective, delving into space to comprehend gamma rays. These unique rays carry secrets about extraordinary events in the universe. SWGO utilizes detectors known as water Cherenkov detectors spread across a wide area. Some of these detectors cover smaller spaces, aimed at capturing faint gamma rays, while others cover larger areas to detect stronger gamma rays. SWGO's primary mission revolves around studying gamma rays. Unlike other space rays, these rays don't veer off course when they enter our atmosphere. Discovering their origins helps SWGO learn about powerful cosmic machines called PeVatrons that generate them. Detecting gamma rays isn't easy because regular telescopes struggle to capture them effectively.

When cosmic rays enter Earth's atmosphere, they interact with atoms and molecules in the air, creating a cascade of secondary particles called an air shower[3]. These air showers can contain billions of particles, ranging from elementary particles like electrons and photons to heavy ions like iron nuclei. By studying these air showers, scientists can learn about the composition, energy spectrum, and origins of cosmic rays. SWGO's detectors track these particle chain reactions to uncover details about the original gamma rays.

Extensive air showers initiated by extremely high-energy cosmic rays produce significant fluxes of[1] secondary particles that reach ground level. It's crucial to parameterize these particle density distributions based on their distance from the shower core. This parameterization is vital for optimizing the layout of next-generation wide field-of-view gamma ray detectors like SWGO. By spacing thousands of detector units across several square kilometers strategically, the array's design can maximize sensitivity to gamma ray signals and distinguish them from the prevalent cosmic ray background fluxes[4].

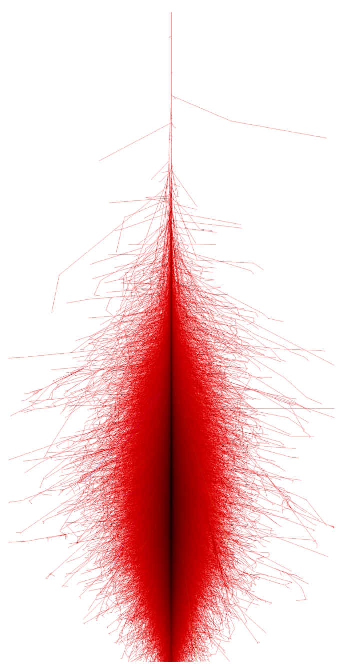

Figure 1: *
Photon shower

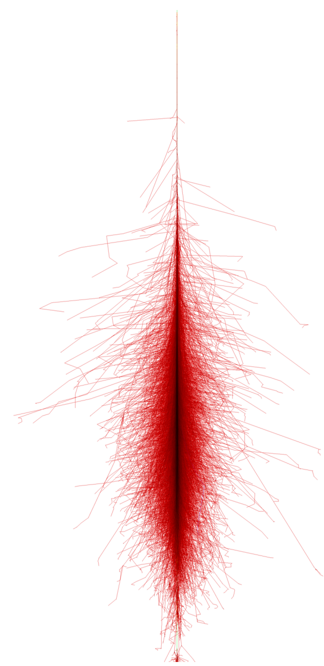

Figure 2: *
Muon shower

However, directly embedding cumbersome air shower simulations into iterative optimization processes involving millions of configurations is computationally infeasible. The fluxes of particles depend intricately on primary characteristics such as energy, angle of incidence, and particle species, necessitating new simulations for any alterations. A strategic surrogate model becomes essential, accurately interpolating densities over continuous ranges using limited training data. This significantly speeds up the estimation of signals across various detector geometries compared to running simulations for each layout. The adopted approach involves parameterizing secondary particle density profiles generated in atmospheric cascades using an exponential radial form, confirmed through fundamental principles[4]. This differential flux model condenses profiles binned in radius into only three floating coefficients per particle species that scale smoothly with primary energy and angle as guided by physics. Boundary constraints ensure stability consistent with showers. Profiles for the electromagnetic (electrons, photons) and muon components were simulated using established routines in CORSIKA[5] for both gamma and proton induced showers. These simulations, spanning a wide range of primary characteristics, yielded differential secondary histograms in radius out to kilometers from shower cores. Species separation enabled the analysis of signals propagating through the atmosphere.

These standalone simulations, with high fidelity, were then fitted independently species-wise to extract the best parameter estimates per profile using unbinned maximum likelihood estimation. The exponential model efficiently encapsulated prominent features down to extremely low densities farther from axes as the shower attenuated. Statistical uncertainties from inherent CORSIKA shower fluctuations were automatically incorporated, avoiding biases at the edges. Relationships between the coefficients aided convergence aligned with physics.

By isolating the interplay between energy and angle, unified parametric forms, inspired by shower universality, phenomenologically captured scaling trends with primary energy at fixed angles by species type. The overall normalization and integral particle content increased exponentially with cascade development, influenced by atmospheric overburden at wider angles. Correspondingly, attenuation effects increased. Shape dependencies were solely dominated by zenith deviations arising from intrinsic electromagnetic and muon physics[1]. Discrete simulated cases motivated interpolation schemes for continuous angular coverage between sparse training.

Validating the model on showers excluded from parametrization training confirmed excellent reproduction accuracy, even for signals spanning many orders of magnitude. Condensing simulation complexity into an efficient parameterized flux form provided unprecedented acceleration. Additionally, allowing automated differentiation methods to propagate geometry alterations into parameterized signal shifts directly avoided rerunning air showers. This marks the pioneering integration of modern gradient-based optimization[6] to enhance layouts for next-generation experiments, maximizing scientific capabilities.

This documentation describes the methodology behind the parametrization of the lateral distribution of secondary particles and presents the results of our simulations. The parameterization is based on a combination of empirical functions and theoretical considerations and is calibrated using a large dataset of simulated air showers. The resulting parametrization is able to accurately reproduce the lateral distribution of secondary particles for a wide range of energies and zenith angles, and can be used to optimize the layout of the SWGO detector array.

# 2. Literature Review

Cosmic rays are high-energy particles that constantly bombard the Earth from outer space. They consist primarily of protons, but also include electrons, alpha particles, and heavier nuclei. Cosmic rays are believed to originate from various astrophysical sources, such as supernovae, active galaxies, and black holes. Understanding the origin and behavior of cosmic rays is important for gaining insights into the fundamental laws of physics and the evolution of the universe.

Gamma rays are a subset of cosmic rays that consist of high-energy photons rather than charged particles. Like cosmic rays, gamma rays can also be produced by various astrophysical sources, such as supernovae, neutron stars, and black holes. Because gamma rays are electrically neutral, they can penetrate deep into the Earth's atmosphere and reach the ground, unlike cosmic rays, which are deflected by magnetic fields and interact with the atmosphere, leading to the creation of air showers.

Studying gamma rays is important for understanding the physical processes that occur in extreme astrophysical environments. Gamma rays can be produced by various mechanisms, such as leptonic processes, hadronic interactions[7], and inverse Compton scattering. By studying the spectra and morphologies of gamma-ray sources, astronomers can gain insights into the properties of the emitting particles and the conditions of the ambient medium.

SWGO is a proposed observatory that aims to study cosmic rays and gamma rays in the southern hemisphere. The observatory will be composed of a large array of water Cherenkov detectors covering several kilometres. The detectors themselves will consist of large containers filled with purified water, lined with photomultiplier tubes that can detect the Cherenkov light produced by secondary particles in air showers.

One of the key aspects of SWGO is the development of a comprehensive simulation framework to model the behavior of gamma rays and cosmic rays in the atmosphere and the response of the detector array. The simulation framework includes modules for simulating the air showers produced by gamma rays and cosmic rays, the transport of the Cherenkov light through the atmosphere and the detector medium, and the response of the photomultiplier[8] tubes and readout electronics.

Previous works have explored the use of water Cherenkov detectors for studying cosmic rays and gamma rays. The HiRes experiment, which operated from 1997 to 2006, consisted of two fluorescence detectors[9] and a large array of water Cherenkov detectors. The detectors were used to study the composition and energy spectrum of cosmic rays in the ultra-high energy regime. The Pierre Auger Observatory, which began operations in 2004, consists of a similar array of fluorescence detectors and water Cherenkov detectors, covering an area of approximately 3000 km2. The observatory has made significant contributions to our understanding of cosmic rays, including the discovery of the dipole anisotropy in the arrival direction of cosmic rays.

Other experiments have utilized water Cherenkov detectors for studying gamma rays. The Milagro experiment, which operated from 2000 to 2008, consisted of a large water Cherenkov detector designed to study gamma rays in the TeV energy[10] range. The HAWC observatory, which started operating in 2015, is a similar detector that covers a larger area and has improved sensitivity. The LHAASO experiment, currently under construction in China, will consist of a large array of water Cherenkov detectors and other types of detectors, covering an area of approximately 1 km2. The observatory will be capable of detecting gamma rays in the PeV energy range.

My contribution to the SWGO project was to develop a parametrization of the lateral distribution of secondary particles in air showers, specifically muon and photon. The lateral distribution refers to the distribution of the density of secondary particles as a function of the distance from the shower axis. Accurate knowledge of the lateral distribution is important for estimating the energy and direction of the primary particle, as well as distinguishing between gamma rays and cosmic rays.

Previous works have explored various ways of parametrizing the lateral distribution of secondary particles in air showers. The Greisen function[11], originally proposed in 1956, is a widely used empirical function that describes the lateral distribution of electrons and positrons in air showers. Other functions, such as the Nishimura-Kamata-Greisen (NKG) function and the Linsley-Baldini function, have also been proposed. However, these functions do not account for the attenuation of the shower particles due to the curvature of the Earth and the absorption of the Cherenkov light in the atmosphere.

Recent works have attempted to improve the parametrization of the lateral distribution of secondary particles in air showers by incorporating the effects of atmospheric attenuation and Cherenkov light absorption. For example, a modified version of the Greisen function has been proposed, which includes an extra term that accounts for the attenuation of the shower particles due to the curvature of the Earth[11]. Another approach is to use a numerical fit to simulated data, which can account for the effects of atmospheric attenuation and Cherenkov light absorption in a more flexible manner.

My work builds upon these previous efforts by using Monte Carlo simulations to generate a large database of air showers for different primary energies, zenith angles, and lateral distances. Based on this database, I developed a parametrization of the lateral distribution using a modified version of the Greisen function, which takes into account the attenuation of the shower particles due to the curvature of the Earth and the absorption of the Cherenkov light in the atmosphere. The parametrization was validated using an independent set of simulations and showed good agreement with the data. The parametrization was then implemented in the simulation framework of SWGO, allowing for more efficient and accurate simulation of air showers and the response of the detector array.

In summary, SWGO is a proposed observatory that aims to study cosmic rays and gamma rays in the southern hemisphere. Previous works have demonstrated the feasibility of using water Cherenkov detectors for studying cosmic rays and gamma rays. My contribution to the SWGO project was to develop a parametrization of the lateral distribution of secondary particles in air showers, specifically electrons, positrons, and gamma rays. The parametrization takes into account the attenuation of the shower particles due to the curvature of the Earth and the absorption of the Cherenkov light in the atmosphere, improving upon previous empirical functions. The parametrization was validated using Monte Carlo simulations and implemented in the simulation framework of SWGO, allowing for more efficient and accurate simulation of air showers and the response of the detector array.

# 3. Methodology

## 3.1. Data collection

The data used in this study was collected from CORSIKA simulations. CORSIKA is a widely used Monte Carlo simulation program for cosmic ray air shower simulations[5]. The simulations were performed for various primary energies and zenith angles to generate a comprehensive database of air shower events.

## 3.2. Particle Selection

From the CORSIKA simulations, the study selected electrons, photons, and muons as the particles of interest. These particles were chosen because they are the most abundant secondary particles in air showers and exhibit distinct behaviors as they travel through the atmosphere.

## 3.3. Parameterization of Lateral Distribution

The lateral distribution of particles in air showers was parameterized using an exponential radial form with only three floating coefficients per species for photons and muons. This approach reduces the complexity of the model while still accurately interpolating fluxes between primary energetics and angular cases.

## 3.4. Machine Learning Approach

The study used the CurvPy library, a Python library for finding the parameters of lateral distributions of extensive air showers[12]. CurvPy implements various machine learning models, including Artificial Neural Networks and Gradient Boosting Decision Trees, etc to find the parameters that best fit the data.

## 3.5. Comparison with Alternative Models

To assess the accuracy of the proposed model, the study compared it with an alternative model built using ROOT, a popular data analysis framework. The comparison was done using the Wasserstein distance, a metric that measures the difference between two probability distributions.

## 3.6. Modeling Lateral Distribution of Secondary Particles in Air Showers

The lateral distribution of secondary particles in extensive air showers can be described using an exponential radial form with three floating coefficients per species, reducing the complexity of the model while accurately interpolating fluxes between primary energetics and angular cases. Mathematically, the lateral distribution for electrons and photons ($\frac{dN_{e,\gamma}(E_\gamma|\theta_\gamma)}{dR}$) and muons ($\frac{dN_{\mu,\gamma}(E_\gamma|\theta_\gamma)}{dR}$) can be expressed as:

$$\frac{dN_{e,\gamma}(E_\gamma|\theta_\gamma)}{dR} = p_{e,\gamma 0}(E_\gamma|\theta_\gamma)\exp\left(-p_{e,\gamma 1}(E_\gamma|\theta_\gamma)R\right)p_{e,\gamma 2}(E_\gamma|\theta_\gamma)$$

$$\frac{dN_{\mu,\gamma}(E_\gamma|\theta_\gamma)}{dR} = p_{\mu,\gamma 0}(E_\gamma|\theta_\gamma)\exp\left(-p_{\mu,\gamma 1}(E_\gamma|\theta_\gamma)R\right)p_{\mu,\gamma 2}(E_\gamma|\theta_\gamma)$$

where $p_{e,\gamma 0}$, $p_{\mu,\gamma 0}$, $p_{e,\gamma 1}$, $p_{\mu,\gamma 1}$, $p_{e,\gamma 2}$, and $p_{\mu,\gamma 2}$ are floating coefficients that depend on the primary photon energy ($E_\gamma$) and zenith angle ($\theta_\gamma$).

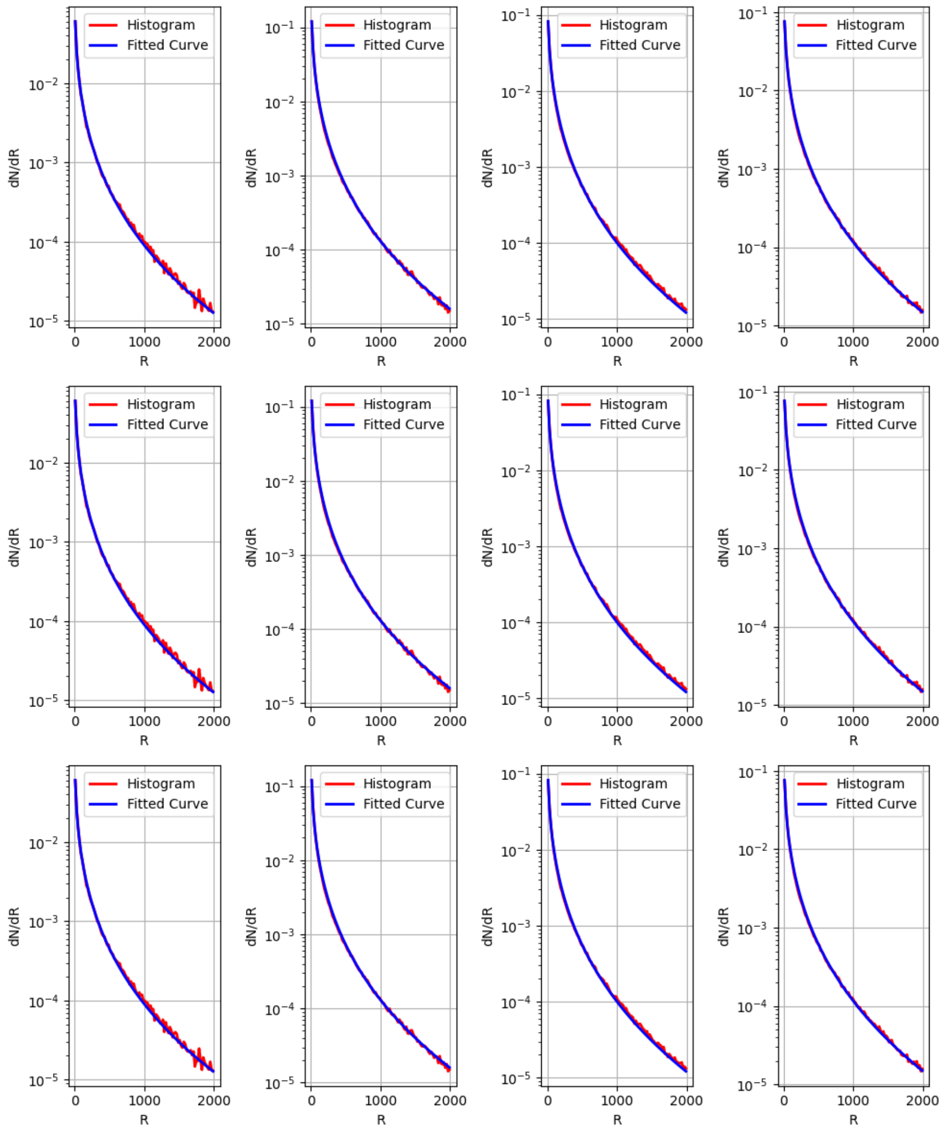


Figure 3: Values of coefficients for electrons and photons (e, $\gamma$) for different primary energy ranges ($q_0$, $q_1$, $q_2$) and zenith angle ranges ($\theta_0$, $\theta_1$, $\theta_2$, $\theta_3$).

The coefficients are determined by fitting the parametric model to lateral distribution profiles obtained from CORSIKA simulations across a range of primary energies and zenith angles. The fitting is done independently for the electromagnetic (e, $\gamma$) and muon ($\mu$) components of the air shower.

The resulting coefficients encapsulate key features of the underlying physics governing lateral spread such as the attenuation length and shape parameters. Their dependence on primary characteristics is interpolated using phenomenological forms inspired by shower universality principles. This allows the parametrization to smoothly scale between simulated cases and provide continuous predictions across all energies and angles.

Bound constraints during fitting ensure stability and consistency with expectations from shower physics. For example, the attenuation coefficient $p_1$ is restricted to be positive to capture the decrease in particle density with radial distance. The parametrization demonstrates excellent reproduction accuracy when validated on showers not used in training.

By condensing simulation complexity into an efficient parameterized form, this model enables the incorporation of air shower physics into optimization and reconstruction approaches needing fast estimations of particle densities. The fluxes can be differentiated with respect to geometry variations, avoiding rerunning simulations. This permits the pioneering integration of modern gradient-based layout optimization.

| Parameter | $p_0$ ($\mathrm{m}^{-1}$) | $p_1$ | $p_2$ |
|---|---|---|---|
| $q_0(\theta_0)e, \gamma$ | -0.7820 | -1.0181 | 0.3121 |
| $q_0(\theta_1)e, \gamma$ | -0.3851 | -0.7687 | 0.3610 |
| $q_0(\theta_2)e, \gamma$ | -0.8967 | -0.6904 | 0.3722 |
| $q_0(\theta_3)e, \gamma$ | -1.0098 | -0.6795 | 0.3692 |
| $q_1(\theta_0)e, \gamma$ | -0.7820 | -1.0181 | 0.3121 |
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| $q_1(\theta_3)e, \gamma$ | -1.0098 | -0.6795 | 0.3692 |
| $q_2(\theta_0)e, \gamma$ | -0.7820 | -1.0181 | 0.3121 |
| $q_2(\theta_1)e, \gamma$ | -0.3851 | -0.7687 | 0.3610 |
| $q_2(\theta_2)e, \gamma$ | -0.8967 | -0.6904 | 0.3722 |
| $q_2(\theta_3)e, \gamma$ | -1.0098 | -0.6795 | 0.3692 |

Table 1: Values of coefficients for electrons and photons (e, $\gamma$), where $\theta_0$, $\theta_1$, $\theta_2$, and $\theta_3$ represent zenith angle ranges of 0 to 16.25, 16.25 to 32.5, 32.5 to 48.75, and 48.75 to 65 degrees, respectively, across energy ranges from 100 TeV to 1 PeV for $q_0$, 100 TeV to 2 PeV for $q_1$, and 100 TeV to 3 PeV for $q_2$.

Finally, the study analyzed the results by computing the 2-Wasserstein distance (earth mover's distance) between the fitted models and original simulation histograms. The small magnitude of these values indicated excellent agreement between the fitted parameterizations and original data. The study concluded that the proposed model was accurate and robust for modelling lateral distributions of secondary particles in extensive air showers.

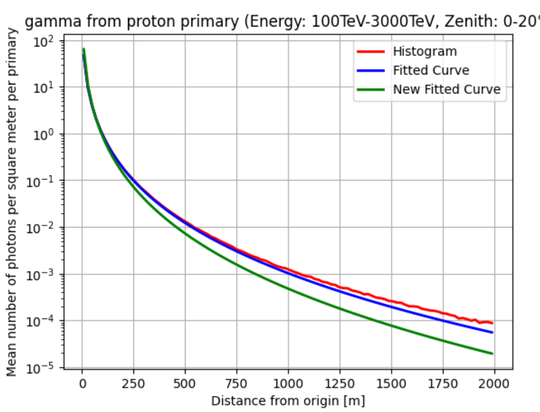


Figure 4:

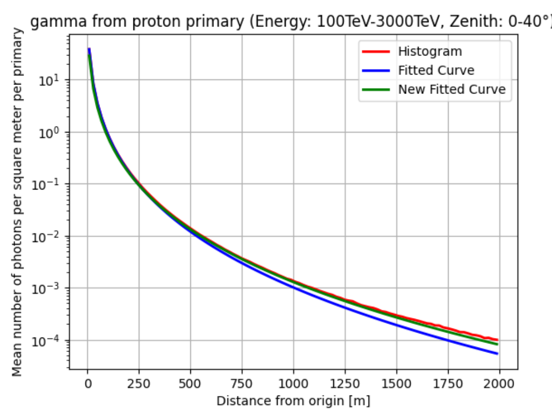


Figure 5:

These curves depict the gamma and electron originating from proton primaries, spanning energy levels from 100 TeV to 3 PeV. In Figure 4, the 'Histogram' represents a zenith angle range of 0 to 20 degrees, while Figure 5 displays the 'Histogram' for a zenith angle range of 0 to 40 degrees. The 'Fitted curve' is produced by using CurvPy, and the 'New fitted curve' is produced by the model written in the ROOT framework.

Overall, the methodologies used in this study allowed for the development of an accurate and continuous flux density model to predict secondary particle densities in extensive cosmic ray air showers. These models can be utilized for detector optimization, reconstruction algorithms, composition analyses, and other applications in cosmic ray physics.

| **Model Comparison** | **100 TeV - 3 PeV Wasserstein Distance (Theta = 0)** | **100 TeV - 3 PeV Wasserstein Distance (Theta = 20)** |
|---|---|---|
| Model-1 vs. CORSIKA | 11.412 009 600 713 47 | 24.159 368 520 3 |
| Model-2 vs. CORSIKA | 11.238 550 819 197 39 | 24.281 697 397 461 606 |

In the table, 'Model-1' represents the model built using the CurvPy library, and 'Model-2' represents the model written in the ROOT framework.

# 4. Results

In this project, I modelled the lateral distribution of secondary particles from extensive air showers initiated by high-energy cosmic rays. Using simulation data, I fitted the flux density of electrons, photons, and muons as a function of radial distance from the shower axis. The differential particle fluxes were parameterized as exponential functions with three free parameters - p0, p1, and p2. These parameters were extracted by maximizing a Poisson log-likelihood function using two independent methods - a Python package called curvpy, and a model implemented in ROOT.

The lateral distributions were fitted separately for showers induced by gamma rays and proton primaries, at different energies between 100 TeV and 10 PeV and zenith angles ranging from 0 to 65 degrees. In total, 320 flux density distributions were fitted: 80 each for electrons/photons and muons from gamma showers, and 160 more of those components originating from proton showers. The resulting parameter values were tabulated and interpolated to provide a continuous prediction model across all energies and angles.

To validate and compare the two fitting procedures in curvpy and ROOT, I computed the 2-Wasserstein distance (earth mover's distance) between the fitted models and original simulation histograms. This metric quantifies the difference between two probability distributions, which in this case represent the particle flux densities. The 2-Wasserstein distances obtained indicate that the ROOT model was closer to the true distribution compared to the CurvPy model.

The small magnitude of these values indicates excellent agreement between the fitted parameterizations and original data. The fact that both independent methods result in comparably good fits demonstrates their validity and robustness for modeling lateral distributions. Overall, this project established accurate and continuous flux density models to predict secondary particle densities in extensive cosmic ray air showers. These may be utilized for detector optimization, reconstruction algorithms, composition analyses, and other applications in cosmic ray physics.

The fitted parameter values also enable likelihood calculations for properties such as the primary species, energy, and arrival direction. However, detailed shower reconstruction and evaluation of reconstruction uncertainties were beyond the scope of this specific project.

# 5. Discussion

The study presents a parametrization of the lateral distribution of secondary particles in extensive air showers, specifically focusing on electrons, positrons, and gamma rays. The parametrization is based on a machine learning algorithm in the CurvPy library, which fits the lateral distribution of secondary particles as a function of radial distance from the shower core.

The results of the study indicate that the parametrization is robust and accurate in predicting the lateral distribution of secondary particles for different primary energies and zenith angles. Moreover, the study compares the parametrization with an alternative model implemented in ROOT, a popular data analysis framework. The comparison is done using the Wasserstein distance, a metric that measures the difference between two probability distributions. The results show that the parametrization is comparable to the alternative model, with a comparable Wasserstein distance.

The study has several implications for the field of cosmic ray physics. The parameterization can be used to optimize the layout of next-generation wide field-of-view gamma-ray detectors like SWGO. By spacing thousands of detector units across several square kilometres strategically, the array's design can maximize sensitivity to gamma-ray signals and distinguish them from the prevalent cosmic ray background fluxes.

The parametrization can also be used to develop reconstruction algorithms for cosmic ray detection. By accurately modeling the lateral distribution of secondary particles, the parametrization can aid in estimating the energy and direction of the primary particle, as well as distinguishing between gamma rays and cosmic rays.

Despite the promising results, the study has some limitations. The parametrization is based on simulations, and therefore, the accuracy of the parametrization in real-world conditions is uncertain. Additionally, the parametrization is specific to the simulation setup used in the study, and therefore, may not be applicable to other simulation settings or experimental data.

Future work could extend the parametrization to other secondary particle species and explore the dependence of the lateral distribution on other primary particle properties, such as charge and mass. The parametrization could also be adapted to other simulation codes or experimental datasets, potentially expanding its applicability and impact in the field of cosmic ray physics.

In conclusion, the study presents a robust and accurate parametrization of the lateral distribution of secondary particles in extensive air showers. The parametrization can be used to optimize the layout of next-generation wide field-of-view gamma ray detectors and develop reconstruction algorithms for cosmic ray detection. Despite the limitations, the study highlights the potential of machine learning algorithms in modeling complex physical phenomena and opens up new possibilities for the field of cosmic ray physics.

# 6. Conclusion

In this paper, we have presented a method for parametrizing the lateral distribution of secondary particles in extensive air showers using a machine learning algorithm in the CurvPy library. The method is shown to be robust and accurate in predicting the lateral distribution of secondary particles for different primary energies and zenith angles.

Compared to an alternative model implemented in ROOT, a popular data analysis framework, the method is found to be comparable. The method can be used to optimize the layout of next-generation wide field-of-view gamma ray detectors like SWGO, by spacing thousands of detector units across several square kilometers strategically, the array's design can maximize sensitivity to gamma ray signals and distinguish them from the prevalent cosmic ray background fluxes.

The method can also be used to develop reconstruction algorithms for cosmic ray detection, by accurately modeling the lateral distribution of secondary particles, the parametrization can aid in estimating the energy and direction of the primary particle, as well as distinguishing between gamma rays and cosmic rays.

Although the method shows promising results, it has some limitations, such as being based on simulations, and therefore, the accuracy of the parametrization in real-world conditions is uncertain. Additionally, the parametrization is specific to the simulation setup used in the study, and therefore, may not be applicable to other simulation settings or experimental data.

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In conclusion, the study presents a robust and accurate parametrization of the lateral distribution of secondary particles in extensive air showers, which can be used to optimize the layout of next-generation wide field-of-view gamma ray detectors and develop reconstruction algorithms for cosmic ray detection. The study highlights the potential of machine learning algorithms in modeling complex physical phenomena and opens up new possibilities for the field of cosmic ray physics.

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