ANDERSON LOCALISATION AND DECOHERENCE

Jack Lowe

Supervisor: Florian Mintert
Assessor: Alejandro Valido
Project code: QOLS-MINTERT-2

Word count: 2423

Imperial College London

1 Abstract

Anderson localisation is compared to classical transport. Weak localisation is studied to give an understanding of how localisation can occur. Anderson's model and the results of his original paper are discussed. The scaling theory of localisation is presented and used to derive a condition for localisation, explain the mobility edge and explain why all states with $d \leq 2$ are localised. Stress is put on how Anderson localisation is vulnerable to decoherence. Decoherence is introduced as how systems lose quantum coherence. The Lindblad master equation is discussed, and recent results about non-monotonic quantum-to-classical transitions are summarised. Finally, some results are shown when decoherence has been introduced into the Anderson problem .

2 Introduction

Since the pioneering work of P. W. Anderson [1], Anderson localisation (AL) has been an active field of research. AL entails suppression of wave transport and exponential decay of the wave function away from a point. Initially considered in disordered solids, AL has been studied and observed in a number of different settings: acoustic waves [25], light waves [5, 10], microwaves [3], Bose-Einstein condensates (as matter waves) [9] and photonic lattices [15].

The classical model of conductivity in solids is the Drude model: electrons, considered free particles, scatter off relatively immobile heavy ions according to an average distance between collisions, the mean free path l [4]. In solids however, l was found to be far larger than the lattice constant. Bloch waves, a quantum description of electrons in solids, remedy this problem. Due to the translational symmetry of the solid's crystals, the crystal momentum of the electron is conserved, and the Bloch wave travels through the solid at constant group velocity [21]. When the wave encounters an impurity in the lattice it is scattered, since the symmetry is broken. This explains the larger than expected mean free path in solids and also why the Drude model works well in predicting classical transport, e.g. Ohm's law [4][23].

When the number of impurities, or amount of disorder, is sufficiently large this scattering via impurities results in AL [18]. Specifically, it arises due to the interference of the many scattering paths that have random path lengths and therefore random phases [12]. This results in destructive interference. Since AL relies on these interferences, it is logical to study AL when the system is dephased, i.e. subject to decoherence. It is expected that the localisation will be destroyed since there will be no interference, as the coherence length L_ϕ will be small.

The exact nature of the transition between the non-classical AL behaviour and the classical regime is unclear and has recently been the subject of some work [8, 11]. The aim of this project is to investigate how AL disappears as decoherence is introduced.

3 Background material

3.1 Anderson localisation

3.1.1 Weak localisation

To understand how AL may occur, it is instructive to study weak localisation. Weak localisation, as the name suggests, occurs at a weaker disorder [2]. In the path integral formulation of quantum mechanics, an amplitude is calculated by summing all possible trajectories, or paths [16]. Consider the paths of a wave from point A to point B as in figure 1a. The paths, after leaving A, scatter from impurities randomly, and arrive at point B having taken distinct routes of different lengths. It follows that when *all* the possible paths are summed, because the path length and phase of a given path is random at point B, the amplitude from A to B averages to zero in the sum. Now consider figure 1b - a closed loop. There are two paths around the loop, which interfere constructively, so the amplitude of a particle returning to A is enhanced by a factor of two [2, 4]. This only holds in the large L_ϕ regime, otherwise probabilities rather than amplitudes would be considered, then interference is not possible. Weak localisation is also characteristic for being destroyed by very weak magnetic fields [18].

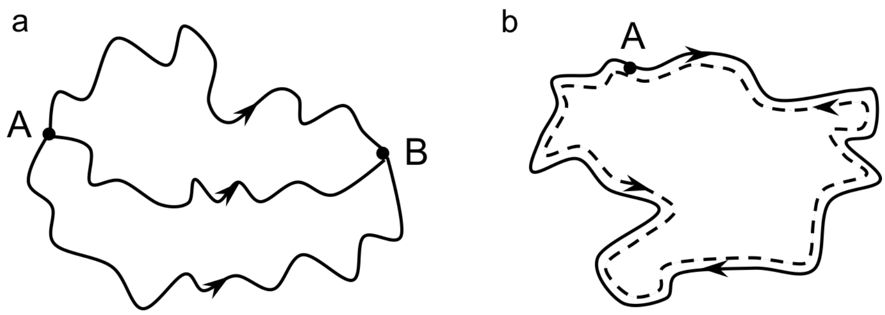

Figure 1: a) three possible scattering paths a particle could take from point A to point B. b) a possible path where the particle loops back on itself.

This enhanced back-scattering is responsible for weak localisation. The correction in conductivity σ due to this effect can be estimated for $d = 2, 3$ [18]

$$\frac{\delta\sigma}{\sigma} \sim -\frac{\lambda^2}{l^2}\left(1-\sqrt{\frac{\tau}{\tau_\phi}}\right) \quad d=3 \tag{1}$$

where λ is the wavelength, τ_ϕ is the timescale over which coherence is lost and τ is the timescale related to l the mean free path. Assuming robust coherence, i.e. $\tau_\phi \to \infty$, equation 1 shows that in three dimensions the correction is finite. For a thin film of thickness a, the correction is

$$\delta\sigma \sim \frac{\ln(\frac{L_\phi}{a})}{a} \quad d=2 \tag{2}$$

so that in two dimensions, taking the limit of infinitesimal thickness of film, $a \to 0$, the correction to conductivity diverges. This can be interpreted as a consequence of the far larger likelihood of self-looping paths in 2D and smaller dimensions than in 3D [2].

3.1.2 Anderson model

In Anderson's original paper [1] he considered a lattice of n sites with energies E_j. E_j is a random variable that has a probability distribution $P(Ej)$ with width w. This represents the system disorder. The specific form of $P(Ej)$ is not important, usually it is a box function [19]. Particles can be labelled by a site number j. Furthermore, there exists a hopping element between nearest neighbour sites t, allowing the particle to propagate through the lattice. The nature of t can be left ambiguous, it may be a random variable or not. The Hamiltonian H of the system can be written

$$H = \sum_i \epsilon_i |i><i| + t\sum_i \left(|i><i+1| + |i+1><i|\right) \tag{3}$$

as in the paper [11]. Anderson's analysis established that in three dimensions, there exist states that are localised with probability unity. He showed that as w, or the strength of the disorder, increases there will be states that are very low in energy compared to their neighbours. In this case the hopping element t is not large enough to cause transport, so the state is strongly peaked around some site j in the lattice. Instead of transport, t enables virtual transitions from to the surrounding sites. This is similar to the way in which localisation may be defined as exponential decay of the wave function ψ away from a site j where it takes value ψ_0

$$\psi = \psi_0 e^{-r/L_{loc}} \tag{4}$$

and from equation 4 a characteristic length scale can be associated with AL, known as the localisation length L_{loc} [19].

3.1.3 Scaling theory of localisation

Using concepts from previous work on critical phenomena, a scaling theory of localisation was developed by Thouless and others in the 1970s [6]. They made progress with the Anderson problem by considering hypercubes, the n dimensional version of a square, of volume L^d. We assume that the problem can be solved exactly in these hypercubes and that a spacing between energy levels dE/dN can be found

$$\frac{dE}{dN} = \frac{1}{NL^d} \tag{5}$$

where N is the number of energy states per unit energy and volume [18].

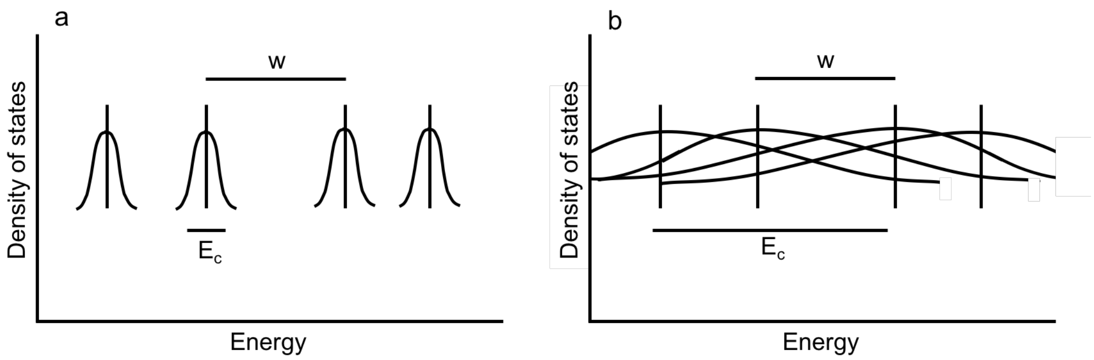

Figure 2: Representation of the spectrum of energy eigenstates. a) regime where $w = dE/dN \gg E_c$, no mixing of individual states. b) regime where $E_c \gg w = dE/dN$. Lots of mixing and the total system eigenstate is extended. Reproduced from [18].

The system is scaled by placing hypercubes together side by side as in figure 3. The eigenstate of the new system will not be exactly the same as the individual eigenstates found before, so a decaying timescale can be associated with the eigenstate. This timescale can be related to an energy width E_c through the Heisenberg uncertainty principle. If the spacing between energy levels, dE/dN is much larger than the width E_c of the eigenstates, $dE/dN \gg E_c$, as in figure 3a, there will be no superposition of neighbouring eigenstates. Then the scaled system eigenstate is described well by the sum of the subsystem eigenstates, which do not overlap, and it is unlikely that a particle could hop between subsystems. This procedure may be continually iterated until the region which a particle would be confined to, say a few subsystems, occupies a small proportion of the full system, producing localisation [6, 18]. In the opposite case $E_c \gg dE/dN$, shown in figure 3b, there is substantial overlap between the subsystem eigenstates. Therefore it is likely that a particle will be transported through the subsystems and so the total system eigenstate is extended.

As the system length L is continually scaled up in this manner, assuming $L > l$, the mean free path, from the above analysis it is clear that the determining quantity for AL is the ratio

$$g(L) = \frac{Ec(L)}{dE(L)/dN} = cG(L) \tag{6}$$

where G is the conductance, c is a constant and g is a dimensionless conductance called the Thouless number [6]. We are now ready to scale the system quantitatively, by an integer factor b. By doing this, it is possible to find the dependence of AL on dimensionality d. Consider separating b^d hypercubes of length L into a group, so that the new length scale is bL. The Thouless number at the length scale bL, $g(bL)$, can be related to $g(L)$ through

$$g(bL) = f(b, g(L)) \tag{7}$$

which allows definition of the function β

$$\beta = \frac{d \ln g}{d \ln L} \tag{8}$$

and can be viewed as the response of the Thouless number, or dimensionless conductance, to an infinitesimal change in the length scale L. The asymptotic behaviour of β can be found through simple considerations of the limiting behaviour of g. If g is large, i.e classical conductivity [6],

$$G(L) = \sigma L^{d-2} \tag{9}$$

so for a thin wire, $d = 1$, conductivity is inversely proportional to length, as expected. Then

$$\lim_{g \to \infty} \beta = d - 2. \tag{10}$$

For small g, it is assumed the system is in the AL regime so g decays exponentially in L [6].

Using these asymptotic results and assuming continuity, β may be plotted as in figure 3. For $d < 2$, Beta is always negative. This means for any dimensionless conductance g an increase in the length scale L, through the blocking procedure outlined previously, always results in a smaller g. This means all states of the system are localised. For $d = 3$ the behaviour is the same until $g = g_c$, wherein β becomes positive and further scaling results in increasing g. As long as $g < gc$ the state is localised, but if $g > gc$ the state is extended. This is the concept of the mobility edge introduced by Mott [20], where all states with an energy above a certain value, corresponding to g_c, are extended and all below are localised. The marginal dimension for AL is therefore $d = 2$, because β tends to zero asymptotically from below as seen in figure 3. Therefore, it is expected that all states are localised in one and two dimensions, whereas in three dimensions there exists a disorder induced transition between localised and extended states.

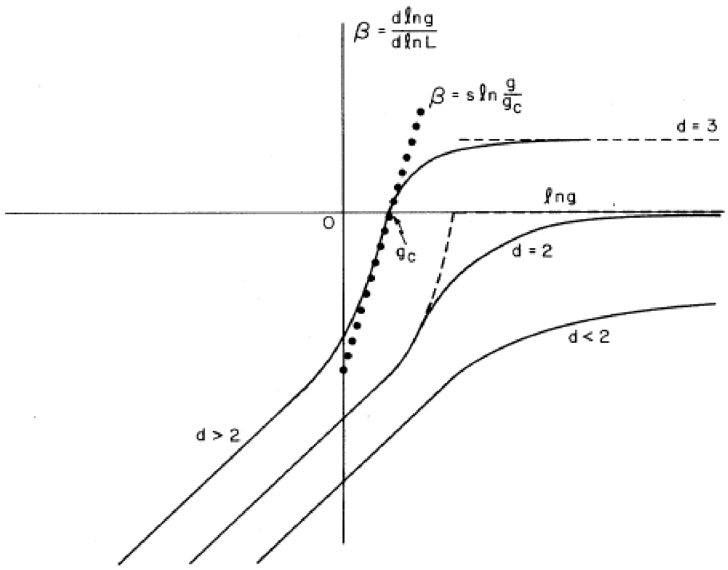

Figure 3: Plot of β vs $\ln g$. For $d \leq 2$ all states are localised, because the beta function is always negative. In 3d, states past the critical value g_c are delocalised and those below are localised. Figure taken from [6].

3.2 Decoherence

Decoherence is the process by which systems lose quantum coherence. A coherent quantum state is termed a pure state and can be defined in terms of the density matrix ρ as $Tr\rho^2 = 1$. The decoherence process transforms a pure state to a mixed state, for which $Tr\rho^2 < 1$ [17].

Physically, decoherence is induced by the interaction of the system with its environment, which suppresses interferences of the system very quickly [22]. Since Schroedinger evolution is unitary, i.e. it preserves coherence, decoherence processes must be modelled differently. A common route is to use a Lindblad master equation , which is the most general Markovian master equation that preserves the defining properties of ρ [7]. It reads

$$\dot{\rho} = \frac{-i}{\hbar}[H, \rho] + \sum_i \gamma_i (L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}) \tag{11}$$

where the first term corresponds to unitary Schroedinger evolution and the second term represents the dissipative dynamics, the L_i are named Lindblad operators. For example, if the L_i are chosen to be projectors onto the eigenbasis of the system Hamiltonian H then the off diagonal elements of ρ, representing interferences, decay while the diagonal elements remain constant [13].

Intuitively, it is expected that as the decoherence is increased, the more classical the system becomes. This picture does not always hold, as shown in a recent experiment involving multiparticle interference [14]. A requirement for interference between multiple particles is their mutual indistinguishability. So, if their indistinguishability is varied, decoherence can be introduced. Here the indistinguishability is varied using the path delay x in a Hong-Ou-Mandel (HOM) system, shown in figure 4. It consists of two input photon modes (a, b), a beam splitter, and a detector. The HOM effect occurs when a photon is incident on each of the input modes $(1, 1)$ and there is complete suppression of the detection event $(1, 1)$. This happens because the paths of both photons being transmitted and both being reflected interfere destructively. In this case the photons are indistinguishable, but by varying x, the photon distinguishability can be controlled.

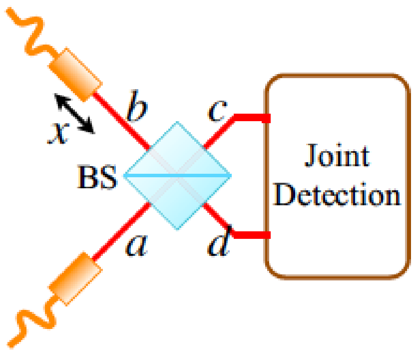

Figure 4: HOM interferometer with path delay x. This allows control of the distinguishability of the photons entering and therefore also the amount of decoherence. Figure taken from [23]

For the (1,1) input mode a monotonic quantum to classical transition was observed. This is because the photons can be decomposed into two orthogonal parts, a fully indistinguishable part analogous to the HOM effect and a fully distinguishable part with no interference [14], because the particles are distinguishable. When two photons are input into each mode, (2,2), there is non-monotonic behaviour. Again, the photons are decomposed into fully indistinguishable and distinguishable parts analogous to the (1,1) input mode. However, there are also terms where the two photons in one mode, say a, are fully indistinguishable with respect to one of the photons in mode b and fully distinguishable with respect to the other [14]. This kind of term is responsible for the non-monotonicity.

As mentioned in the introduction, the effect of decoherence on the Anderson model has been studied recently. It has been shown that introducing dissipation, via local dissipators as Lindblad operators within the master equation formalism, AL can be driven into a mixed steady state localisation which has tunable properties [8]. Furthermore, the author showed that in the regime of strong localisation, i.e. where the width of $P(E_j)$ is large, dissipation of this kind results in a robust delocalisation.

In the discrete one dimensional Anderson model of equation 3, the transition between AL and classical ohmic behaviour has been solved analytically [11]. The work confirms that, as expected, the classical relationship of equation 9 holds even if there is disorder, so long as an appropriate amount of decoherence is introduced. This is quantified by the condition for conductance that the coherence length L_ϕ must obey

$$L_\phi < \frac{1}{1 - \exp\left(-\zeta^{-1}\right)} \tag{12}$$

where is a generalised second order Lyapunov exponent. Generalised Lyapunov exponents relate to the intermittency of a dynamical system, i.e. the jumping between periodic and chaotic motion [24].

References

[1] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492, 1958.

[2] S. Datta. *Electronic transport in mesoscopic systems.* 1995.

[3] A. A. Chabanov et al. Statistical signatures of photon localisation. *Nature*, 404:850, 2000.

[4] A. Lagendijk et al. Fifty years of anderson localisation. *Physics Today*, 62:24, 2009.

[5] D. S. Wiersma et al. Localisation of light in a disordered medium. *Nature*, 390:671, 1997.

[6] E. Abrahams et al. Scaling theory of localisation: absence of quantum diffusion in two dimensions. *Phys. Rev. Lett.*, 42:673, 1978.

[7] H. P. Breuer et al. *The theory of open quantum systems.* 2002.

[8] I. Yusipov et al. Localisation in open quantum systems. *Phys. Rev. Lett.*, 118.

[9] J. Billy et al. Direct observation of anderson localization of matter waves in a controlled disorder. *Nature*, 453:891, 2013.

[10] M. Stoerzer et al. Observation of the critical regime near anderson localisation of light. *Phys. Rev. Lett.*, 96, 2009.

[11] M. Zilly et al. Decoherence-induced conductivity in the discrete 1d anderson model. *Phys. Rev. B*, 85, 2011.

[12] P. E. Lindelof et al. New light on the scattering mechanisms in si inversion layers by weak localisation experiments. *Physica Scripta*, 1986, 1986.

[13] R. A. Bertlmann et al. Decoherence modes of entangled qubits with neutron inferometry. *Phys. Rev. A*, 73, 2006.

[14] R. Young-Sik et al. Nonmonotonic quantum-to-classical transition in multiparticle interference. *PNAS*, 110:1227, 2012.

[15] T. Schwartz et al. Transport and anderson localization in disordered two-dimensional photonic lattices. *Nature*, 453:891, 2008.

[16] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys*, 20:367, 1948.

[17] C. J. Isham. *Lectures on quantum theory.* 1995.

[18] B. L. Altshuler P. A. Lee. Disordered electronic systems. *Physics Today*, 41:36, 1988.

[19] B. Kramer A. MacKinnon. Localisation: theory and experiment. *Rep. Prog. Phys*, 56:1469, 1993.

[20] N. Mott. The mobility edge since 1967. *Journal of Physics C: Solid State Physics*, 20, 1987.

[21] U. Roessler. *Solid state theory.* 2009.

[22] M. Schlosshauer. Decoherence, the measurement problem, and interpretations of quantum mechanics. *Rev. Mod. Phys.*, 76:1267, 2005.

[23] N. Hatano M. Suzuki. *Quantum annealing and other optimisation methods.* Springer, 2005.

[24] G. Paladin A. Vulpiani. Anomalous scaling and generalized lyapunov exponents of the one-dimensional anderson model. *Phys. Rev. B*, 35:2015, 1986.

[25] R. L. Weaver. Anderson localisation of ultrasound. *Wave Motion*, 12:129, 1990.