ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 9: Variance Reduction

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clicker question: back to the circle

generate n rv pairs $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, where all $X_i, Y_i \sim U[-1, 1]$, and compute n observations $Z_i = \mathbb{1}_{\{X_i^2 + Y_i^2 \leq 1\}}$
which of the following is an estimator for π?

(a) $\frac{1}{n} \sum_{i=1}^{n} Z_i$

(b) $\frac{1}{n} \sum_{i=1}^{n} Z_i^2$

(c) $\frac{1}{2n} \sum_{i=1}^{n} Z_i$

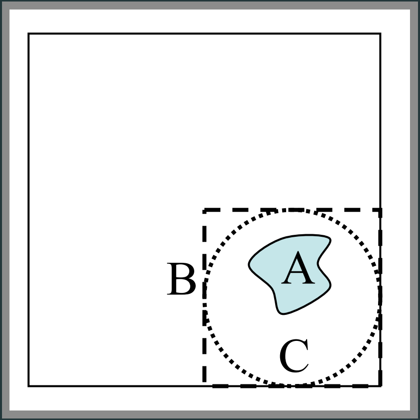
(d) $\frac{2}{n} \sum_{i=1}^{n} Z_i$

(e) $\frac{4}{n} \sum_{i=1}^{n} Z_i$

variance reduction

- construct estimators with lower variance
- fewer replications to build CI of given width
- may need to exploit problem-specific information
  depending on application, this effort can be worthwhile

importance of smaller variance estimators



- aim: compute volume v_A of region A in the unit square
- **method 1**: generate points uniformly over the unit square (outermost box) and compute the fraction of points falling in region A

importance of smaller variance estimators

- let $X_1, \ldots, X_n$ be n points uniformly distributed in $[0,1]^2$
- an estimator of v_A is

$$\widetilde{V}_A =$$

- $Var(\mathbb{I}_{[X_i \in A]}) =$
- $Var(\widetilde{V}_A) =$

importance of smaller variance estimators

- **method 2**: generate n points $Y_1, \ldots, Y_n$ uniformly in square B
- an estimator of v_A is

$$\widehat{V}_A =$$

- $Var(\mathbb{I}_{[Y_i \in A]}) =$
- $Var(\widehat{V}_A) =$
- $Var(\widehat{V}_A) \leq Var(\widetilde{V}_A) \Rightarrow \widehat{V}_A$ gives more accurate estimates

importance of smaller variance estimators

- **method 3**: generate n points $Z_1, \dots, Z_n$ uniformly in circle C
- an estimator of v_A is

$$\bar{V}_A =$$

- $Var(\mathbb{I}_{[Z_i \in A]}) =$
- $Var(\bar{V}_A) =$
- $Var(\bar{V}_A) \leq Var(\widehat{V}_A) \leq Var(\widetilde{V}_A)$

complexity vs. variance reduction

- $\bar{V}_A$ requires points that are uniformly distributed over a circle
- to generate points uniformly in circle centered at $(0,0)$ with radius a:
  1. generate $U_1 \sim U[0,1]$, $U_2 \sim U[0,1]$ and $U_3 \sim U[0,1]$.
  2. set $R = a \max[U_1, U_2]$, $\theta = 2\pi U_3$.
  3. return $(R\cos\theta, R\sin\theta)$.
- requires cosine and sine computations
- faster to generate points uniformly in rectangle

  $\Rightarrow$ more points in same computation time

complexity vs. variance reduction

- although $\bar{V}_A$ has smaller variance than $\widehat{V}_A$, may be better to use $\widehat{V}_A$
- trade-off between reduction in variance and extra computation needed for variance reduction

variance reduction: techniques that help reduce estimator variance

– antithetic variates

– importance sampling

– control variates

– stratified sampling

– common random numbers

running example: Monte Carlo integration

compute $\int_a^b g(x)dx$

- we know how to compute $\mathbb{E}\left[f(U)\right]$, where $U \sim U[0,1]$

running example: Monte Carlo integration

compute $\int_a^b g(x)dx$

$$\begin{aligned}(b-a)\int_a^b g(x)\frac{1}{b-a}dx &= (b-a)\,\mathbb{E}\left[g(Z)\right] \\ &= (b-a)\,\mathbb{E}\left[g(a+(b-a)U)\right] \\ &= \mathbb{E}\left[f(U)\right]\end{aligned}$$

$$f(x) = (b-a)g(a+(b-a)x)$$

antithetic variates

- observation: $X, X' =$ identically distributed random variables

$$\mathbb{E}\left[\frac{X+X'}{2}\right] =$$

$$Var\left(\frac{X+X'}{2}\right) =$$

antithetic variates

- if X and X' are independent,

$$Var\left(\frac{X+X'}{2}\right) = \frac{1}{2}Var(X).$$

- if X and X' are negatively correlated,

$$Var\left(\frac{X+X'}{2}\right) < \frac{1}{2}Var(X).$$

- want simulation model to give two estimates of the performance measure X and X' such that $Cov(X, X') < 0$.

antithetic variates in Monte Carlo integration

- compute $\mathbb{E}[f(U)]$, where $U \sim U[0,1]$
- if $U_1, \ldots, U_{2n} \sim U[0,1]$, the regular MC estimator of $\mathbb{E}[f(U)]$ is

$$\alpha_{reg} =$$

- when U is large, $1 - U$ is small
- $f(\cdot)$ monotone $\Rightarrow f(U)$ and $f(1-U)$ are negatively correlated
- the antithetic variates estimator of $\mathbb{E}[f(U)]$

$$\alpha_a =$$

example: Monte Carlo integration

to see why this works, compute the variance:

$Var(\alpha^r) =$

$Var(\alpha^a) =$

- since $Cov(f(U_i), f(1 - U_i)) \leq 0$, we have $Var(\alpha^a) \leq Var(\alpha^r)$.
- a sufficient condition for antithetic variates to work is that the performance measure is monotone (increasing or decreasing)

clicker question: variance of antithetic estimators

to estimate $\mathbb{E}[f(U)]$, use n uniform rv $U_1, U_2, \ldots, U_n$ to get $2n$ observations:
$Y_1 = f(U_1), Y_2 = f(U_2), \ldots, Y_n = f(U_n)$
$Z_1 = f(1 - U_1), Z_2 = f(1 - U_2), \ldots, Z_n = f(1 - U_n)$
and use these to get the antithetic estimator:

$$M_n = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i + Z_i}{2}$$

this estimator has variance σ^2/n, where σ^2 is approximated by:

(a) sample variance of all $2n$ observations: $\frac{1}{2n-1}\sum_{i=1}^{n}(Y_i - \mu_n)^2 + (Z_i - \mu_n)^2$

(b) sample variance of combined observations: $\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{Y_i+Z_i}{2} - \mu_n\right)^2$

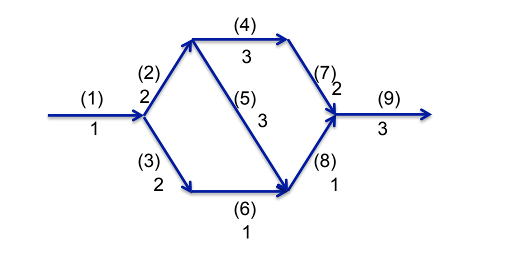
(c) average of $\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \mu_n)^2$ and $\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \mu_n)^2$

(d) all of the above

(e) none of the above

clicker question: variance of antithetic estimators

example: critical paths



- arc length = duration of activity (assume Exp(label))
- activity durations are independent rvs $X_1, \ldots, X_9$
- project duration = length of longest source→sink path
- length of the critical path is

$$C(X_1, \ldots, X_9) = X_9 + \max\left[X_1 + X_2 + X_4 + X_7,\right.$$
$$\left.\max\left[X_1 + X_2 + X_5, X_1 + X_3 + X_6\right] + X_8\right].$$

example: critical paths

- $C(\cdot, \ldots, \cdot)$ is nondecreasing
- want identically distributed samples $\tilde{X}_1, \ldots, \tilde{X}_9$ and $\hat{X}_1, \ldots, \hat{X}_9$ such that when $C(\tilde{X}_1, \ldots, \tilde{X}_9)$ is large, $C(\hat{X}_1, \ldots, \hat{X}_9)$ is small
- suppose $X_i \sim F_i$ for each $i \in \{1, 2, \ldots, 9\}$

example: critical paths

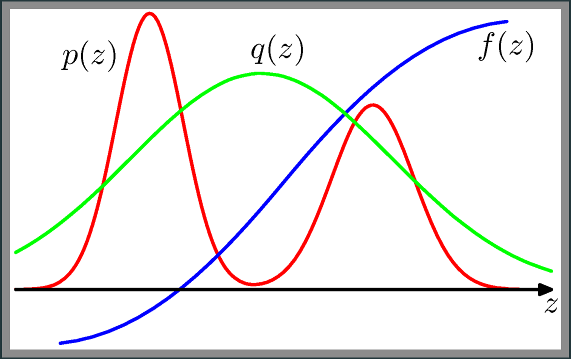
- $(U_1^n, \ldots, U_9^n) =$ 9-dim vector of iid $U[0,1]$ rvs
- $C(F_1^{-1}(U_1^n) \ldots, F_9^{-1}(U_9^n))$ and $C(F_1^{-1}(1 - U_1^n), \ldots, F_9^{-1}(1 - U_9^n))$ are negatively correlated
- the antithetic estimator

$$\widehat{T} = \frac{1}{N} \sum_{n=1}^{N} \frac{\left[C(F_1^{-1}(U_1^n), \ldots, F_9^{-1}(U_9^n)) + C(F_1^{-1}(1 - U_1^n), \ldots, F_9^{-1}(1 - U_9^n))\right]}{2}$$

should have smaller variance than the estimator $\frac{1}{2N} \sum_{n=1}^{2N} C(F_1^{-1}(U_1^n), \ldots, F_9^{-1}(U_9^n))$.

importance sampling

- given function $\phi(\cdot)$, want $\mathbb{E}[\phi(X)]$ where $X \sim P$
- can generate samples $Z \sim Q$ (but not from P)

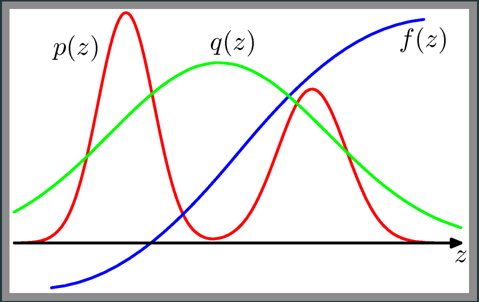


importance sampling

- given function $\phi(\cdot)$, want $\mathbb{E}[\phi(X)]$ where $X \sim P$
- can generate samples $Z \sim Q$

importance sampling

1. generate $Z_1, Z_2, \ldots, Z_L \sim Q$
2. compute $\mathbb{E}[\phi(X)] = \frac{1}{L}\sum_{i=1}^{L} w_i \phi(Z_i)$, where $w_i = p(Z_i)/q(Z_i)$



importance sampling: why does it work?

given function $\phi(\cdot)$, want $\mathbb{E}[\phi(X)]$ where $X \sim P$

importance sampling

1. generate $Z_1, Z_2, \ldots, Z_L \sim Q$
2. compute $\mathbb{E}[\phi(X)] = \frac{1}{L}\sum_{i=1}^{L} w_i \phi(Z_i)$, where $w_i = p(Z_i)/q(Z_i)$

importance sampling: variance

1. generate $Z_1, Z_2, \ldots, Z_L \sim Q$
2. compute $\mathbb{E}[\phi(X)] = \frac{1}{L} \sum_{i=1}^{L} w_i \phi(Z_i)$, where $w_i = p(Z_i)/q(Z_i)$

importance sampling: comments

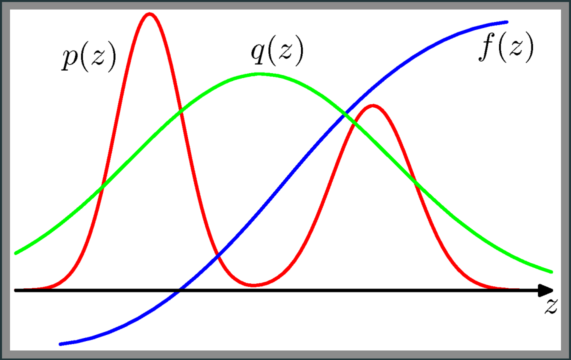
given function $\phi(\cdot)$, want $\mathbb{E}[\phi(X)]$ where $X \sim P$

importance sampling

1. generate $Z_1, Z_2, \ldots, Z_L \sim Q$
2. compute $\mathbb{E}[\phi(X)] = \frac{1}{L}\sum_{i=1}^{L} w_i \phi(Z_i)$, where $w_i = p(Z_i)/q(Z_i)$

(advanced) sampling-importance-resampling (SIR)

what if we want samples from P?



(advanced) sampling-importance-resampling (SIR)

sampling-importance-resampling (SIR)

1. generate $Z_1, Z_2, \ldots, Z_L \sim Q$
2. compute $w_i = p(Z_i)/q(Z_i)$ for each i 3. resample $X_1, X_2, \ldots, X_L$, where $X_i = Z_k$ with probability $\frac{w_k}{\sum_{j=1}^{L} w_j}$