ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 10: Intro to Markov Chains

Sid Banerjee

School of ORIE, Cornell University

random process

random process

indexed collection of rvs $X_t \in \mathcal{S}$, one for each $t \in T$

– $\mathcal{S}$: state space, $\mathcal{T}$: index set

Markov property

random process X_t has the Markov property if the probability of moving to a future state only depends on the present state and not on past states

random process

random process

indexed collection of rvs $X_t \in \mathcal{S}$, one for each $t \in T$

– $\mathcal{S}$: state space, $\mathcal{T}$: index set

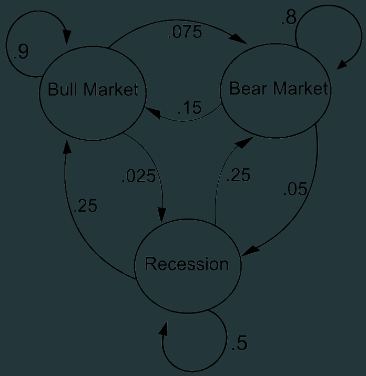
Markov property

random process X_t has the Markov property if the probability of moving to a future state only depends on the present state and not on past states

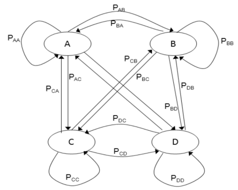
four types

- $\mathcal{S}$ discrete, $\mathcal{T}$ discrete: discrete-time Markov chain (DTMC)
  – random walk
- $\mathcal{S}$ continuous, $\mathcal{T}$ discrete: continuous-time Markov chain (CTMC)
  – Poisson process
- $\mathcal{S}$ discrete, $\mathcal{T}$ continuous: discrete-time Markov process
- $\mathcal{S}$ continuous, $\mathcal{T}$ continuous: Markov process
  – Brownian motion

Markov chains: basic definition



Markov chains: transition-diagram and transition matrix



$$\mathbf{P}=\begin{bmatrix} P_{AA} & P_{AB} & P_{AC} & P_{AD} \\ P_{BA} & P_{BB} & P_{BC} & P_{BD} \\ P_{CA} & P_{CB} & P_{CC} & P_{CD} \\ P_{DA} & P_{DB} & P_{DC} & P_{DD} \end{bmatrix}$$

example: coin tosses and geometric rv.

recall the Geometric rv $p(k) = q^{k-1}(1-q)\,\forall\, k \in \{1, 2, \ldots\}$

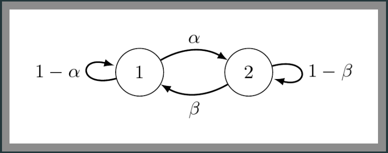
we can view this as a Markov chain as follows:

example: the coupon collector

a brand of cereal always distributes a baseball card in every cereal box, chosen randomly from a set of n distinct cards

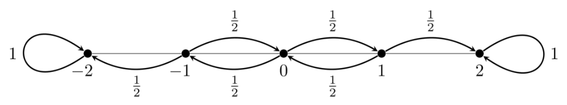
Markov chain model for number of cards owned by a collector:

Markov chains: two viewpoints

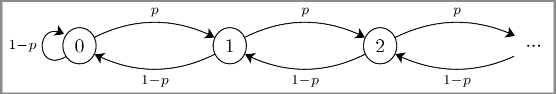


Markov chains: long-term behavior

Markov chains: absorbing chain

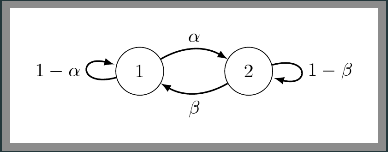


Markov chains: transient/recurrent chain



Markov chains: steady-state behavior

example:



a little diversion (to mark the day...)

Bertrand's ballot problem

two candidates A and B contest an election with n votes, out of whom a majority a people vote for A and $b = n - a < a$ vote for B

if votes are counted in random order, what is the chance A is always in the lead?

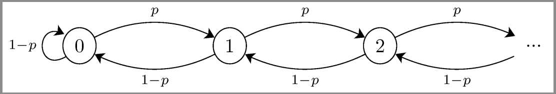
and a little magic!

Bertrand's ballot theorem

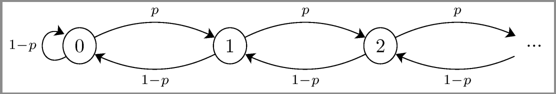
two candidates A and B contest an election with n votes, out of whom a majority a people vote for A and $b = n - a < a$ vote for B

if votes are counted in random order, then $\mathbb{P}[A \text{ is always in the lead}] = \frac{a - n/2}{2n}$

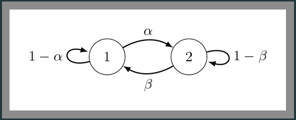
Markov chains: steady-state for infinite chains



Markov chains: reversibility



clicker question: long-term behavior in the flip-flop



in the flip-flop Markov chain, which of the following outcomes is possible for $\pi_t(1) = \mathbb{P}[X_t = 1]$ (for different α, β)?

(a) $\lim_{t\to\infty} \pi_t(1) = 0$

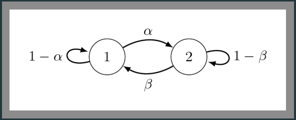
(b) $\lim_{t\to\infty} \pi_t(1) = 1$

(c) $\lim_{t\to\infty} \pi_t(1)$ settles down to a constant $\pi(1) \in (0, 1)$

(d) $\lim_{t\to\infty} \pi_t(1)$ oscillates

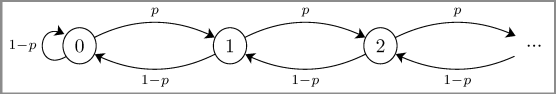
(e) all of these

clicker question: long-term behavior in the flip-flop



in the flip-flop Markov chain, which of the following outcomes is possible for $\pi_t(1) = \mathbb{P}[X_t = 1]$ (for different α, β)?

Markov chains: the ergodic theorem



DTMC: applications and problems

from discrete to continuous time

Markov property

random process X_t has the Markov property if the probability of moving to a future state only depends on the present state and not on past states

- $\mathcal{S}$ discrete, $\mathcal{T}$ discrete: discrete-time Markov chain (DTMC)
  - random walk
- $\mathcal{S}$ continuous, $\mathcal{T}$ discrete: continuous-time Markov chain (CTMC)
  - Poisson process

exponential distribution cheat sheet

can extend DTMCs to continuous time using special properties of the exponential distribution/poisson process

suppose $T \sim$exponential(λ), then:

- pdf: $f_T(t) =$
- cdf: $F_T(t) = \mathbb{P}[T \leq t] =$
- (memorylessness): cdf of T knowing it is bigger than t?

  $\mathbb{P}[T \leq t + x | T > t] =$

understanding exponential distributions

suppose $T_1, T_2, \ldots, T_n$ are all exponentially distributed, with $T_i \sim$exponential(λ_i).

- (minimum of exponentials): let $T_{\text{min}} = \min\{T_i | i \in \{1, 2, \ldots, n\}\}$; distribution of T_{min}?

  $T_{\text{min}} \sim$

- (first arrival): let $T_{\text{min}} = \arg\min_i \min\{T_i | i \in \{1, 2, \ldots, n\}\}$; distribution of T_{min}?

  $T_{\text{min}} \sim$

clicker question: spreading a rumor

we model rumor spreading among n people using a Markovian model:
– each pair of people (i, j) independently meet after Exponential$(1/\tau)$ time
– when a person in the know meets someone who is unaware, then the rumor spreads
suppose at time t, there are $N(t)$ people who know the rumor
what is the distribution of the time T after which the number of people in the know increases to $N(t) + 1$?

(a) Exponential$(N(t)/\tau)$

(b) Exponential$(N(t)\tau)$

(c) Poisson$(N(t)^2/\tau)$

(d) Exponential$(N(t)^2/\tau)$

(e) Exponential$(N(t)(N(t) + 1)/2\tau)$

Poisson process cheat sheet

given Poisson processes $X(t) \sim PP(\lambda)$

- (inter-arrival times): let $\{A_1, A_2, A_3 \ldots\}$ be the arrival times of the agents; then

  $T_i = A_i - A_{i-1} \sim$

- (splitting): suppose we probabilistically split arrivals from $X(t)$ to $Y(t)$ with probability p, else to $Z(t) = X(t) - Y(t)$

  $Y(t) \sim$

  $Z(t) \sim$

- (time-varying rate): a time-varying rate of $\lambda(t) \in [0, \lambda^\star]$ is equivalent to a $PP(\lambda^\star)$ for which arrivals at time t are thinned with probability $p(t) = \lambda(t)/\lambda^\star$

Poisson process cheat sheet (contnd)

given independent Poisson processes
$X_1(t) \sim PP(\lambda_1), X_2(t) \sim PP(\lambda_2), , X_3(t) \sim PP(\lambda_3)$

- (superposition): suppose $S(t) = X_1(t) + X_2(t) + X_3(t)$

  $S(t) \sim$

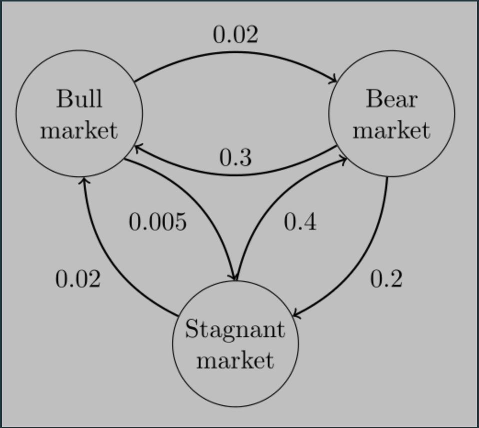
- (first arrival): let $I_{\min}$ = the identity (i.e., $\{1, 2, 3\}$) of the first arrival among the three processes; distribution of $I_{\min}$?

  $I_{\min} \sim$

continuous-time Markov chains

CTMC: continuous-time Markov processes on discrete state-space

eg. modeling the financial market in continuous time



CTMCs

used as a model for many applications:

- queueing and service systems
- transportation networks
- epidemiology
- communications and computer networks
- agent choice models

advantages

- easy to analyze (in some cases)
- easier to simulate than general discrete-event simulation

problems

- need all inter-event times to be exponentially distributed
- can give spurious insights, hide critical issues

simulating a CTMC

example: queueing

the single-server M/M/1 queue

- number of servers: 1
- capacity: infinite
- service discipline: first-come-first-served (FCFS or FIFO)
- interarrival times: exponential(λ)
- service times: exponential(μ)
  (independent interarrival and service times)

example: simulating an $M/M/1$ queue

example: epidemics

want to model the spread of the latest influenza strain among the population
– there is a population of n people
– at any time t, each person i is either susceptible (denoted S or 0) or infected (denoted I or 1)
– infected people get cured on average after time τ, becoming susceptible to future infection. – each pair of people (i, j) independently meet each other with average rate λ
– when a susceptible person meets an infected person, the susceptible person becomes infected
– when two susceptible people or two infected people meet, nothing happens

example: SIS epidemic (contnd)

what assumptions do we need to make this Markovian?

example: SIS Epidemic (contnd)

how do we simulate it?

example: SIS Epidemic (contnd)