**ORIE 4580/5580: Simulation Modeling and Analysis**

**ORIE 5581: Monte Carlo Simulation**

Unit 4: Generating Random Numbers

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## generating uniformly distributed random variables

random number: a sample from $U[0, 1]$

### the 'fundamental theorem' of simulation

can 'transform' a stream of $U[0, 1]$ to any other random variable
– arbitrary probability distribution
– arbitrary correlations
– complex processes
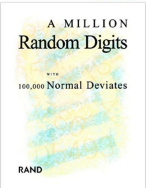
# where can we find random numbers?

## A Million Random Digits with 100,000 Normal Deviates 0th Edition
by The RAND Corporation (Author)

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★★★☆☆ **Weirdest sudoku book ever**
By John Peter O'connor on October 6, 2012
Format: Paperback

This has got to be the most useless set of sudoku puzzles ever.

In my copy of the book, all of the puzzles were already filled in which I find really annoying and what is worse, most of them have been filled in wrongly.

I have been through the whole book really carefully and only found seven puzzles that had been filled out correctly! Yes, just seven.

Well, making the best of a bad job, I am now going through the book trying to correct all of the faulty puzzles and I will then submit my corrections.

Perhaps a second edition will be more useful.

I did find last week's winning lottery numbers on page 18 though.

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★★☆☆☆ **Too unpredictable**
By pontifex on January 24, 2011
Format: Paperback

The book is too hard to follow, the author randomly shifts from one number to another without any prior warning.

‣ 1 comment    408 people found this helpful. Was this review helpful to you? [ Yes ] [ No ] Report abuse

★☆☆☆☆ **Not really random**
By TDB on September 26, 2012
Format: Paperback

# physical Methods

manual methods: coin toss, dice throw, drawing from an urn

objects that appear random: computer clock

physical devices: circuit noise, gamma-ray detectors

## advantage

"true" random numbers (critical for cryptographic applications)
- for example, check out Radiolab podcast on launching a cryptocurrency

## drawbacks

- **slow** (if generated as needed)
  **expensive** (if precomputed and stored in memory)

- **bias** may exist in the device
  for example, see Persi Diaconis on coin-tossing

- **hard to replicate** the random input sequence

## 'pseudo-random' generators

- mid-square method (von Neumann, 1949)

$$8234 \times 8234 = 67(7987)56$$
$$7987 \times 7987 = 63(7921)69$$
$$7921 \times 7921 = 62(7422)41$$
$$7422 \times 7422 = 55(0860)84 \ldots.$$

- **objection**: random numbers are not random at all!
  – this criticism applies to all pseudo-random number generators
  – need tests to determine if algorithm produces "valid" outputs

## linear congruential generators

$$X_{n+1} = (aX_n + c) \bmod m,$$

- (fixed) parameters: modulus $m$, multiplier $a$, increment $c$
- seed: $X_0$ (the first input) is typically supplied by the user
- each $X_n$ is an integer in the set $\{0, 1, \ldots, m-1\}$.
- to get pseudorandom number $U_n \in (0, 1)$, set:

## LCG: example

$m = 32$,   $a = 11$,   $c = 0$,   different seeds.

| $n$ | $X_n$ | $X_n$ | $X_n$ |
|-----|-------|-------|-------|
| 0 | 1 | 2 | 4 |
| 1 | 11 | 22 | 12 |
| 2 | 25 | 18 | 4 |
| 3 | 19 | 6 | 12 |
| 4 | 17 | 2 | 4 |
| 5 | 27 | 22 | 12 |
| 6 | 9 | 18 | 4 |
| 7 | 3 | 6 | 12 |
| 8 | 1 | 2 | 4 |
| 9 | 11 | 22 | 12 |
| ⋮ | ⋮ | ⋮ | ⋮ |

## LCG: properties

- an LCG produces periodic output. period $\leq m$
  1. if period $= m$ with a seed $X_0$, then period $= m$ *for any* seed
  2. if period $< m$, then it may depend on the seed

- full period is desirable:
  1. one should never use the whole period of a LCG, otherwise dependencies between the random numbers will occur.
  2. not have full period $\implies$ gaps in the output sequence.

- full period $\implies$ granularity $= 1/m$.
  not a problem when $m$ is large.

## LCG: theory

the period of LCGs is well understood.

**Theorem: LCGs with full period**

an LCG($m, a, c$) has full period if all of the following are true

1. $m$ and $c$ are relatively prime.

2. if $q$ is a prime number that divides $m$, then $q$ divides $a - 1$.

3. if 4 divides $m$, then 4 divides $a - 1$.

**Corollary:**

an LCG with $m = 2^b$ has full period if $c$ is odd and 4 divides $a - 1$.

*example –  $m = 8$,   $a = 5$,   $c = 3$*

## multiplicative generators

$$X_{n+1} = (aX_n) \bmod m$$

- $X_n = 0 \implies X_n = X_{n+1} = X_{n+2} = \ldots = 0$
- for an MG, Full period $\implies \{1, \ldots, m-1\}$.

### Theorem: MGs with full period

an MG has full period if all of the following are true

1. $m$ is prime      2. $a^{m-1} - 1$ is divisible by $m$.

3. there is no $j < m-1$ such that $a^j - 1$ is divisible by $m$.

### Theorem: Sufficient conditions for good MGs

the largest possible period for a MG with $m = 2^b$ is $m/4$

this is achieved when $X_0$ is odd and $a$ is of the form:
$$a = 3 + 8k \quad \text{or} \quad a = 5 + 8k,$$
for some positive integer $k$.

## deficiencies of LCGs

- LCG's possess **theoretical deficiencies**
  (any deterministic generator must have deficiencies.)
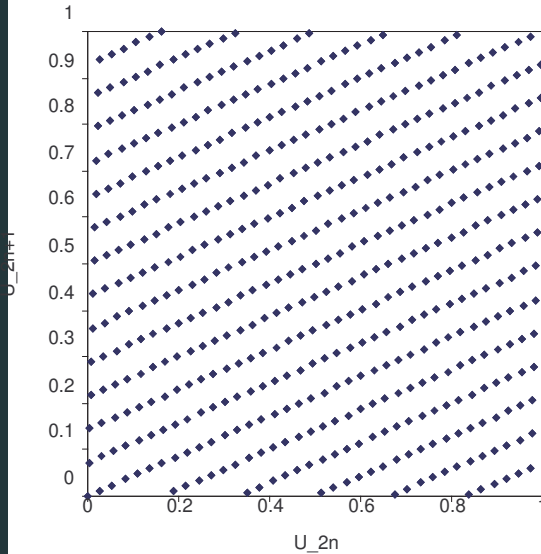- if $U_0, U_1, \ldots$ are iid $U[0, 1]$, then the points

$$(U_0, U_1), \ (U_2, U_3), \ (U_4, U_5), \ldots$$

  should lie uniformly over the square $[0, 1] \times [0, 1]$.
- suppose $U_0, U_1, \ldots$ be generated by a LCG:
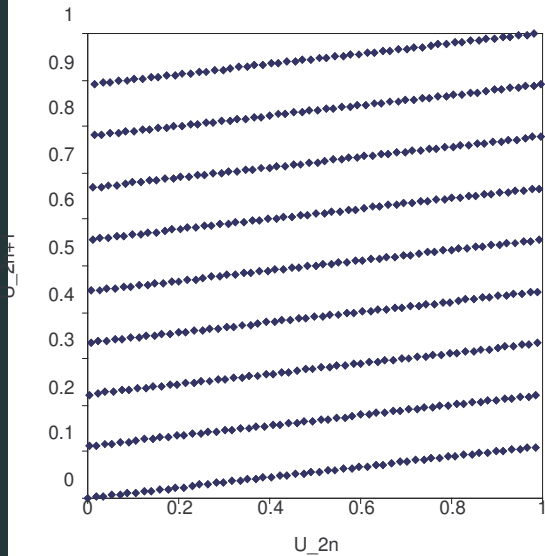  How do the points $(U_0, U_1), \ (U_2, U_3), \ (U_4, U_5), \ldots$ behave?

## deficiencies of LCGs



$m = 2^{10},\ a = 37,\ c = 1$

## deficiencies of LCGs



$$m = 2^{10}, \ a = 57, \ c = 1$$

## deficiencies of LCGs

- the points

$$(U_0, U_1), \ (U_2, U_3), \ (U_4, U_5), \dots$$

  lie on a relatively small number of parallel lines!

- in general, the points

$$(U_0, U_1, U_2, \dots, U_{d-1}), \ (U_d, U_{d+1}, U_{d+2}, \dots, U_{2d-1}), \dots$$

  lie on parallel $(d-1)$-dimensional hyperplanes!

- problematic in simulations of geometric phenomena.
  OK for discrete-event simulations.

## deficiencies of LCGs for generating other rvs

let $N_0, N_1, N_2, \ldots$ be samples from $\mathcal{N}(0,1)$ generated using the Box-Muller method using $U_0, U_1, U_2, \ldots$ from an LCG

– then the pairs $(N_0, N_1), (N_2, N_3), (N_4, N_5), \ldots$ lie on a spiral in two-dimensional space. E.g., $a = 9, m = 2^{21}, c = 1$
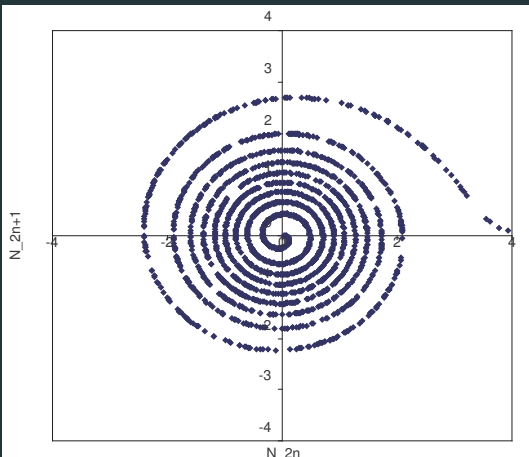
## combining generators

- $m = 2^{31} - 1$ is popular, but period is only about 2 billion.
- not sufficient! E.g. traffic simulators need lots of random numbers. (10s of 000s of vehicles, 1000s of random disturbances, lots of replications).
- shouldn't use anywhere near the full period - maybe $\leq 1\%$
- to generate longer period, take two MG's

$$X_{n+1} = (a_1 X_n) \bmod m_1 \qquad , \qquad Y_{n+1} = (a_2 Y_n) \bmod m_2$$

and set

$$Z_n = (X_n + Y_n) \bmod m_3.$$

- period can be on the order of $m_1 m_2$. For example, set $a_1 = 40014$, $a_2 = 40692$, $m_1 = 2147483563$, $m_2 = 2147483399$ and $m_3 = m_1$.
- can combine more than two.

## streams and substreams

- useful to divide the numbers produced by a PRNG into streams and substreams
- stream = simulation replication
  substream = source of randomness
- useful for debugging and for variance-reduction techniques

## tl;dr

- hyperplane/spiral problems are well understood (and avoided)
- current generators have been carefully tested, pass lots of statistical tests (but must fail at least one test...)

**the last word**

modern PRNGs are
- random enough for your sim answer to be correct
- deterministic enough (by setting seed) for your sim to be repeatable