ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 15: Markov chain Monte Carlo

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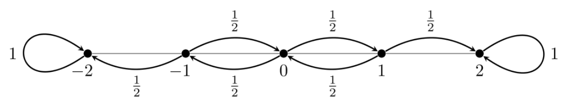
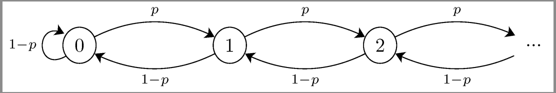
Markov chains: long-term behavior

Markov chains: absorbing chain

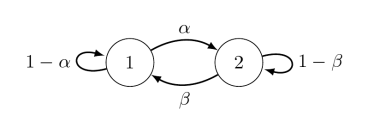


Markov chains: transient/recurrent chain

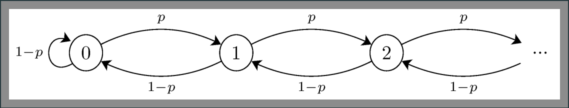


Markov chains: steady-state behavior

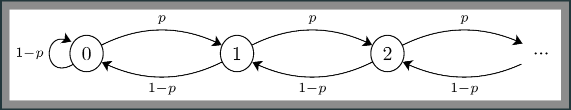
example:



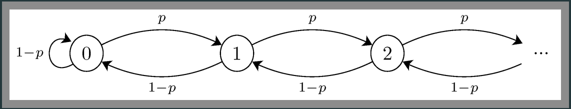
Markov chains: steady-state for infinite chains



Markov chains: reversibility



Markov chains: the ergodic theorem



Markov-chain monte carlo

the Metropolis algorithm

- target distribution $P(x) = \widetilde{P}(x)/Z$
- proposal distribution(s) $Q(x|y)$, with $Q(x|y) = Q(y|x) \, \forall \, x, y$

Metropolis sampling

1. choose initial Z_0
2. to obtain sample t, generate $Y_t \sim Q(\cdot|Z_{t-1})$
3. accept $Z_t = Y_y$ with probability $A(Y_t, Z_{t-1}) = \max\left\{1, \frac{\widetilde{P}(Y_t)}{\widetilde{P}(Z_{t-1})}\right\}$
   else reject and set $Z_t = Z_{t-1}$

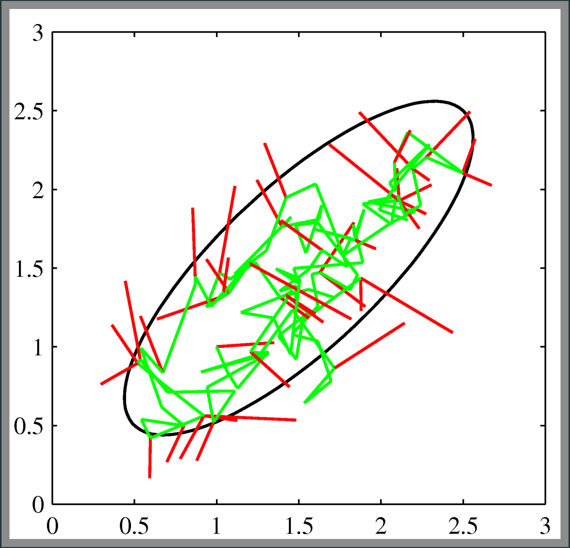
Metropolis-Hastings:example

generate r.v. $X \in \{1, 2, \ldots, n\}$ s.t. $\mathbb{P}[X = i] = 1/n$ for each $i \in [n]$

Metropolis-Hastings:example

generate r.v. $X \in \{1, 2, \ldots, n\}$ s.t. $\mathbb{P}[X = i] = p_i$ for each $i \in [n]$

Metropolis for 2-d Gaussian



Metropolis algorithm: proof of correctness

Metropolis-Hastings

- target distribution $P(x) = \widetilde{P}(x)/Z$
- proposal distribution(s) $Q(x|y)$

Metropolis-Hastings sampling

1. choose initial Z_0
2. to obtain sample t, generate $Y_t \sim Q(\cdot|Z_{t-1})$
3. accept $Z_t = Y_y$ with prob $A(Y_t, Z_{t-1}) = \max\left\{1, \frac{\widetilde{P}(Y_t)Q(Z_{t-1}|Z_t)}{\widetilde{P}(Z_{t-1})Q(Z_t|Z_{t-1})}\right\}$
   else reject and set $Z_t = Z_{t-1}$

Gibbs sampling

- target distribution $P(x(1), x(2), \ldots, x(n))$

Gibbs sampling

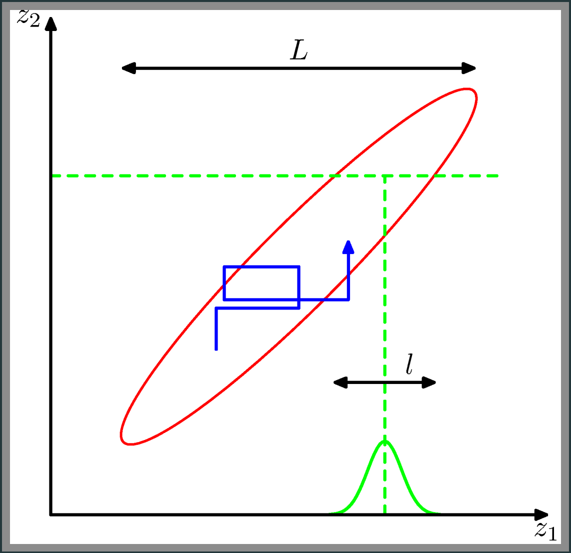
1. choose initial $X_0 = (X_0(1), X_0(2), \ldots, X_0(n))$
2. to obtain sample t:

   pick I_t uniformly at random

   set $X_t(i) = X_{t-1}(i)$ for $i \neq I_t$

   set $X_t(I_t) \sim P(\cdot | X_{t-1} \setminus X_{t-1}(I_t))$

Gibbs sampling for 2-d Gaussian



Gibbs sampling: proof of correctness