ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 6: Generating Random Vectors

Sid Banerjee

School of ORIE, Cornell University

review and roadmap

generating random variables

we have seen how to:

- generate pseudorandom $U[0,1]$ samples np.random.rand
- transform $U[0,1]$ samples to another rv using ('fundamental thm')
  - inversion $X = F^{-1}(U)$
  - acceptance-rejection
    - time taken is random
    - 'always' applicable

two special cases

- multivariate Normal rvs (random vectors $(X_1, X_2, \ldots, X_d)$)
  - generating correlated vectors (X_1, X_2) s.t $X_1 \not\perp X_2$
- Exponential rvs and the Poisson process
  - generating time-indexed processes 'canonical' process over time

memorylessness

$(X_t;\ t \in \{1, 2, \ldots\})$

$(X_t;\ t \in \mathbb{R})$

generating Normal rvs

can be done via several methods

- (numerical) inversion $(\Phi^{-1}(U))$
- Box-Muller method
- acceptance-rejection (using *Exp*(1))

generating correlated random variables

variance and covariance

- variance: $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ ≥ 0
- X and Y are independent if $\underbrace{\mathbb{P}[X \leq x, Y \leq y]}_{\text{Joint cdf}} = F_X(x)F_Y(y)$ for all $x, y \in \mathbb{R}^2$
- covariance: $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \Rightarrow Cov(X, Y) = 0$ (independent implies uncorrelated)
- however, uncorrelated rvs can be dependent

$Cov(X,X) = Var(X)$, $Cov(X,Y)$ need not be ≥ 0

Eg –

$\begin{cases} a & \text{wp } 1/2 \\ b & \text{wp } 1/2 \end{cases}$ (?)

correlation

for any rvs X, Y, their correlation coefficient is

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

properties:

$\rho \in [-1, 1]$

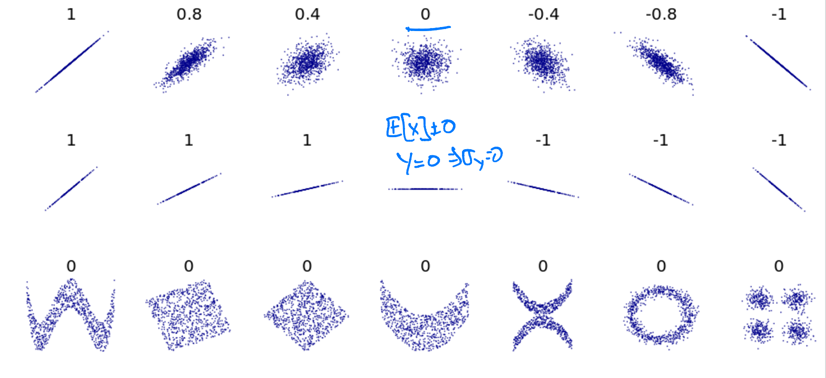
Pf - Cauchy-Schwartz Ineq - For any $(x_1 \dots x_n)$ $(y_1 \dots y_n)$

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

equality iff $y_i = c x_i \ \forall i$

correlation: examples

wikipedia



$E[X] \neq 0$

$Y = 0 \Rightarrow \sigma_Y = 0$

clicker question: independence and correlation for Normal rvs

- $Z \sim \mathcal{N}(0,1)$, $B \sim Ber(1/2)$
- $X = Z$, $Y = (2B-1)Z$

Rademacho $\begin{cases} -1 & \text{wp } 1/2 \\ 1 & \text{wp } 1/2 \end{cases}$

(a) X and Y are correlated and dependent

27

(b) X and Y are uncorrelated and dependent

44

(c) X and Y are uncorrelated and independent

13

(d) X and Y are correlated and independent

15

<u>Moral</u>

X, Y uncorrelated

$+$ X Gaussian, Y Gaussian

$\not\Rightarrow X \perp\!\!\!\perp Y$

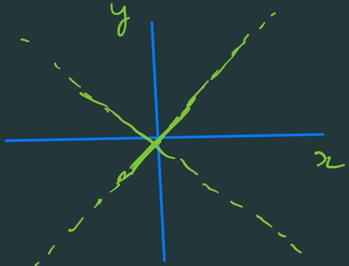
clicker question: independence and correlation for Normal rvs

- $Z \sim \mathcal{N}(0,1)$, $B \sim Ber(1/2)$
- $X = Z$, $Y = (2B-1)Z$ $\longrightarrow$ $Y = \begin{cases} X & \text{wp } 1/2 \\ -X & \text{wp } 1/2 \end{cases} \Rightarrow XY = \begin{cases} X^2 & \text{wp } 1/2 \\ -X^2 & \text{wp } 1/2 \end{cases}$

- X and Y are dependent

If $X = x$, then $Y = \begin{cases} x & \text{wp } 1/2 \\ -x & \text{wp } 1/2 \end{cases}$

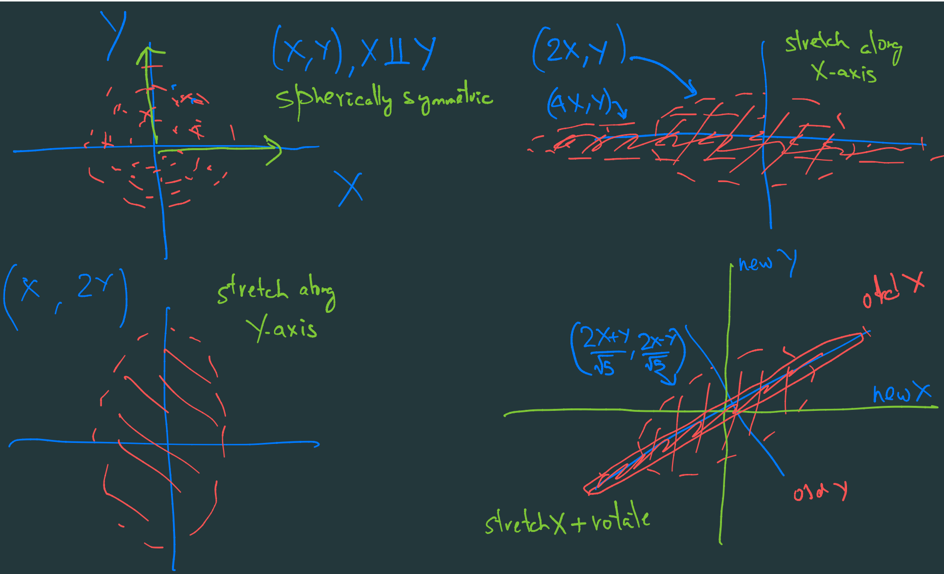
(However $Y \sim N(0,1)$)



- $\mathbb{E}[XY] = \frac{1}{2}\mathbb{E}[X^2] + \frac{1}{2}\mathbb{E}[-X^2] = 0$

$\mathbb{E}[X] = \mathbb{E}[Y] = 0 \Rightarrow \mathrm{Cov}(X,Y) = 0$

multivariate Normal rvs via linear combinations



multivariate Normal rvs via linear combinations

General linear combination – Given $X, Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$,

consider $\boxed{A = aX + bY, \quad B = cX + dY}$ A, B are Gaussian!

1) $E[A] = aE[X] + bE[Y] = 0, \quad E[B] = 0$

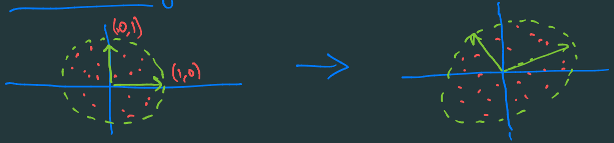
2) $Var(A) = a^2 Var(X) + b^2 Var(Y) = a^2 + b^2, \quad Var(B) = c^2 + d^2$

3) $$Cov(A,B) = E[AB] - E[A]E[B] = E[(aX+bY)(cX+dY)]$$

$$= ac\underbrace{E[X^2]}_{=Var(X)} + bd\underbrace{E[Y^2]}_{Var(Y)} + (ad+bc)\underbrace{E[XY]}_{=E[X]E[Y]=0}$$

$$= ac + bd$$

Geometrically



(Search online for 'linear transformation of circle' for nice demos)

eg – geogebra.org/m/Tx5VkpKP

multivariate Normal rvs via linear combinations

More generally, given $X \in \mathbb{R}^d$, $X_i \sim N(0,1)$, iid $\left(\text{ie, } N(0, I_d)\right)$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}$$

Then $Y = \underset{d\times d \text{ matrix}}{A} X + \underset{\in \mathbb{R}^d}{b} \Rightarrow Y_i \sim$ Gaussian

- $\mathbb{E}[Y_j] = b_j$
- $\mathrm{Var}(Y_j) = \sum_j A_{ij}^2$ (denoted σ_{jj})
- $\mathrm{Cov}(Y_i, Y_j)$ (denoted σ_{ij}) $= \mathbb{E}\left[\left(\sum_k A_{ik} X_k\right)\left(\sum_{k'} A_{jk'} X_{k'}\right)\right] = \sum_{k=1}^{d} A_{ik} A_{jk} = \mathrm{Cov}(Y_j, Y_i)$
- $$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & & \ddots & \vdots \\ \sigma_{d1} & \cdots & \cdots & \sigma_{dd} \end{bmatrix} = \mathbb{E}[Y Y^T] = \mathbb{E}[AX(AX)^T]$$
  $$= A^T \underbrace{\mathbb{E}[XX^T]}_{=I} A = A^T A$$

- Now we want to do this in reverse: Given μ, Σ, generate $Y = A\underset{N(0, I_d)}{X} + b$

multivariate normal rvs

- given a sample $X \sim \mathcal{N}(0,1)$, can generate $Y \sim \mathcal{N}(\mu, \sigma^2)$ as

$$X = \frac{Y-\mu}{\sigma} \Rightarrow \boxed{Y = \sigma X + \mu}$$

$E[Y] = 0 + \mu$

$Var(Y) = \sigma^2$

- *d*-dimensional multivariate normal $\iff$ vector $(X_1, X_2, \ldots, X_d)$
  - each $X_i \sim \mathcal{N}(\mu_i, \sigma_{ii})$ (i.e., normally distributed with mean μ_i and variance σ_{ii})
  - covariance between X_i and X_j is $Cov(X_i, X_j) = \sigma_{ij}$
  - covariance between X_j and X_i is $Cov(X_j, X_i) = \sigma_{ji}$
  - conditions:
    - $\sigma_{ii} \geq 0$
    - $\sigma_{ij} = \sigma_{ji}$ (ie, Σ is symmetric)
    - Σ is positive semi-definite

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

random vectors and covariance

consider any random vector $(X_1, X_2, \dots, X_n)$

(not necessarily Gaussian)

- vector of means $\mu = (\mu_1, \mu_2, \dots, \mu_d)^T$
- covariance matrix: $\Sigma = \mathbb{E}[(X-\mu)(X-\mu)^T]$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \dots & \sigma_{dd} \end{bmatrix}$$

- Σ always positive definite

- Consider $Y = c^T(X-\mu) \in \mathbb{R}$

$$= \sum_{i=1}^{d} c_i (X_i - \mu_i)$$

$$\mathrm{Var}(Y) = \mathbb{E}\left[(c^T(X-\mu))(c^T(X-\mu))^T\right]$$

$$\Rightarrow \mathrm{Var}(Y) = c^T \mathbb{E}\left[(X-\mu)(X-\mu)^T\right] c$$

$((AB)^T = B^T A^T)$

$$= c^T \Sigma c \geq 0 \quad \left(\text{as } \mathrm{Var}(Y) \geq 0\right)$$

$$\Rightarrow \underbrace{\forall c \in \mathbb{R}^d, \ c^T \Sigma c \geq 0}$$

$\Leftrightarrow \Sigma$ is PSD

multivariate normal rvs

$Y_1 = \mu_1 + \sum_i \sigma_{1i} X_1$

- given a sample $X \sim \mathcal{N}(0,1)$, can generate $Y \sim \mathcal{N}(\mu, \sigma^2)$ as

$Y_2 = \mu_2 + \sum_i \sigma_{2i} X_i$

$$Y = \mu + \sigma X \leftarrow$$

Multivariate Gaussian

$$Y = \mu + \Sigma X, \quad \mu \in \mathbb{R}^d, \ \Sigma : d \times d$$

$$Y, X \in \mathbb{R}^d$$

- *d*-dimensional multivariate normal $\iff$ vector $(X_1, X_2, \ldots, X_d)$
  - each $X_i \sim \mathcal{N}(\mu_i, \sigma_{ii})$ (i.e., normally distributed with mean μ_i and variance σ_{ii})
  - covariance between X_i and X_j is $Cov(X_i, X_j) = \sigma_{ij}$
  - covariance between X_j and X_i is $Cov(X_j, X_i) = \sigma_{ji}$
  - conditions: $\sigma_{ij} = \sigma_{ji}$, $\sigma_{ii} = \sigma_i^2$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$ is positive semidefinite (PSD)

How do we generate this?

bivariate Normal rvs

$\mu_1, \mu_2 \in \mathbb{R}$

$\left(\sigma_{11} > 0, \sigma_{22} > 0, \; \rho \in (-1,1)\right)$

- start with the case $d = 2$.
- $\sigma_1^2 = \underline{\sigma_{11}} = Var(X_1)$, $\quad \sigma_2^2 = \underline{\sigma_{22}} = Var(X_2)$,

$$\boxed{\rho} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{\sigma_{21}}{\sigma_1 \sigma_2}.$$

$\rho\sqrt{\sigma_{11}\sigma_{22}} = \rho\sigma_1\sigma_2$

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \underline{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

$\rho\sqrt{\sigma_{11}\sigma_{22}} = \rho\sigma_1\sigma_2$

- want to generate samples of X_1 and X_2, where

$$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), \; X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2),$$

$$Cov(X_1, X_2) = \sigma_{12} = \sigma_{21} = \rho \underbrace{\sigma_1 \sigma_2}_{\sqrt{\sigma_{11}\sigma_{22}}}$$

generating correlated bivariate Normal rvs

- take $N_1, N_2 \sim \mathcal{N}(0,1)$ and independent. $\to \Sigma_{N_1 N_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- set $X_1 = \mu_1 + \sigma_1 N_1$, $\leftarrow$ ie, $E[X_1] = \mu_1$, $Var(X_1) = \sigma_1^2$ ✓
- set $X_2 = \mu_2 + aN_1 + bN_2$ $\leftarrow$ $E[X_2] = \mu_2$ ✓
- we need to have

$\because N_1 \perp N_2$

$$\sigma_{22} = \sigma_2^2 = Var(X_2) = a^2 Var(N_1) + b^2 Var(N_2) = \boxed{a^2 + b^2 = \sigma_2^2}$$

$$\sigma_{12} = Cov(X_1, X_2) = Cov(\cancel{\mu_1} + \sigma_1 N_1, \cancel{\mu_2} + aN_1 + bN_2) =$$

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2} \qquad \sqrt{\sigma_{11}\sigma_{22}}$$

$$= E[\sigma_1 N_1 (aN_1 + bN_2)] - \underbrace{E[\sigma_1 N_1] E[aN_1 + bN_2]}_{=0}$$

$$= \sigma_1 a E[N_1^2] + \sigma_1 b E[N_1 N_2]$$

$$\Rightarrow \boxed{\rho \sigma_1 \sigma_2 = \sigma_1 a} \Rightarrow a = \rho \sigma_2$$

- $\boxed{a^2 + b^2 = \sigma_2^2, a\sigma_1 = \rho\sigma_1\sigma_2} \implies \boxed{(a, b) = \left(\frac{\sigma_{12}}{\sigma_1}, \sigma_2\sqrt{1 - \frac{\sigma_{12}^2}{\sigma_1^2 \sigma_2^2}}\right)}$

generating correlated bivariate Normal rvs

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ \frac{\sigma_{12}}{\sigma_1} & \sigma_2\sqrt{1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

$$X = \underline{\mu} + \underline{L} \quad N$$

$$\mathrm{Cov}(X) = \boxed{LL^T = \Sigma}$$

$$E[X] = E[\mu + LN] = \mu + L\,E[N] = \mu$$

$$\mathrm{Cov}(X) = \mathrm{Cov}(X-\mu) = E[(X-\mu)(X-\mu)^T] = L\,E[NN^T]\,L^T$$

$$= LL^T$$

generating correlated multivariate Normal rvs

- this method works when $d > 2$ as well
- write X as $X = \mu + LN$, where N is a d-dimensional vector whose components are independent $\mathcal{N}(0,1)$.
- to connect the matrix L to Σ, observe that

$$\Sigma = \mathbb{E}\left[(X-\mu)(X-\mu)^T\right] =$$

(last page)

$LL^T = \Sigma \Rightarrow L$ is "square root" of Σ

generating correlated Normal rvs

- L is a "square root" of Σ.
- writing Σ as

$$\Sigma = LL^T$$

  is called the Cholesky factorization of Σ.
- once we can compute the Cholesky factor of Σ, we are done!

- Not the unique square-root. Can also use the 'Principle square-root' or any other L s.t $LL^T = \Sigma$

copulas

Let $(X_1, X_2, \ldots, X_d)$ be any random vector such that

i) $\mathbb{P}[X_i \le x_i] = F_i(x_i)$ (marginal distributions)

ii) $\mathbb{P}[X_1 \le x_1, X_2 \le x_2, \ldots, X_d \le x_d] = F_X(x_1, x_2, \ldots, x_d)$ (joint distr)

– by inversion, we know $U_i = F_i(X_i) \sim U[0,1]$

Copula of $X \equiv \boxed{C(u_1, u_2, \ldots, u_d) = \mathbb{P}[F(X_1) \le u_1, F(X_2) \le u_2, \ldots, F(X_d) \le u_d]}$

$(u_i \in [0,1])$

copulas

- (Sklar's Thm) Given (continuous) marginal CDFs $\{F_i(\cdot)\}_{i=1}^{d}$ + copula $C : [0,1]^d \to [0,1]$ defines unique joint distrib F for random vector $X = (X_1, \ldots, X_d)$

- How to use in simulation to generate $X = (X_1, \ldots, X_d)$?

  1) Generate $(U_1, U_2, \ldots, U_d)$ s.t. $\mathbb{P}[U_1 \le u_1, \ldots, U_d \le u_d] = C(u_1, \ldots, u_d)$

  2) Set $X_i = F^{-1}(U_i) \quad \forall i$

Step 1 may be hard to do exactly! Can only approximate (see assignment)

(or can use acceptance-rejection!)

correlated rvs beyond multivariate Normal

FELIX SALMON 02.23.09 12:00 PM

Recipe for Disaster: The Formula That Killed Wall Street

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