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Problem 1: Practice with DP (Inventory control)

Consider the problem of controlling the inventory of a product over the time periods $\{1, \ldots, T\}$. At each time period t, the following sequence of events take place: (i) the inventory position x_t is observed, (ii) a quantity of $u_t \geq 0$ is ordered if necessary, and is immediately received and (iii) a random demand of D_t (of known distribution F_t) is realized. If there is excess inventory after covering the demand, then a holding cost of h per unit per time period is incurred. If the demand is exceeds the available inventory level, then it is *backlogged*, whereupon the inventory position becomes negative; for each backlogged unit, we incur a backlogging cost of b per time period. The purchasing cost for the product is c per unit, with $b > c$ (try to reason what would happen if we dropped this assumption). Our objective is to minimize the total expected cost over T time periods.

Part (a)

Write the problem of optimal ordering via a dynamic programming formulation (dynamics/state/actions/transition/reward and the Bellman equation) to find the optimal policy.

Part (b)

Suppose that the value functions at each time t are convex functions of the inventory position. Show that there exists a scalar b_t^* for each time period $t = 1, \ldots, T$ such that it is optimal to raise the inventory position to b_t^* after making the replenishment decision at time period t.

Part (c)

Show by induction that the value function at any time period t is a convex function of the inventory position.

Problem 2: Revenue maximization under dynamic seller bidding

Assume that we have C units of a particular product that we want to sell over the time periods $\{1, \ldots, T\}$. At each time period, there is one customer arrival. The customer arriving at time period t makes a bid for the product; assume that each customer makes an i.i.d bid from a set of prices $\{1, \ldots, p_n\}$, where price p_j is chosen with probability σ_j (with $\sum_{j=1}^{n} \sigma_j = 1$). If the customer offers price p_j and we accept this offer, then we generate a revenue of p_j and consume one unit of inventory; else, if we reject the offer, the customer departs. The objective is to maximize the total expected revenue over T time periods.

Part (a)

Write down a dynamic programming formulation to find the optimal policy.

Part (b)

Assume that the value functions are concave functions of the remaining inventory. Show that there exists a scalar y^*_{jt} for each possible price level $j = 1, \ldots, n$ and time period $t = 1, \ldots, T$ such that if the remaining capacity at the beginning of time period t is above y^*_{jt}, then it is optimal to accept an offer of price p_j at time period t. Otherwise, it is optimal to reject.

Part (c)

Assume that $p_1 \leq p_2 \leq \ldots \leq p_n$. Show that $y^*_{jt} \geq y^*_{j+1,t}$, that is, if the remaining inventory at the beginning of time period t is x and $x > y^*_{jt}$ so that it is optimal to accept an offer of price p_j, then we also have $x > y^*_{j+1,t}$, which means that it is also optimal to accept an offer of price p_{j+1}.

Part (d)

Show via induction that the value function at wach time period is a concave function of the remaining inventory.

Problem 3: Static posted prices and prophet inequalities

We again consider the previous setting, where we have C units of a product which we want to sell in periods $1, 2, \ldots, T$, and wherein one customer arrives in each period. Now however, the customer in period t makes a bid B_t according to some (known) distribution F_t. Moreover, instead of using the DP solution, we want to use a single posted price threshold for accepting bids.

Part (a)

Consider the case for $C = 1$: Argue that the optimal revenue is bounded by $\mathbb{E}[\max_{t \in [T]} B_t]$. Moreover, show that:

$$\mathbb{E}\left[\max_{t \in [T]} B_t\right] \leq p + \sum_{t=1}^{T} \mathbb{E}[(B_t - p)^+]$$

Hint: Add and subtract p in the LHS.

Part (b)

Now suppose we use a single threshold p, and accept the first bid greater than p (keeping the item if no bid exceeds the threshold). Let $\Pi(p)$ denote the revenue earned under this policy, and define $q(p) \triangleq \mathbb{P}[\cap_{t=1}^{T}\{B_t < p\}]$ to be the probability that the item goes unsold. Now argue that:

$$\begin{aligned}
\mathbb{E}[\Pi(p)] &\geq p(1 - q(p)) + \sum_{t=1}^{T} \mathbb{E}[(B_t - p)^+]\mathbb{P}[B_k < p \,\forall\, k < t] \\
&\geq p(1 - q(p)) + q(p) \sum_{t=1}^{T} \mathbb{E}[(B_t - p)^+]
\end{aligned}$$

Part (c)

Show that by appropriately choosing p, we can ensure that $\mathbb{E}[\Pi(p)]$ is at least 1/2 the maximum possible reward over all policies.

Part (d)

Extend the above result to the case where the seller has C units to sell.

Problem 4: Overbooking

Consider the problem of selling C seats on a flight, where customers can reserve a seat at a price p. The total demand is given by random variable $D \sim F$. Moreover, each customer with a reservation does not show up independently with probability $1 - q$, at which time they are refunded their reservation cost. To counter this loss in revenue, we can *overbook* by allowing up to b reservations, where b can exceed capacity C. However, each customer who is denied admission is refunded an amount $\theta > p$.

Part (a)

Given booking limit b, let $S(b)$ denote the (random) number of reservations who show up; Similarly, let $R(b)$ be the total expected revenue. Argue that $S(b+1) = S(b) + X\mathbb{1}_{\{D \geq b+1\}}$, where $X \sim Ber(q)$.

Part (b)

Show that $R(b) = p\sum_{j=1}^{\infty} \mathbb{P}[S(b) \geq j] - \theta \sum_{k=c+1}^{\infty} \mathbb{P}[S(b) \geq k]$. Moreover, show that $\Delta R(b) = R(b+1) - R(b) = q\mathbb{P}[D \geq b+1] \cdot (p - \theta \cdot \mathbb{P}[Bin(b,q) \geq C])$ (where $Bin(b,q)$ is the binomial distribution), and use this to characterize the optimal booking level b^*.

Next, we consider overbooking with capacity C and n fare classes $\{(p_i, D_i)\}$ $(p_1 \leq p_2 \leq \ldots \leq p_n)$, which arrive sequentially in the order 1 to n. Each accepted reservation shows up independently with probability q. If a booked reservation for fare class j does not show up at the departure time, then we give this customer a refund of h_j. For each booked reservation that shows up and is denied boarding, we incur a penalty cost of $\theta > p_n$.

Part (c)

Let x_j^k be the number of accepted reservations for fare class k just before making the decisions for fare class j (with $x_j^k = 0$ for $k \geq j$). Using the n-dimensional vector $x_j = (x_j^1, x_j^2, \ldots, x_j^n)$ as the state variable just before making the decisions for fare class j, formulate a dynamic program that maximizes the expected profit. *Make sure to charge the no-show refunds at departure time.*
Hint: Let e_j be the j-th unit vector; accepting one request for fare class j results in state change from x_j to $x_j + e_j$. What are the boundary conditions in this case?

Part (d)

Suppose instead of refunding no-shows, we discount each customers fare by the *expected* refund cost. In other words, at the time of accepting a reservation, the fare associated with fare class j is $p_j - (1-q)\,h_j$. Let z_j be the total number of accepted reservations that we have on-hand just before making the decisions for fare class j. Using the scalar z_j as your state variable, formulate a dynamic program that maximizes the expected profit.

Part (e)

Denote the value function in Part (c) as $V_j(x_j)$ and the value function in Part (d) as $J_j(z_j)$. Use induction to show that $V_j(x_j) = J_j(\sum_{k=1}^{n} x_j^k) - \sum_{k=1}^{n}(1-\rho)\,h_k\,x_j^k$.