A coupling of a Markov Chain was (Q,P)

is a pain of processes $(x_t, Y_t)_{t=0}^{\infty}$ const.

i) both X_t , Y_t are MC on Ω with transition miliar. P.

ii) $X_t = Y_t = X_{t+1} = X_{t+1}$ (coakscence)

. If $X_0 = \infty$, $Y_0 = y$, then we denote $P_{x,y}$, $E_{x,y}$ to be with the probability space on which X_t , Y_t are jointly defined.

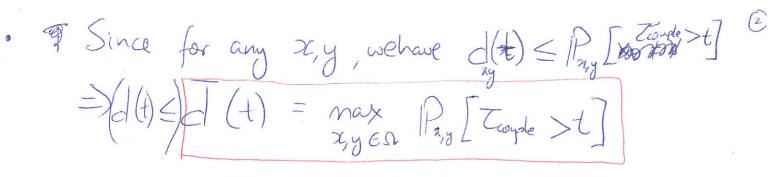
Then Given Markoving coupling (Xt, Yt) with Xo = x, Yo = y. Let $T_{couple} = \min\{t \mid Xt = Yt\}$ Then $\|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV} \le \|P_{x,y}\| [T_{couple} > t]$ $\frac{d_{xy}(t)}{d_{xy}(t)}$

Pf - Pt(x,z) = Px,y [Xt=z], Pt(g,z) = Pa,y [Yt=z] \ \tau \text{260}

- This means that Xt, Yt form a coupling for Pt(z,), Pb;

- Thus |Pxy [Xt ≠ Yt] ≥ || Pt(x,)-Pt(y,)||TV

- Finally, note Pary [Xt + Yt] = Pary [Tecouple >t]



Eg- Lazy RW on {0,13" P = WP 0.5, \$Xt+1 = Xt
WP 0.5, Pick was coordinate, flip bit - (Check) Pis inheducible, appriblic

(Check) TT = Uniform over £0,13" - To bound Emix (1/2e): Consider foll's coupling i) Pick coordinate C was ii) If $X_t[c] = Y_t[c]$, then $\frac{stay}{L} X_{th}[c] = Y_{th}[c] = X_t[c] =$ (Check) This is a valid coupling (i,e, XOD, X ~ P) Zouple = Coupon collector problem with n bins => IP_ Trouple > n long [n(n)+cn] < e^-c =) + mix(2e) = n(m(n) + ln(2e)) = O(nln(a))

- Coupling for this chain = let dH(Xt, Xt) = Hamming distance below Xt, Xt

ii) Ipodocopados Pich i E {0,1,..., n} uar

ii) If i=1, do hothing to both Xt, YE

in) If Xe[i] = Ye[i], do nothing

iv) If $Xt[i] \neq Xt[i]$, then flip Xt[i] and Yt[f(i)], where fli) is an (arbitrary) cyclic permitation on disagreeing bits of X_t , Y_t

v) If $d(x_t, x_t) = 1$ and x_t and x_t disagree on io - If x_t picks io, x_t picks 0 (and vice versa)

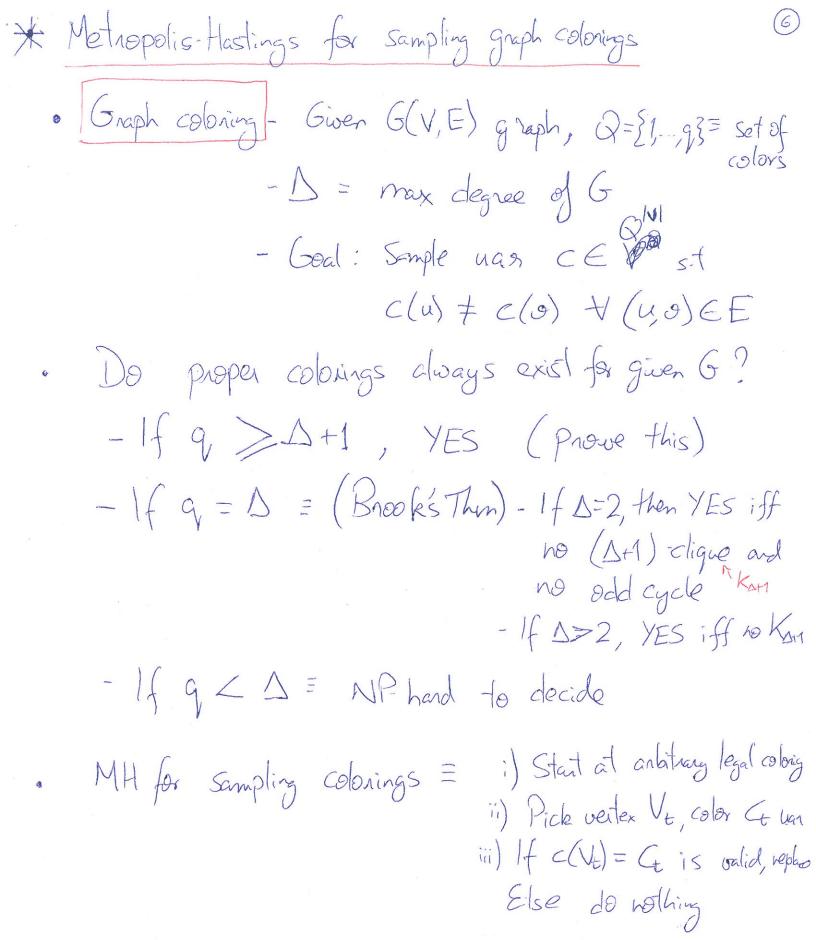
- d Harring (Xt, Yt) is non-incleasing.

- Equiv to coupon collecte on n/2 coupons! $t_{mix}(\chi_e) \leq \frac{1}{2} h \ln(n) + O(n)$

· Mixing time for handom transposition shuffle - Pick 2 sands at randon and Swap - Equivalently = pick and c, position p E {1,2,..., n} wan - exchange and a with card in position p - Propling (Xt, Yt) = Pick same C, Pinall steps - Let d(XE, YE) = the 2d # of positions in which XE, YE differ Dynamics of det (Xt, Yt) = i) If cat same pos" in Xt, Yt =) dt = dt ii) If cat diff posts in Xt, YE - If could at pos" P same >d+1=d+ - If could at pos" P diff =) deri-d+1 $\Rightarrow \mathbb{P}\left[d_{th} \leq d_{t}-1\right] = \left(\frac{d_{t}}{n}\right) \cdot \left(\frac{d_{t}}{n}\right) \Rightarrow \mathbb{E}\left[\operatorname{Time} \text{ for } d_{t} \rightarrow d_{t}-1\right] \leq \left(\frac{n^{2}}{d_{t}}\right)^{2}$ $=) \mathbb{E}\left[\mathsf{T}_{couple}\right] \leq \mathsf{E}\left[\frac{n}{d_t}\right]^2 \leq \frac{6n^2}{712}$ $\exists \quad |P[T_{couple}>t] \leq \frac{6n^2}{\pi^2 t} \Rightarrow t_{nix}(\frac{1}{2}e) \leq \frac{3n^2}{\pi^2 e} = O(n^2)$

- The true mixing time is O(nlogn)

Ghard Coupling - Common source of randomness used to construid coupling for every starting state $x \in \Omega$ ins. — the previous example was a grand coupling - Special case of Markowian coupling - Joint transition natrix Ω defines a MC over $\Omega \times \Omega$ s.t i) $\forall x, y, x', \Sigma \Omega((x,y), (x,y')) = P(x,x')$, ii) Sane for x,y,y'



· This is clearly reversible (symmetric), aperiodic. Not irreducible if 9=11 (consider Korr). Doublands

· Claim - MH chain i Meducible if 9 > 1 +1 (check) · Than - If q > 41+1, then mixing time is O(nlgm) Pf - Coupling (Xt, Xt) defined as followsi) Pick (Vt, Ct) nan and apply to (Xt, Xt) - Let $d_t = d(X_t, Y_t) = H$ of vertices whose colons disagree in X_t and Y_t Good Moves = dt >dt-1 · Happens if we Vt has different colons in Xt, Yt and Ct is a valid color (i.e., not in nbd of Vt) - P[Good Move] > (9-21). dt - Bal Moves $\equiv dt \rightarrow dt + 1$ · Happens if Vt is agreeing, and Gt is such that it is valid for Xt, invalid for It, or vice en - [P [Bad Hove] < 1. Oboon 2. Adt bound on ha Choose Xt Hof rection in or Xt nbd of disagreeing

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$$\mathbb{E}\left[d_{t+1} \mid X_{t}, Y_{t}\right] \leq d_{t} - \frac{d_{t}(q-2\Delta)}{q_{n}} + \frac{d_{t} \cdot 2\Delta}{q_{n}}$$

$$= d_{t} \left(1 - q_{t} + \Delta \Delta\right)$$

$$= \sum_{i} \mathbb{E}\left[d_{t} \mid X_{0}, Y_{0}\right] \leq d_{0} \left(1 - \frac{q-4\Delta}{qn}\right)^{t}$$

$$\leq n \left(1 - \frac{q-4\Delta}{qn}\right)^{t} \leq n \exp\left(\frac{4\Delta - q}{qn}\right)^{t}$$

$$=) t_{mix}(\varepsilon) > (9 - 40) n (log n + log 1/\varepsilon)$$

$$=) \quad t_{\text{mix}} \left(\frac{1}{2e} \right) \leq \otimes \left(\frac{9}{9-40} \right) n \left(\ln \left(\ln \frac{9}{9} + \ln \left(2e \right) \right) \right)$$

(8)