Single-Parameter Environments

We now generalize the single-item auction to a more general setting of single-parameter environments

- n bidders, each with private valuations
- For bidder i, private value v_i is value 'per-unit of stuff' that it receives
- Mechanism decides an allocation $(x_1, x_2, \ldots, x_n)$, where x_i = 'amount of stuff' given to bidder i.
- X is the set of feasible allocations $(X \subseteq \mathbb{R}_+^n)$
- Sealed bid mechanism
  - collect bids $\underline{b} = (b_1, b_2, \ldots, b_n)$
  - (Allocation Rule) choose allocation $(x_1, \ldots, x_n) \in X$ as fn of bids
  - (Payment Rule) choose payments $\underline{p}(b)$ as fn of bids

- Given allocation and payment rules $(\underline{x}, \underline{P})$, the utility of bidder i is (under bids $\underline{b}$)

$$u_i(\underline{b}) = v_i \cdot x_i(\underline{b}) - P_i(\underline{b})$$

- We focus on payments satisfying $P_i(b) \in [0, b_i \cdot x_i(b)]$
- Using this notation, we can define IC, IR:

$\boxed{\text{DSIC}}$ - For every bidder i and vector b_{-i}, the allocation and payment rules $x(b), P(b)$ obey

$$\boxed{u_i(b_i, b_{-i}) \leq u_i(v_i, b_{-i})} \quad \forall\, b_i$$

$\boxed{\text{IR}}$ - For every bidder i, we have

$$\boxed{u_i(v_i, b_{-i}) \geq 0}$$

- Possible objectives of interest
  1) Revenue: $R = \sum_i P_i(b)$
  2) Welfare: $W = \sum_i x_i(b)\, v_i$

- Examples of single-parameter environments

i) Single-item auction: $x_i \in \{0,1\}$, X = set of vectors in $\{0,1\}^n$ such that $\sum_{i=1}^{n} x_i \leq 1$

ii) k identical items: Assuming each person wants at most one item, we have $X \subseteq \{0,1\}^n$ s.t. $\sum_{i=1}^{n} x_i \leq k$

iii) Knapsack auction: Suppose we want to sell ad-time on a TV show. Each bidder i has an ad with private value v_i, public run time w_i. If we have ad-time of size at most W, then $x_i \in \{0,1\}$ and $X = \{ x \in \{0,1\}^n \mid \sum_{i=1}^{n} x_i w_i \leq W \}$

iv) Sponsored search: - k slots $\{1, 2, \ldots, k\}$, where slot j has a click-through-rate (CTR) of α_j (i.e., $\mathbb{P}[\text{Visitor clicks on slot } j \text{ ad}] \propto \alpha_j$); assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$

- Each bidder i has private value v_i, public quality β_i; $\mathbb{P}[\text{Visitor clicks on ad for bidder } i \text{ in slot } j] = \beta_i \alpha_j$. If visitor clicks, then bidder gets value v_i

- $X \equiv$ bipartite matchings between bidders, slots

Myerson's Lemma

In order to design DSIC mechanisms, we want to first characterize what such mechanisms look like.

- Implementable Allocation Rule - An allocation rule $\underline{x}(\underline{b})$ is implementable if there is a payment rule $\underline{p}(\underline{b})$ such that $(\underline{x}, \underline{p})$ is DSIC

Eg - For single-item auctions, $x_i = \mathbb{1}\{i = \arg\max_j b_j\}$ (i.e., allocate to maximum bidder) is implementable (via the second price auction)

- What about allocate item to second highest bidder?

- Monotone Allocation Rule - An allocation rule is monotone if for every bidder i, bids $\underline{b}_{-i}$, the allocation $x_i(z, b_{-i})$ is non-decreasing in z.

Eg - Allocate to highest bidder is monotone. Allocate to second highest bidder is not. (check)

- **Theorem** (Myerson '81) For a single-parameter setting

  a) An allocation rule $\underline{x}(\cdot)$ is implementable iff monotone

  b) If $\underline{x}(\cdot)$ is monotone, then there is a unique payment rule $\underline{p}(\cdot)$ such that $(\underline{x}, \underline{p})$ is DSIC (and $\underline{p}$ has an explicit formula)

Note - the second statement needs a condition that $p_i(b)=0$ if $b_i=0$

Proof -- Suppose $(\underline{x}, \underline{p})$ is DSIC. Then for every bidder i, every value v_i, and every bid vector $\underline{b}_{-i}$, we have for every z

$$(*) \quad \underbrace{v_i\, x_i(v_i, \underline{b}_{-i}) - p(v_i, \underline{b}_{-i})}_{u_i(v_i, \underline{b}_{-i})} \geq \underbrace{v_i\, x_i(z, \underline{b}_{-i}) - p_i(z, \underline{b}_{-i})}_{u_i(z, \underline{b}_{-i})}$$

For short hand, let $x_i(z) = x_i(z, \underline{b}_{-i})$, $p_i(z) = p_i(z, \underline{b}_{-i})$

- (Swapping trick) for any z_1, z_2, the equation $(*)$ holds for $v_i = z_1$, $z = z_2$ and $v_i = z_2$, $z = z_1$.

- For $v_i = z_1$, we have

$$z_1 x_i(z_1) - p_i(z_1) \geq z_1 x_i(z_2) - p_i(z_2)$$

For $v_i = z_2$, we have

$$z_2 x_i(z_2) - p_i(z_2) \geq z_2 x_i(z_1) - p_i(z_1)$$

- Re-arranging, we get (assume $0 \leq z_2 < z_1$)

$$z_2 \left(x_i(z_1) - x_i(z_2)\right) \leq p_i(z_1) - p_i(z_2) \leq z_1 \left(x_i(z_1) - x_i(z_2)\right)$$

~~Moreover, by monotonicity, $x_i(z_1) - x_i(z_2)$~~

- Since $z_1 > z_2$, we must have
$x_i(z_1) - x_i(z_2) \geq 0 \Rightarrow x_i(z)$ is monotone (non-decreasing).

- Now we want to find a pricing rule. For this, we assume henceforth that $\underline{x}$ is monotone, and piecewise constant; we then generalize to when $\underline{x}$ is continuous.

- First consider the equation

$$z_2\left[x_i(z_1)-x_i(z_2)\right]\le P_i(z_1)-P_i(z_2)\le z_1\left[x_i(z_1)-x_i(z_2)\right]$$

If we take $z_1 \searrow z_2$, then both the left and right sides become 0 if $x_i(z_2)=x_i(z_2^+)$ (i.e, no jump at z_2). If however there is a jump at z_2, of size h, then both sides tend to $z_2 . h$

$\Rightarrow$ (jump in P_i at z) $= z$. (jump in x_i at z)

- Finally, if $P_i(0)=0$, then we have

$$\boxed{P_i(z, \underline{b}_{-i}) = \sum_{j=1}^{l} z_j \cdot (\text{jump in } x_i(\cdot,\underline{b}_{-i}) \text{ at } z_j)}$$

where $z_1, z_2, \dots, z_l$ are breakpoints of $x_i(\cdot,\underline{b}_{-i})$ in $[0,z]$

- If instead $x_i(\cdot)$ is continuous, we take $z_1 \searrow z_2$ to get

$$\frac{dP_i(z)}{dz} = z \cdot \frac{dx_i(z)}{dz}$$

$$\Rightarrow \boxed{P_i(b_i, \underline{b}_{-i}) = \int_0^{b_i} z \cdot x_i'(z) \cdot dz}$$

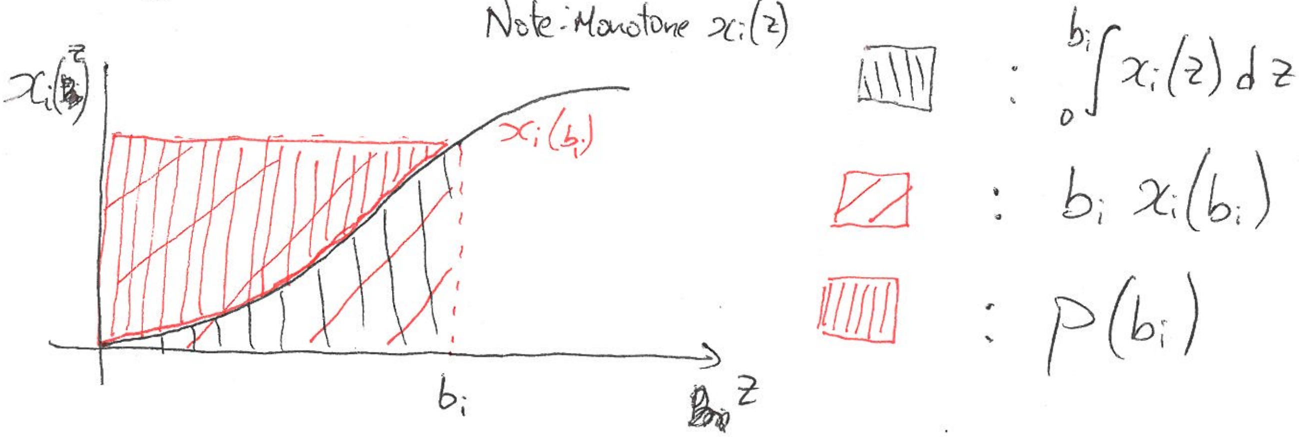
- By integrating by parts, we can get an easier form: (Recall $\int f \cdot g' = fg - \int f'g$)

$$P_i(b_i, \underline{b}_{-i}) = \int_0^{b_i} \underbrace{z}_{f} \cdot \underbrace{\left[\frac{dx_i(z)}{dz}\right]}_{g} \cdot dz$$

$$= z\, x_i(z)\Big|_0^{b_i} - \int_0^{b_i} x_i(z)\,dz$$

$$= b_i\, x_i(b_i) - \int_0^{b_i} x_i(z)\,dz$$

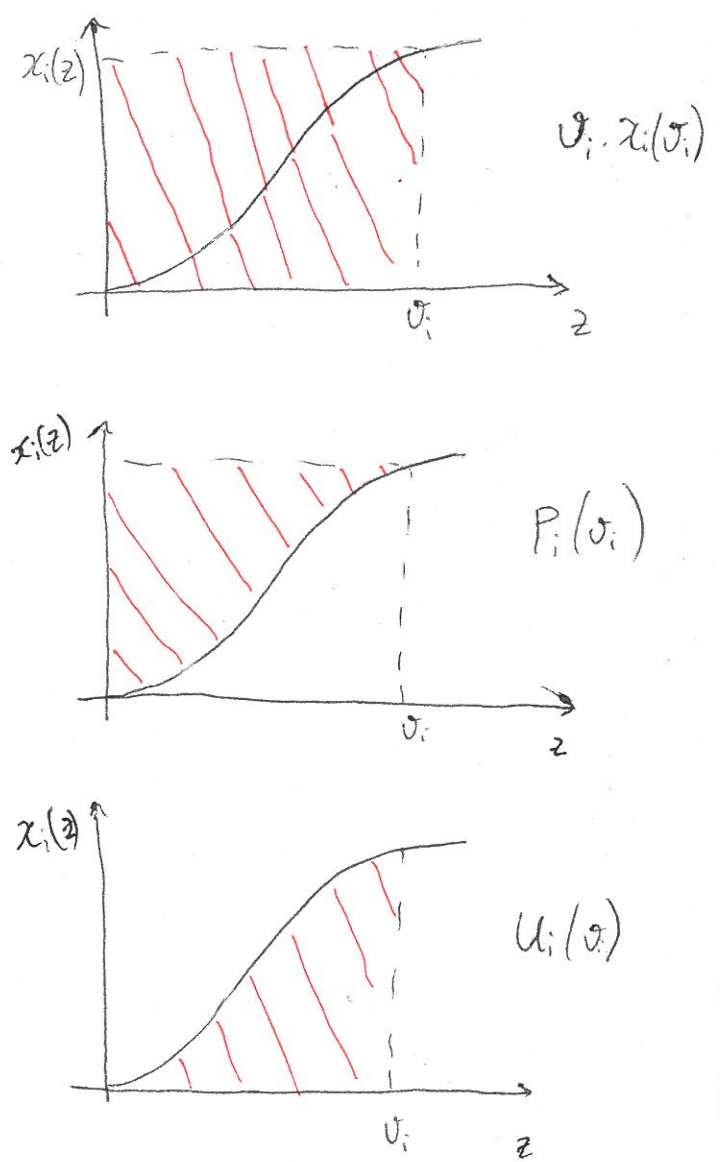
Pictorially, this can be depicted as follows (Assume $\underline{b}_{-i}$ is fixed)



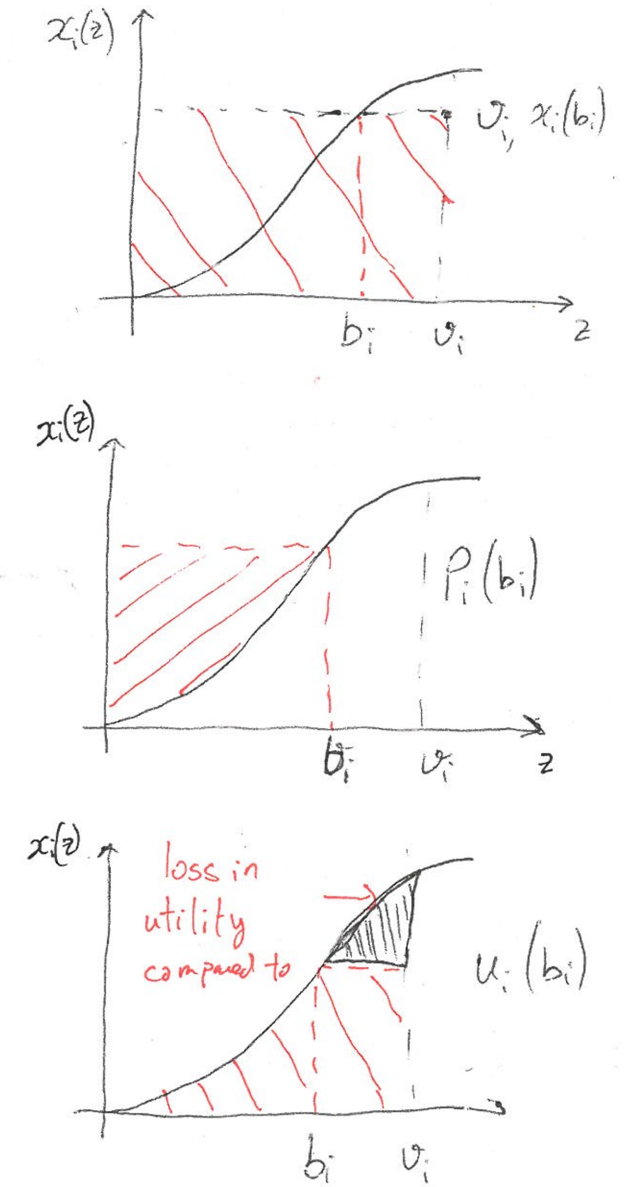
Thus the unique payment (assuming $P_i(0)=0$) is always the area of the rectangle $b_i\, x_i(b_i)$ which is above the curve $x_i(z)$. Now, using this fact, we can easily prove that this price is DSIC, IR.

- We can show DSIC via pictures. Recall $p_i(b_i) = b_i x_i(b_i) - \int_0^{b_i} x_i(z)\,dz$

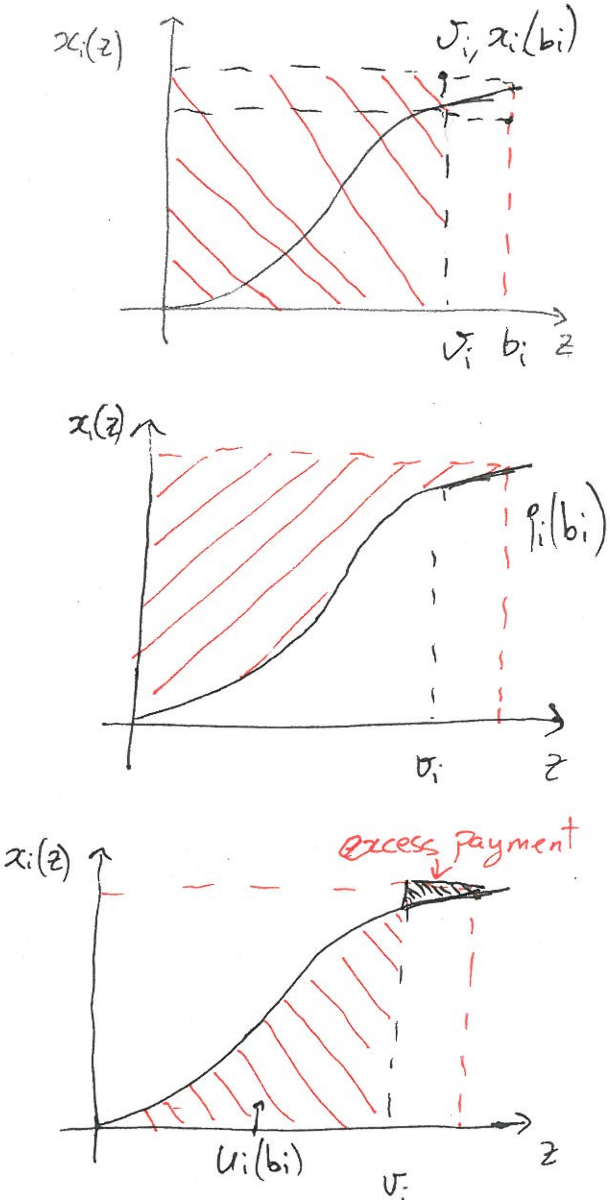
<u>Case 1</u> - $b_i = v_i$



<u>Case 2</u> - $b_i < v_i$



<u>Case 3</u> - $b_i > v_i$



The above proof also works when $x_i(v_i)$ is discrete (i.e., has discontinuous jumps)

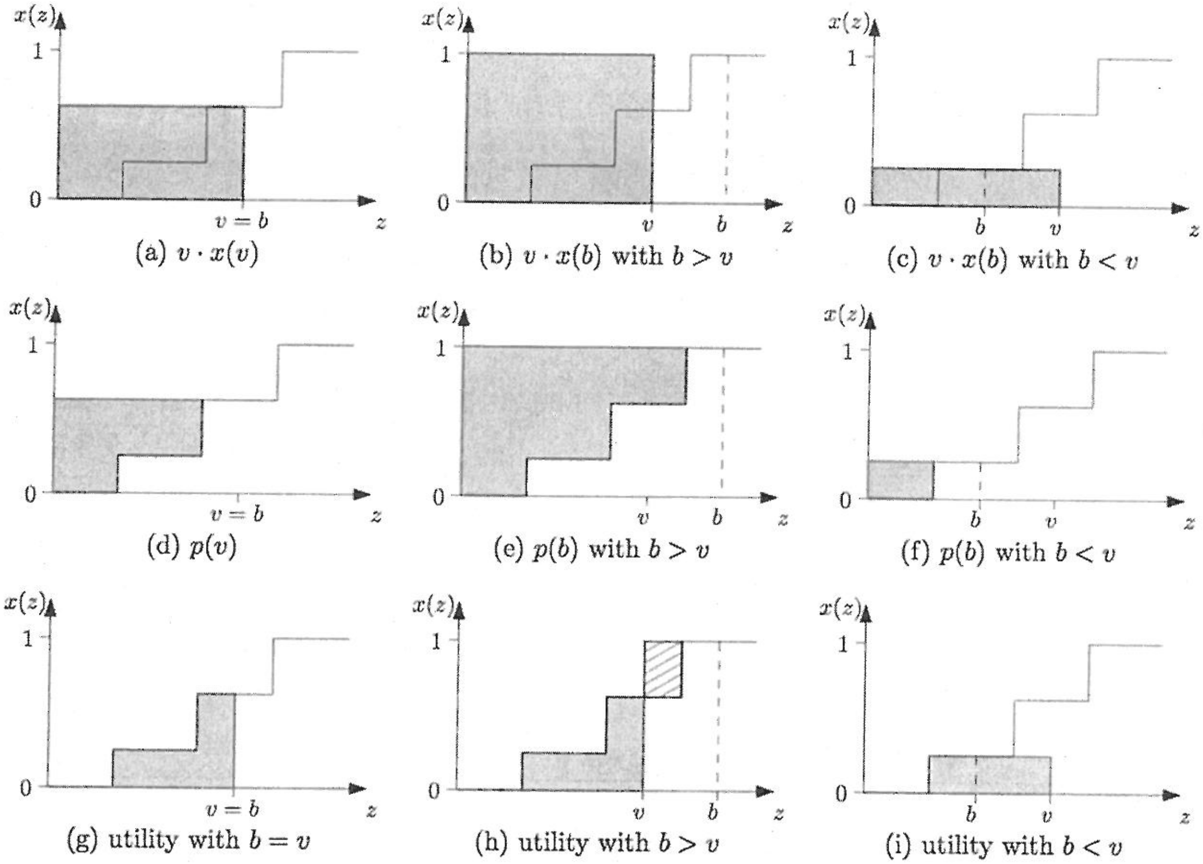


Figure 2: Proof by picture that the payment rule in (6), coupled with the given monotone and piecewise constant allocation rule, yields a DSIC mechanism. The three columns consider the cases of truthful bidding, overbidding, and underbidding, respectively. The three rows show the surplus $v \cdot x(b)$, the payment $p(b)$, and the utility $v \cdot x(b) - p(b)$, respectively. In (h), the solid region represents positive utility and the lined region represents negative utility.

(Courtesy: Tim Roughgarden, Jason Hartline)