Revenue Maximization in Single Parameter Environments

- Given single parameter mechanism design environment
  - n bidders; bidder i has private value $v_i \sim F_i$
  - set of feasible allocations X

We now want to design a mechanism that chooses an allocation so as to maximize revenue.

- We will restrict ourselves to only consider mechanisms $(\underline{x}, \underline{p})$ which are DSIC (i.e., we ask agents for bids, and it is a dominant strategy for agent i to reveal $b_i = v_i$) and IR (i.e., $u_i(v_i; \underline{b}_{-i}) \geq 0 \; \forall i, v_i, \underline{b}_{-i}$)

- From Myerson's Lemma, we know that:
  i) We can only use $\underline{x}(\underline{b})$ such that $x_i(z, \underline{b}_{-i})$ is monotone
  ii) We must set $p_i(z, \underline{b}_{-i}) = z\, x_i(z) - \int_0^z x_i(y)\, dy$

Our aim is to choose $\underline{x}(\underline{b}) \in X$ to maximize $R = \sum_{i=1}^{n} p_i(\underline{b})$

- More particularly, since we choose DSIC mechanisms, we have $b_i = v_i$, and hence we want to choose a mechanism $(\underline{x}, \underline{p})$ to maximize

$$\mathbb{E}[R] = \mathbb{E}\left[\sum_{i=1}^{n} P_i(\underline{V})\right] = \sum_{i=1}^{n} \mathbb{E}\left[P_i(V_i, \underline{V}_{-i})\right]$$

- Consider any bidder i, and fix $\underline{V}_{-i} = \underline{v}_{-i}$. We have

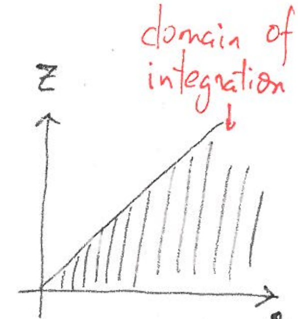
$$\mathbb{E}\left[P_i(V_i, \underline{v}_{-i})\right] = \mathbb{E}\left[V_i\, x_i(V_i, \underline{v}_{-i}) - \int_0^{V_i} x_i(z, \underline{v}_{-i})\, dz\right]$$

(from Myerson's payment rule)

- Moreover: $\mathbb{E}\left[\int_0^{V_i} x_i(z)\,dz\right] = \int_0^{\infty} \left[\int_0^{v} x_i(z)\,dz\right] f_i(v)\,dv$

$$= \int_0^{\infty} \left[\int_z^{\infty} f_i(v)\,dv\right] x_i(z)\,dz$$

$$= \int_0^{\infty} \left(1 - F_i(z)\right) x_i(z)\,dz$$



- Substituting, we get

$$\mathbb{E}_{F_i}\left[P_i(V_i, \underline{v}_{-i})\right] = \int_0^{\infty} z\, x_i(z)\, f_i(z)\,dz - \int_0^{\infty} \left(1 - F_i(z)\right) x_i(z)\,dz$$

$$= \int_0^{\infty} \left(z - \frac{1 - F_i(z)}{f_i(z)}\right) x_i(z)\, f_i(z)\,dz$$

- Let $\phi_i(v) \triangleq v - \frac{1-F_i(v)}{f_i(v)}$ $\left(\boxed{\text{virtual value}} \text{ of bidder } i\right)$

Then we can rewrite the above expression as

$$\mathbb{E}\left[P_i(V_i, \underline{v}_{-i})\right] = \mathbb{E}\left[\phi_i(V_i)\, x_i(V_i, \underline{v}_{-i})\right]$$

and thus summing over all $\underline{v}_i$ ($\because V_i$ are independent)

$$\mathbb{E}[R] = \mathbb{E}\left[\sum_{i=1}^{n} \phi_i(V_i)\, x_i(\underline{V})\right]$$

Note - this looks similar to welfare $W = \sum_{i=1}^{n} v_i\, x_i(\underline{v})$.

This suggests that a natural algorithm to maximize revenue is to choose $\underline{x} \in X$ to maximize the 'virtual welfare' $\sum_{i=1}^{n} \phi_i(V_i)\, x_i(\underline{V})$

- One concern however is if this allocation rule is monotone. Recall that for welfare, we know choosing $\underline{x} \in X$ to maximize $\sum_{i=1}^{n} v_i\, x_i(\underline{v})$ is a monotone rule. Now, if $\phi_i(x)$ is non-decreasing in x, we again have that this is monotone

- To summarize - for any DSIC mechanism

1) $\mathbb{E}[R] = \mathbb{E}\left[\sum_{i=1}^{n} \phi_i(v_i)\, x_i(v_i, v_{-i})\right]$

2) If we knew the v_i, we could choose $\underline{x} \in \mathcal{X}$ to maximize $\sum_{i=1}^{n} \phi_i(v_i) x_i(\underline{v})$

3) If $\phi_i(v_i)$ is non-decreasing in v_i, then the above allocation rule is monotone

4) We can then use the Myerson payment rule to get ~~the~~ a DSIC mechanism that maximizes $\mathbb{E}[R]$

Thus - $\boxed{\text{Maximizing } R \text{ (if } \phi_i \text{ non-decreasing)} \equiv \text{Maximizing virtual } \cancel{\text{value}} \text{ welfare } \sum_{i=1}^{n} \phi_i(v_i) x_i}$

Note - We already know that $\phi_i(v)$ is non-dec[g] if F_i is regular (in fact, that is how we defined it; see HW 1).

- One thing which is unclear from the above discussion is what are the Myerson payments (and more generally, what do these optimal-revenue auctions look like). We will see this via some examples

Eg 1 - Single-item, single bidder with $V_i \sim F_i$

- $\phi_i(V_i) = V_i - \dfrac{1-F_i(V_i)}{f_i(V_i)}$

- Optimal allocation is $x^* = \arg\max_{x \in \{0,1\}} x \cdot \phi_i(V_i)$
$$= \begin{cases} 1 \;; & \phi_i(V_i) > 0 \\ 0 \;; & ow \end{cases}$$

If $x_i^* = 1$,

- The Myerson payment p_i^* corresponds to the 'critical bid', i.e., the smallest ~~value~~ bid needed to win the item

$\Rightarrow p_i^*$ is such that $\phi_i(p_i^*) = 0 \Rightarrow \boxed{p_i^* = \phi_i^{-1}(0)}$

- This is a posted-price mechanism with $p = \phi_i^{-1}(0)$ (Note - You already saw this in HW1 !)

Eg - Single item, n bidders with iid values (i.e, each bidder i has $V_i \sim F$)

- Now, $\phi_i(V_i) = V_i - \frac{1-F(V_i)}{f(V_i)} = \phi(V_i)$

(same virtual-value function for all bidders!)

- Optimal allocation: $\underline{x}^* = \arg\max \sum_{i=1}^{n} \phi_i(V_i)\, x_i$

$$\text{s.t } \sum_{i=1}^{n} x_i \leq 1, \; x_i \in \{0,1\}$$

- In words, the optimal allocation corresponds to sorting bids, and awarding item to highest bid $V^{(1)}$ as long as $\phi(V^{(1)}) \geq 0$

- For winning bidder, critical bid (hence, Myerson payment) P_i^* is such that $\phi(P_i^*) = \max\{0, \phi(V^{(2)})\}$

2nd highest bid

$$\Rightarrow P_i^* = \max\{\phi^{-1}(0), V^{(2)}\}$$

- This is a <u>2nd price auction with reserve price</u> $\phi^{-1}(0)$!

- In general, ~~as~~ as long as all F_i are regular, we have a simple recipe for the optimal DSIC mechanism -

  1) Ask bidder i for value, and compute virtual value $\phi_i(v_i)$
  2) Find $\underline{x}^* \in X$ that maximizes virtual welfare $\sum_{i=1}^{n} x_i^* \phi_i(v_i)$
  3) Charge bidder i a price p_i^* such that $\phi_i(p_i^*)$ is the 'virtual Myerson price' (i.e., the price you would charge if the values were truly $\phi_i(v_i)$)

- This is an amazingly general result! However it has 2 big problems:

  1) If the ϕ_i are different, then the resulting mechanism is strange (see HW 5)
  2) We need to know F_i to find ϕ_i

- For the special case of iid bidders (i.e., $V_i \sim F$ for all bidders i), the optimal mechanism corresponds to the optimal welfare mechanism, with an additional reserve $\phi^{-1}(0)$ (as we saw for the single item auction)
- Moreover, in this setting, ~~as~~ it turns out that the loss from not knowing F_i (and hence having incorrect reserve prices) ~~as~~ can be remedied by attracting additional bidders!

- Let $R_{Welfare}(n)$ denote the revenue of the welfare maximizing auction with n bidders, and $R^*(n)$ denote the ~~welfare~~ revenue of the optimal auction.

__Thm__ (Bulow-Klemperer '96). For a single-item auction with iid ~~ag~~ bidders (and F regular)

$$\mathbb{E}\left[R^*(n)\right] \leq \mathbb{E}\left[R_{welfare}(n+1)\right]$$

- In words - the Bulow-Klemperer result shows that running the second-price (the optimal welfare) auction with $n+1$ bidders gives higher revenue than the optimal auction with n bidders. This is great as the second-price auction does not need F!

Pf - Consider a third mechanism M on $n+1$ bidders

1) Run optimal mechanism on n bidders
2) If item unsold (because highest bid is lower than $\phi^{-1}(0)$), then give it to $(n+1)^{st}$ bidder for free!

- Clearly $\mathbb{E}[R_M(n+1)] = \mathbb{E}[R^*(n)]$
- On the other hand, we know $\mathbb{E}[R]$ for any mechanism on $n+1$ bidders is $\mathbb{E}\left[\sum_{i=1}^{n} \phi(v_i)\, x_i\right]$.

Thus, the highest revenue of a mechanism that always allocates the item is that of the second-price mechanism (which awards item to highest bidder)