Autoregressive Large Language Models are Computationally Universal

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Abstract

We show that autoregressive decoding of a transformer-based language model can realize universal computation, without external intervention or modification of the model's weights. Establishing this result requires understanding how a language model can process arbitrarily long inputs using a bounded context. For this purpose, we consider a generalization of autoregressive decoding where, given a long input, emitted tokens are appended to the end of the sequence as the context window advances. We first show that the resulting system corresponds to a classical model of computation, a Lag system, that has long been known to be computationally universal. By leveraging a new proof, we show that a universal Turing machine can be simulated by a Lag system with 2027 production rules. We then investigate whether an existing large language model can simulate the behaviour of such a universal Lag system. We give an affirmative answer by showing that a single system-prompt can be developed for gemini-1.5-pro-001 that drives the model, under deterministic (greedy) decoding, to correctly apply each of the 2027 production rules. We conclude that, by the Church-Turing thesis, prompted gemini-1.5-pro-001 with extended autoregressive (greedy) decoding is a general purpose computer.

1 Introduction

The emergence of large language models has raised fundamental questions about their computational capability relative to classical models of computation. Several works have investigated the computational abilities of large language models, for example, by considering the expressiveness of transformer architectures for representing circuits [Pérez et al., 2019, Bhattamishra et al., 2020, Wei et al., 2022a] and for representing sequential versions of such circuits [Feng et al., 2023, Merrill and Sabharwal, 2024] under bounded chain of thought extensions [Wei et al., 2022b]. In this paper, we consider the more general question of whether a large language model can support universal computation when applying unbounded chain

of thought. Recently, it has been established that a large language model can be augmented with an external memory to enable simulation of a universal Turing machine via prompting [Schuurmans, 2023]. However, such a result is weakened by the use of external control mechanisms—in particular, regular expression parsers—that offload computational responsibility from the language model. The question of whether an unaided large language model can be Turing universal has not previously been resolved.

We provide an affirmative answer by showing that an unaided large language model can simulate a universal Turing machine. Achieving such a result requires a more general view of autoregressive decoding that allows processing of arbitrarily long input strings. In particular, we consider a natural generalization of autoregressive decoding where emitted tokens are appended to the end of the sequence after processing each successive context, which reduces to standard autoregressive decoding whenever the input fits within the context window.

The main result in this paper is established in a series of steps. First, in Section 2, we introduce the more general perspective on autoregressive decoding that accommodates long input strings, subsequently showing in Section 3 that the proposed extension drives a language model to realize a restricted form of Lag system [Wang, 1963]—a variant of one of the earliest general models of computation [Post, 1943]. We then show in Section 4 that a Lag system, which is able to organize memory as a circular queue, can also provide bidirectional control over memory access. After relevant background on finite memory simulation of Turing machines in Section 5, Section 6 then proves that any Turing machine can be simulated by a restricted Lag system with a context length of 2. Although the universality of Lag systems has been known since [Wang, 1963], the alternative proof presented in this paper is more direct and enables the subsequent argument. Next, in Section 7, we apply the reduction technique to a specific universal Turing machine, $U_{15,2}$ [Neary and Woods, 2009], and obtain a universal Lag system that is defined by a set of 2027 production rules over an alphabet of 262 symbols. Finally, Section 8 develops a single system-prompt that is able to drive a particular large language model, gemini-1.5-pro-001, to correctly apply each of the 2027 rules under greedy decoding. This outcome allows us to conclude that gemini-1.5-pro-001 with extended autoregressive (greedy) decoding can exactly simulate the execution of $U_{15,2}$ on any input, hence it is a general purpose computer.

2 Autoregressive decoding

Given a finite alphabet $\Sigma = \{\sigma_1, ..., \sigma_n\}$, a *string* is defined to be a finite sequence of symbols $s_1...s_k$ such that $k \in \mathbb{N}$ and $s_i \in \Sigma$ for $1 \le i \le k$. Note that every string has a finite length but there is no upper bound on the length of a string. We let |s| denote the length of a string s, let Σ^* denote the set of all strings, and let $H \subset \Sigma$ denote a set of halt symbols.

A language model expresses a conditional distribution $p(s_{n+1}|s_1...s_n)$ over a next symbol $s_{n+1} \in \Sigma$ given an input string $s_1...s_n$. Any such model can be extended to a conditional distribution over an output sequence $s_{n+1}...s_{n+k}$ by the chain rule of probability

$$p(s_{n+1}...s_{n+k}|s_1...s_n) = p(s_{n+1}|s_1...s_n) p(s_{n+2}|s_1...s_{n+1}) \cdots p(s_{n+k}|s_1...s_{n+k-1}).$$
 (1)

$$s_1...s_k s_{k+1}s_{k+2}...s_{k+N} s_{k+N+1}...s_{n+k} s_{n+k+1}$$

 $s_1...s_k s_{k+1} s_{k+2}...s_{k+N}s_{k+N+1} ...s_{n+k}s_{n+k+1} s_{n+k+2}$

Figure 1: Generalized autoregressive decoding when the length of the input sequence n exceeds the context length N. Here k is the number of output symbols that have already been appended. The figure depicts the generation of the (k+1)st output symbol conditioning on the context of length N starting at index k+1, followed by the generation of the (k+2)nd output symbol conditioning on the context of length N starting at index k+2. This process reduces to standard N-Markov autoregressive decoding when $n \leq N$.

Therefore, an output sequence $s_{n+1}...s_{n+k}$ can be generated from the conditional distribution on the left hand side of (1) by sampling each successive symbol from the corresponding conditional distribution on the right hand side, i.e., $s_{n+1} \sim p(\cdot|s_1...s_n)$, ..., $s_{n+k} \sim p(\cdot|s_1...s_{n+k-1})$. This process is referred to as *autoregressive decoding*. In practice, decoding typically proceeds until a halt symbol in H is generated or a maximum emission length is reached.

Note that a key restriction of any transformer based language model is that it employs a bounded context that limits the length of the input string to at most N symbols for some $N < \infty$. However, to capture any universal notion of computation we will need to consider computing over arbitrarily long inputs, while also allowing for the possibility of arbitrarily long output sequences. Therefore, we require a more general perspective on autoregressive decoding that is able to accommodate long input and output sequences.

First, if we assume the input is shorter than the context, a long output sequence can be generated simply by adopting the standard N-Markov assumption

$$p(s_{n+k}|s_1...s_{n+k-1}) = \begin{cases} p(s_{n+k}|s_{n+k-N}...s_{n+k-1}) & \text{if } n+k > N+1\\ p(s_{n+k}|s_1...s_{n+k-1}) & \text{if } n+k \le N+1 \end{cases}$$
 (2)

for all $s_1...s_{n+k-1}$; that is, the conditioning can be truncated to the most recent N symbols. Under this assumption, autoregressive decoding is able to generate output sequences by successively sampling $s_{n+k} \sim p(\cdot|s_{n+k-N}...s_{n+k-1})$ for arbitrarily large k, which is a default assumption in the literature.

The key question remains of how to perform autoregressive decoding when the input is longer than the context length. Such an extension is not typically considered in the literature, but necessary in our case. For this purpose, we consider a simple generalization of N-Markov autoregressive decoding, generalized autoregressive decoding, where the next generated symbol is appended to the end of the sequence, as shown in Figure 1. In particular, given an input of length n, if we start from an initial context window at index 1 and let n+k denote the length of the entire sequence after k symbols have been emitted, then assuming n > N, k + 1 will index the start of the current context and the (k + 1)st output symbol will be generated according to $s_{n+k+1} \sim p(\cdot|s_{k+1}...s_{k+N})$; see Figure 1. Note that if $n \leq N$ the process reduces to standard N-Markov autoregressive decoding (2). Although such an autoregressive decoding mechanism might seem peculiar, given that the language model was

$$s_1...s_k s_{k+1}s_{k+2}...s_{k+N} s_{k+N+1}...s_{n+\ell} s_{n+\ell+1}s_{n+\ell+2}$$

Figure 2: Extended autoregressive decoding when two output symbols are generated for a given context. Here ℓ is the number of output symbols that have already been appended after k iterations, with $\ell \geq k$, and it is assumed the length of the input sequence n exceeds the context length N.

trained to predict s_{k+N+1} rather than s_{n+k+1} from the context $s_{k+1}...s_{k+N}$, we will see that this generalization allows the language model to exhibit rich computational behaviour.

To compare the computational ability of a language model to classical models of computation, we furthermore need to assume the conditional distribution is deterministic; that is, there exists a function $M: \Sigma^N \to \Sigma$ such that, for all strings $s_1...s_N$, $p(\sigma|s_1...s_N) = 1$ if $\sigma = M(s_1...s_N)$ and $p(\sigma|s_1...s_N) = 0$ if $\sigma \neq M(s_1...s_N)$. Such deterministic behaviour can be achieved by setting the temperature parameter of the language model to zero and fixing any additional random seeds affecting its output. Thus, the (k+1)st output symbol is generated according to $s_{n+k+1} = M(s_{k+1}...s_{k+N})$ then appended to the end of the sequence.

Finally, to accommodate a special edge case that we will have to elaborate below, the language model will be required to emit a pair of output symbols rather than just a single symbol in certain contexts, as shown in Figure 2. In this case, we use ℓ to denote the number of output symbols that have already been appended after k iterations, with $\ell \geq k$. Note that such a mechanism does not conform to the generalized autoregressive decoding mechanism depicted in Figure 1, because $s_{n+\ell+2}$ will need to depend on s_{k+1} and not s_{k+N+1} , even after $s_{n+\ell+1}$ is generated, as shown in Figure 2. Therefore, we need to consider another form of decoding, extended autoregressive decoding, to achieve such behaviour. To realize this extension we leverage the existence of an implicit halt token h outside the base alphabet Σ (disjoint from the explicit halt tokens H), and decode the language model given a context $s_{\ell+1}...s_{\ell+N}$ by generating symbols from $\Sigma \cup \{h\}$ until h is encountered (then discarded). This technicality will be revisited when it becomes relevant, but briefly it is the differentiator between being able to simulate a general Turing machine versus a linear bounded automaton.

In summary, we consider deterministic autoregressive decoding of a language model with context length N according to Algorithm 1.¹

3 Lag systems

The first key observation is that autoregressive decoding of a large language model can be replicated by a Lag system. Lag systems were introduced by [Wang, 1963] as a simple variation of one of the earliest formal models of general computation, the Tag systems studied in [Post, 1943].

A Lag system consists of a finite set of rules $x_1...x_N \to y$, where N is the length of the context, $x_1...x_N \in \Sigma^N$ denotes a sequence of symbols to be matched, and $y \in \Sigma^*$ is

¹Note that the language model does not need to maintain the count ℓ of the number of symbols appended; we include it simply to keep an end-of-sequence index $n + \ell$ that will be helpful to refer to in some proofs.

Algorithm 1 Extended autoregressive decoding of model M with context length N

```
\begin{array}{l} \textbf{Input: } s \leftarrow s_1...s_n \\ \ell \leftarrow 0 \\ \textbf{for } k{=}0,\!1,\!...\,\textbf{do} \\ & c \leftarrow s_{k+1}...s_{k+N} \\ & y \leftarrow M(c) \\ & s \leftarrow s_1...s_{n+\ell}\,y \\ & \ell \leftarrow \ell + |y| \\ & \textbf{if } y \cap H \neq \emptyset \textbf{ then} \\ & \bot \text{ halt} \\ \end{array} \right. \\ \triangleright get \ next \ context \\ \triangleright get \ model \ response \\ \triangleright append \ response \ to \ overall \ sequence \\ \triangleright track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \ appended \\ \Rightarrow track \ number \ of \ symbols \
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a corresponding output. For a deterministic Lag system, each pattern $x_1...x_N$ is unique, hence the Lag system defines a partial function $L: \Sigma^N \to \Sigma^*$ that maps a pattern $x_1...x_N$ to a corresponding output y. The computation of a Lag system is defined by operating over a memory string, where in each iteration, a rule is matched to the prefix of the memory string, then the result appended to the string, after which the first symbol is deleted; see Algorithm $2.^2$

Algorithm 2 Lag system L with context length N

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Input: s \leftarrow s_1...s_n
\ell \leftarrow 0
for k=0,1,... do

if L contains a rule x_1...x_N \rightarrow y such that x_1...x_N = s_{k+1}...s_{k+N} then

s \leftarrow s_{k+2}...s_{n+\ell}y
delete first symbol then append the response to sequence else

halt
\ell \leftarrow \ell + |y|
formula formula for the probability of the symbols appended if <math>formula formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols appended if <math>formula for the probability of the symbols ap
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Like autoregressive decoding, we introduce an explicit set of halt symbols $H \subset \Sigma$ so that computation terminates whenever a halt symbol is generated. Computation also terminates when there is no rule matching the current context $s_{k+1}...s_{k+N}$, which additionally covers the case when the current memory string is shorter than N.³

Note that, by inspection, the computation of a Lag system specified by Algorithm 2 is similar to that of autoregressively decoding a language model in Algorithm 1. We formalize this observation in the following proposition.

²Once again, the Lag system does not need to maintain the count ℓ ; we include it simply to keep an explicit end-of-sequence index $n + \ell$.

 $^{^3}$ Classically, Lag systems allow the output y to be empty, hence the string s being tracked in Algorithm 2 can contract. However, we will not make use of this possibility in this paper.

Proposition 1 For any deterministic language model M with context length N, there exists a deterministic Lag system L such that, for any input string s with $|s| \geq N$, the execution of Algorithm 2 with L simulates extended autoregressive decoding of M with Algorithm 1, in the sense that, for every iteration $k \in \mathbb{N}$, the two algorithms will encounter the same context $s_{k+1}...s_{k+N}$ and generate the same next output y.

Proof: Given a language model M with context length N, construct a Lag system L where

for every string $x_1...x_N$ of length N we add a rule $x_1...x_N \to M(x_1...x_N)$ to L. Consider an input string $s_1...s_n$ where $n \ge N$. Let $s^{M,k} = s_1^{M,k}...s_{n+\ell^M}^{M,k}$ denote the string maintained by M at the start of iteration k, and let $s^{L,k} = s_{k+1}^{L,k}...s_{n+\ell^L}^{L,k}$ denote the string maintained by L at the start of iteration k. We prove by induction that, for all $k \in \mathbb{N}$,

$$s_{k+1}^{M,k}...s_{n+\ell^M}^{M,k} = s_{k+1}^{L,k}...s_{n+\ell^L}^{L,k}$$
 (3)

at the start of iteration k, where $\ell^M = \ell^L = \ell$ denote the total number of symbols that have previously been appended by the respective algorithms.

For the base case, when k=0, we have that $s^{M,0}=s^{L,0}=s_1...s_n$, which immediately satisfies (3).

Next, assume both algorithms have produced the respective strings $s^{M,k} = s_1^{M,k}...s_{n+\ell}^{M,k}$ and $s^{L,k} = s^{L,k}_{k+1}...s^{L,k}_{n+\ell}$ satisfying (3) at the start of the kth iteration. Then the N context symbols $s^{M,k}_{k+1}...s^{M,k}_{k+N} = s^{L,k}_{k+1}...s^{L,k}_{k+N}$ are identical in both algorithms, hence $L(s^{L,k}_{k+1}...s^{L,k}_{k+N}) = M(s^{M,k}_{k+1}...s^{M,k}_{k+N})$ by construction, implying that both algorithms will generate the same $y = M(s^{M,k}_{k+1}...s^{M,k}_{k+N})$ as output. This means that at the start of the next iteration k+1, Algorithm 1 will be maintaining the string $s^{M,k+1}=s_1^{M,k}...s_{n+\ell}^{M,k}y$ and Algorithm 2 will be maintaining the string $s^{L,k+1} = s^{L,k}_{k+2}...s^{L,k}_{n+\ell}y$, so the pair still satisfies (3).

Thus, the computational capacity of a large language model under extended autoregressive decoding (Algorithm 1) does not exceed that of a Lag system with the same context length N. Proposition 1 clearly motivates our interest in understanding the computational expressiveness of Lag systems, since the processing of a Lag system is closely related to (and ultimately simulable by) extended autoregressive decoding of a language model.

We will find it useful to characterize subclasses of Lag systems, restricted (N, K)-Lag systems, by their context length N and by the maximum length K of any output y in the right hand side of a rule. Of particular interest will be restricted (N,1) and (N,2)-Lag systems. One of the key lemmas below will be to establish that any Turing machine can be simulated by a restricted (2,2)-Lag system, while any linear bounded automaton [Myhill, 1960] can be simulated by a restricted (2, 1)-Lag system.

Although Lag systems have been known to be computationally universal since [Wang, 1963]. the main arguments in this paper require a more succinct construction based on a direct reduction. To achieve such a reduction, we first need to understand how a Lag system can enable sufficiently flexible memory access even though it operates solely through Algorithm 2.

4 Memory access control with a Lag system

Demonstrating that any Turing machine can be simulated by a Lag system requires a series of developments. The first key step is understanding how a Lag system can exercise sufficient control over memory access to simulate the bidirectional memory access patterns of a Turing machine. In this section, we demonstrate how bidirectional memory access control can be implemented in a Lag system.

First, note that a Lag system can naturally operate a circular queue over its memory string. Consider the Lag system defined by the simple set of rules $x \to x \ \forall x \in \Sigma$. In this system, for any input string $s_1s_2s_3...s_n$, Algorithm 2 will produce an update sequence on successive iterations that behaves as

$$s_1 s_2 s_3 \dots s_n \Rightarrow s_2 s_3 \dots s_n s_1 \Rightarrow s_3 \dots s_n s_1 s_2 \Rightarrow \dots; \tag{4}$$

that is, each iteration will rotate one symbol from the start (dequeue) to the end of the string (enqueue), thus perpetually rotating the string in a circular queue. It is known that a finite state machine that operates on a memory queue can simulate any Turing machine [Zaiontz, 1976, Kudlek and Rogozhin, 2001], but the challenge here is to achieve the same capability without leveraging an external finite state controller, since a Lag system (or autoregressive decoding of a language model) only performs updates based on the current memory context without access to any external state machine.

The remainder of this section shows how a Lag system can implement bidirectional access control over its memory, separating the two cases of moving a control location one position left (counterclockwise) and one position right (clockwise).

4.1 Counterclockwise position control

For a base alphabet Σ , consider an extended alphabet over pairs $\Sigma_p = \Sigma \times \{ \bot, p \}$, where in each pair $(\sigma, p) \in \Sigma_p$, we use the second coordinate $p \in \{ \bot, p \}$ as a position control symbol and the first coordinate $\sigma \in \Sigma$ as the original "data". We develop a Lag system that can move a targeted control location one step left (counterclockwise).

Consider a data string $s_1...s_{n-2}s_{n-1}s_n \in \Sigma^n$ with $n \geq 3$. Imagine we formulate a corresponding string of augmented pairs, $(s_1, _)...(s_{n-2}, _)(s_{n-1}, p)(s_n, _) \in \Sigma_p^n$, such that a position control symbol p has been associated with the (n-1)st symbol and all other position control symbols are assigned "blank". Assume we would like to move the position control symbol one step to the left (counterclockwise); that is, we would like to associate p with s_{n-2} . This can be achieved in a Lag system by the following set of rules

$$R_{p} = \begin{cases} (x, \square)(y, \square) \to (x, \square) & \forall x, y \in \Sigma \\ (x, \square)(y, p) \to (x, p) & \forall x, y \in \Sigma \\ (x, p)(y, \square) \to (x, \square) & \forall x, y \in \Sigma \end{cases} \qquad \begin{array}{c} \square \square \longrightarrow \square \\ \square p \longrightarrow p \\ p \square \longrightarrow \square \end{array}$$
(5)

Proposition 2 Define a Lag system by R_n . Consider any string

$$(s_1, \bot)...(s_{n-2}, \bot)(s_{n-1}, p)(s_n, \bot)$$

 $n \geq 3$. Then after n-1 iterations Algorithm 2 will render the memory string

$$(s_n, _)(s_1, _)...(s_{n-2}, p)(s_{n-1}, _)$$

thus moving the control token p one step counterclockwise.

Proof: Whenever the position control symbols in the first two pairs (the "context") are both blank, the first pair will be rotated unaltered to the end of the string, by the rules corresponding to the pattern $__ \to _$. Therefore, for the first n-3 iterations, the first pair will simply be rotated to the end unmodified. Then, on iteration n-2, the context encountered will be $(s_{n-2}, _)(s_{n-1}, p)$, implying that (s_{n-2}, p) will be rotated to the end according to $_p \to p$. On iteration n-1, the next context will be $(s_{n-1}, p)(s_n, _)$, so $(s_{n-1}, _)$ will be rotated to the end by $p_\to _$, yielding the stated result.

Ideally, we would also like to distinguish an initiating control symbol L from a terminating control symbol 1, so we can differentiate the beginning of a move from its conclusion. This can be achieved by a minor alteration of the rule set.

$$R_{\text{left}} = \left\{ \begin{array}{ll} (x, \square)(y, \square) \to (x, \square) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{L}) \to (x, \mathbb{1}) & \forall x, y \in \Sigma \\ (x, \mathbb{L})(y, \square) \to (x, \square) & \forall x, y \in \Sigma \end{array} \right\} \quad \begin{array}{ll} \square \square \longrightarrow \square \\ \square \square \longrightarrow \square \\ \square \square \longrightarrow \square \\ \square \square \longrightarrow \square \end{array}$$
 (6)

Proposition 3 Define a Lag system by R_{left} . Consider any string

$$(s_1, \square)...(s_{n-2}, \square)(s_{n-1}, \mathsf{L})(s_n, \square)$$

 $n \geq 3$. Then after n-1 iterations Algorithm 2 will render the memory string

$$(s_n, _)(s_1, _)...(s_{n-2}, 1)(s_{n-1}, _)$$

changing the control token from L to 1 and rotating the control position one step counter-clockwise.

Proof: The first n-3 iterations simply rotate the first pair unmodified to the end. Then, on iteration n-2, the encountered context will be $(s_{n-2}, _)(s_{n-1}, L)$, implying that $(s_{n-2}, 1)$ will be rotated to the end by $_L \to 1$. On iteration n-1, the next context will be $(s_{n-1}, L)(s_n, _)$, implying that $(s_{n-1}, _)$ will be rotated to the end by $L _ \to _$, rendering the stated result.

Intuitively, the presence of the control token 1 informs s_{n-2} that it has become the next control point, after which appropriate updates can be subsequently triggered. Thus, a Lag system can easily and naturally move control tokens in a counterclockwise direction around a circular queue, incurring only O(n) iterations to move a control token and realign the queue.

4.2 Clockwise position control

Moving a control location one step right (clockwise) is also achievable, but it is far more involved. Such functionality can be achieved by a construction that takes $O(n^3)$ iterations. It will take a series of developments to arrive at a suitable rule set in an understandable way.

First, consider an example string $(s_1, \square)(s_2, \square)...(s_{n-1}, p)(s_n, \square)$ where we would like to move the control token p one step clockwise (right) rather than counterclockwise (left); that is, we would like to associate p with s_n . To understand how such a clockwise move might be possible, consider the previous rule set R_p but note that if we run Algorithm 2 for a larger number of iterations, the pulse token p will continue to move around the queue in a counterclockwise manner until it eventually lands on s_n .

Proposition 4 For the Lag system defined by R_p and any string

$$(s_1, \bot)(s_2, \bot)...(s_{n-1}, p)(s_n, \bot)$$

 $n \geq 3$, after $(n-1)^2$ iterations Algorithm 2 will render the memory string

$$(s_2, _)...(s_{n-1}, _)(s_n, p)(s_1, _)$$

effectively rotating the control token p one step clockwise.

Continuing for another n-1 iterations will restore the memory string to its initial state, so the overall behaviour is periodic with period n(n-1).

Proof: By Proposition 2 we know that after n-1 iterations the control token p will have moved one position counterclockwise to s_{n-2} . Therefore, repeating this process n-2 times, i.e., (n-2)(n-1) iterations in total, the memory string will become $(s_3, _)...(s_n, _)(s_1, p)(s_2, _)$. From this string, another n-1 iterations will lead to $(s_2, _)...(s_{n-1}, _)(s_n, p)(s_1, _)$, yielding the first stated claim after $(n-2)(n-1)+n-1=(n-1)^2$ iterations.

Continuing from this string for another n-1 iterations will result in the original string, using a total of $(n-1)^2 + n - 1 = n(n-1)$ iterations.

Therefore, after every block of n(n-1) iterations the token p will complete an orbit of the memory string. We will refer to each block of n-1 iterations as a pass, and each block of n passes (i.e., n(n-1) iterations) as a cycle.

The key consequence of Proposition 4 is that, if we had remembered s_{n-1} as the initial control position, then after $(n-1)^2 + n - 2$ iterations we would encounter the context $(s_{n-1}, _)(s_n, p)$, at which point the first symbol s_{n-1} would be able to "see" that the pulse p has reached its clockwise neighbour s_n , and we could potentially notify s_n that it has become the new control point. Unfortunately, the information that s_{n-1} was the original control position has been lost, so we need to augment the construction to retain this.

To retain a memory of the initial control position, consider the extended alphabet $\Sigma_t = \Sigma_p \cup (\Sigma \times \{t, w\})$ and the extended rule set

$$R_{t} = R_{p} \cup \left\{ \begin{array}{ll} (x, \square)(y, \mathbf{t}) \to (x, \mathbf{p}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbf{w}) \to (x, \square) & \forall x, y \in \Sigma \\ (x, \mathbf{t})(y, \square) \to (x, \mathbf{w}) & \forall x, y \in \Sigma \\ (x, \mathbf{w})(y, \square) \to (x, \mathbf{w}) & \forall x, y \in \Sigma \\ (x, \mathbf{w})(y, \mathbf{p}) \to (x, \mathbf{t}) & \forall x, y \in \Sigma \\ (x, \mathbf{p})(y, \mathbf{w}) \to (x, \square) & \forall x, y \in \Sigma \end{array} \right\} \quad \begin{array}{ll} \square \mathbf{t} \to \mathbf{p} \\ \square \mathbf{w} \to \square \\ \mathbf{t} \sqcup \to \mathbf{w} \\ \mathbf{w} \sqcup \to \mathbf{w} \\ \mathbf{w} \sqcup \to \mathbf{w} \\ \mathbf{w} \mathbf{p} \to \mathbf{t} \\ \mathbf{p} \mathbf{w} \to \square \end{array}$$
 (7)

Here we have introduced two new "stationary" tokens t and w that will remain at the original location s_{n-1} to remember the initial control point, while still allowing the pulse token p to orbit the memory string unimpeded. In particular, w serves as the stationary marker at s_{n-1} . The second token t is needed to represent the situation when p arrives at location s_{n-1} , which occurs once during each orbit. Whenever this happens, we combine p and w into t to remember the control location and also represent the current location of p. On the subsequent pass, t will re-initiate p in the next counterclockwise position, s_{n-2} , then switch back to w. Such a mechanism allows p to continue its synchronous orbit around the memory queue, while w retains knowledge of the initial location s_{n-1} . We formalize this process in the following proposition.

Proposition 5 For the Lag system defined by R_t and any string

$$(s_1, \bot)...(s_{n-2}, \bot)(s_{n-1}, t)(s_n, \bot)$$

 $n \geq 3$, after n-1 passes $((n-1)^2$ iterations) Algorithm 2 will render the memory string

$$(s_2, \llcorner)...(s_{n-1}, \mathtt{w})(s_n, \mathtt{p})(s_1, \llcorner)$$

so that the control symbol w remembers the initial position and the pulse token p has been placed one step clockwise from the initial position of t.

Moreover, after an additional pass (n-1 iterations) the memory string will be restored to its initial state, yielding periodic behaviour with period n(n-1) (i.e., one cycle).

Proof: The first pass behaves as follows. Note that the first n-3 iterations simply rotate the first pair to the end unaltered. On iteration n-2, the encountered context will be $(s_{n-2}, _)(s_{n-1}, t)$, implying that (s_{n-2}, p) will be rotated to the end by the rule $_t \to p$. On iteration n-1, the encountered context will be $(s_{n-1}, t)(s_n, _)$, so (s_{n-1}, w) will be rotated to the end by $t_\to w$. Therefore, at the conclusion of the first pass (n-1) iterations the memory will consist of $(s_n, _)(s_1, _)...(s_{n-2}, p)(s_{n-1}, w)$.

On the second pass, the next n-3 iterations will rotate the first pair unaltered. The subsequent iteration will encounter the context $(s_{n-3}, \bot)(s_{n-2}, p)$ and rotate (s_{n-3}, p) to the end, while the following iteration will encounter $(s_{n-2}, p)(s_{n-1}, w)$ and rotate (s_{n-2}, \bot) to the end by the rule $p w \to \bot$. Therefore, at the conclusion of the second pass the memory will consist of $(s_{n-1}, w)(s_{n-1})(s_{n-1}, \bot)...(s_{n-3}, p)(s_{n-2}, \bot)$.

Each of the next n-3 passes will result in the symbol \mathbf{w} remaining at s_{n-1} , by the rule $\mathbf{w}_{-} \to \mathbf{w}$, while \mathbf{p} will be advanced one position counterclockwise, as established in Proposition 4. These claims follow because \mathbf{p} and \mathbf{w} will both be followed by blanks. Moreover, the position control symbol associated with s_{n-2} remains blank because the context $(s_{n-2}, _{-})(s_{n-1}, _{\mathbf{w}})$ will rotate $(s_{n-2}, _{-})$ to the end by the rule $_{-}\mathbf{w} \to _{-}$. Therefore, at the conclusion of n-1 passes the memory will consist of $(s_2, _{-})...(s_{n-1}, _{\mathbf{w}})(s_n, _{\mathbf{p}})(s_1, _{-})$, which establishes the first claim after $(n-2)(n-1)+n-1=(n-1)^2$ iterations.

Continuing from the memory string $(s_2, _)...(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})(s_1, _)$, the next n-3 iterations will rotate the first pair unaltered, resulting in the context $(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})$. The next iteration will rotate (s_{n-1}, \mathbf{t}) to the end, by the rule $\mathbf{w} \, \mathbf{p} \to \mathbf{t}$, leaving the context $(s_n, \mathbf{p})(s_1, _)$. Finally, $(s_n, _)$ will be rotated to the end unaltered, resulting in the original string after a total of $(n-1)^2 + n - 1 = n(n-1)$ iterations, or n passes.

Therefore, at the end of $(n-1)^2 + n - 2$ iterations, we will encounter the context $(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})$, making it possible for the Lag system to recognize that the pulse \mathbf{p} has arrived at the correct clockwise neighbour s_n . At this point, to move the control locus clockwise to s_n , all we will need to do is somehow "freeze" an appropriate control token on s_n while clearing the other position control tokens. This leads to the final construction for clockwise rotation control.

To achieve the ability to freeze an orbiting token once a desired position is detected, we introduce a two time-scale message passing scheme. First, as above, a fast pulse signal p will orbit the memory queue once every cycle (every n passes, or n(n-1) iterations), visiting each successive counterclockwise position after each pass (n-1) iterations. We also retain the same stationary tokens w and v to remember the initial control position in the same manner as above. On top of these mechanisms, we add another, slower pulsed-hold signal v that also propagates counterclockwise around the queue, but only advances when "pulsed" by v p. That is, v will only advance one position counterclockwise for every full cycle completed by v p. (This is analogous to a clock, where v behaves like a second hand and v like a minute hand, both rotating counterclockwise.) When the pulsed-hold symbol v finally reaches the target location v it will be visible from the initial control location v at that point, a final "victory" pulse v will be sent around the queue to inform v which is still holding the control token v that it is indeed the next control location.

This scheme is realized by the extended alphabet $\Sigma_{\text{right}} = \Sigma_t \cup (\Sigma \times \{R, d, g, z, v, r\})$ and

the extended rule set

ed rule set
$$R_{right} = R_{t} \cup \begin{cases} (x, \square)(y, \mathbb{R}) \to (x, \mathbb{d}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{d}) \to (x, \square) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \square)(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(y, \mathbb{g}) \to (x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})(x, \mathbb{g}) & \forall x, y \in \Sigma \\ (x, \mathbb{g})$$

Here the new symbols R and r are used to represent the beginning and end of the rightward (clockwise) rotation respectively. The symbols d and v represent the new pulse signals being added to the system, one slow and one fast, as explained above. In particular, the symbol d serves as a pulsed-hold that advances one position counterclockwise whenever the fast pulse p passes through it. The final two symbols g and z are used to implement the desired interaction between p and d. In particular, g represents an d symbol that has absorbed p but not yet shifted from its previous position, while z represents an d symbol that has absorbed p and has already shifted one step counterclockwise. Overall, the rule set R_{right} implements a scheme where the pulsed-hold symbol d advances one step counterclockwise in each cycle (every n passes or n(n-1) iterations), eventually appearing to the right of the original control position after n-1 such cycles (equivalently n(n-1) passes or $n(n-1)^2$ iterations). At that point, the victory pulse v is sent to freeze the control position at the desired location.

Lemma 6 For the Lag system defined by R_{right} and any string

$$(s_1, \sqcup)...(s_{n-2}, \sqcup)(s_{n-1}, \mathtt{R})(s_n, \sqcup)$$

 $n \geq 3$, after $n(n-1)^2 + n + 1$ iterations Algorithm 2 will render the memory string

$$(s_2, \mathrel{\reflectbox{\rotatebox{$\scriptstyle -$}}})...(s_{n-1}, \mathrel{\reflectbox{\rotatebox{$\scriptstyle -$}}})(s_n, \mathsf{r})(s_1, \mathrel{\reflectbox{\rotatebox{$\scriptstyle -$}}})$$

changing the control symbol from R to r and rotating the control position one step clockwise.

Proof: We break the proof into a series of claims that will be proved separately. It will be convenient to assume $n \geq 5$ in the following.⁴

Claim 1 Starting from the initial string stated in the lemma, after n iterations Algorithm 2 will produce the memory string $(s_1, _)...(s_{n-3}, _)(s_{n-2}, \mathsf{d})(s_{n-1}, \mathsf{t})(s_n, _)$.

Claim 2 After an additional cycle (another n(n-1) iterations) the memory string will consist of $(s_1, _)...(s_{n-4}, _)(s_{n-3}, \mathtt{d})(s_{n-2}, _)(s_{n-1}, \mathtt{t})(s_n, _)$, hence the symbol \mathtt{t} continues to be associated at s_{n-1} but the pulsed-hold token \mathtt{d} has been shifted one position to the left.

Claim 3 After each of the next n-4 cycles, the memory string will continue to hold t at position s_{n-1} , but the pulsed-hold symbol d will be shifted one further position to the left, with all other positions left blank.

Claim 4 Therefore, at the end of Claims 1, 2 and 3 the memory string will consist of $(s_1, \mathbf{d})(s_2, \mathbf{u})...(s_{n-2}, \mathbf{u})(s_{n-1}, \mathbf{t})(s_n, \mathbf{u})$, with \mathbf{t} at position s_{n-1} and the pulsed-hold token \mathbf{d} at position s_1 .

Claim 5 After one more cycle (another n(n-1) iterations) the memory string will consist of $(s_1, _)...(s_{n-2}, _)(s_{n-1}, w)(s_n, d)$, so s_{n-1} can now see that s_n should be the next control location.

Claim 6 A final cycle plus an additional iteration (another n(n-1)+1 iterations) will result in the final memory string $(s_2, _)...(s_{n-1}, _)(s_n, \mathbf{r})(s_1, _)$, satisfying the stated outcome. The lemma then follows since the total number of iterations taken is $n + n(n-1) + n(n-1)(n-3) + n(n-1) + n(n-1) + 1 = n(n-1)^2 + n + 1$.

Proof of Claim 1: Starting from the initial string, the first n-3 iterations of Algorithm 2 will simply rotate the first pair unaltered, according to the rule $__ \to _$. The next context will be $(s_{n-2}, _)(s_{n-1}, R)$, which causes (s_{n-2}, d) to be rotated by the rule $_R \to d$. The subsequent context will be $(s_{n-1}, R)(s_n, _)$, which cases (s_{n-1}, t) to be rotated by the rule $R _ \to t$. One final iteration rotates the first pair $(s_n, _)$ unaltered, yielding the result stated in Claim 1. Note that the symbol R can never be generated again.

Note The remaining claims all characterize how the major cycles work in the system. Recall from the proof of Proposition 4 that p orbits the memory once every cycle (n(n-1)) iterations). Such a periodic orbit is sustained in this new rule set. In particular, during each of the cycles asserted by Claims 2–5, the symbol p will visit each memory location s_i once, in a counterclockwise order, residing for a single pass (n-1) iterations on each location, where for the first pass p will appear implicitly within a t symbol (i.e., absorbed into a w, as in Proposition 4), then for zero or more passes p will appear as itself, then after it encounters d, p will appear implicitly in a g symbol (i.e., absorbed into d) for a pass, after which p will appear implicitly as a z (i.e., still absorbed into d but shifted counterclockwise one position) for another pass, before finally appearing as itself again for any remaining passes in the cycle.

Proof of Claim 2: First pass within the cycle: p appears implicitly as t at location s_{n-1} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-2}, \mathbf{d})(s_{n-1}, \mathbf{t})$, so the pair (s_{n-2}, \mathbf{g}) is rotated by rule $\mathbf{dt} \to \mathbf{g}$. The next context is $(s_{n-1}, \mathbf{t})(s_n, \ldots)$, so

⁴The same proof also holds for n=4 and n=3 with minor modifications. For n=4, the resulting string in Claim 2 is $(s_{n-3}, \mathbf{d})(s_{n-2}, \mathbf{u})(s_{n-1}, \mathbf{t})(s_n, \mathbf{u})$, with the claims otherwise the same. For n=3, the resulting string in Claim 1 is $(s_{n-2}, \mathbf{d})(s_{n-1}, \mathbf{t})(s_n, \mathbf{u})$, after which Claims 2–4 are skipped, and Claims 5 and 6 are the same.

the pair (s_{n-1}, \mathbf{w}) is rotated by rule $\mathbf{t}_{-} \to \mathbf{w}$. The first pass concludes with the string $(s_n, _)(s_1, _)...(s_{n-2}, \mathbf{g})(s_{n-1}, \mathbf{w})$.

Second pass: p appears implicitly as g at location s_{n-2} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-3}, _)(s_{n-2}, g)$, so the pair (s_{n-3}, z) is rotated by rule $_g \to z$. The next context is $(s_{n-2}, g)(s_{n-1}, w)$, so the pair $(s_{n-2}, _)$ is rotated by rule $g w \to _$. The second pass concludes with the string $(s_{n-1}, w)(s_n, _)(s_1, _)...(s_{n-3}, z)(s_{n-2}, _)$.

Third pass: p appears implicitly as z at location s_{n-3} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-4}, _)(s_{n-3}, z)$, so the pair (s_{n-4}, p) is rotated by rule $_z \to p$. The next context is $(s_{n-3}, z)(s_{n-2}, _)$, so the pair (s_{n-3}, d) is rotated by rule $z _ \to d$. The third pass yields the string $(s_{n-2}, _)(s_{n-1}, w)(s_n, _)(s_1, _)...(s_{n-4}, p)(s_{n-3}, d)$.

Next n-4 passes: p appears as itself, followed by d or a blank. Therefore, p moves one position counterclockwise after each pass. The token w at s_{n-1} and d at s_{n-3} are also followed by blanks and therefore stay at the same location by the rules $\mathbf{w}_{-} \to \mathbf{w}$ and $\mathbf{d}_{-} \to \mathbf{d}$ respectively. At the end of the next n-4 passes the memory string will be $(s_2, _)...(s_{n-3}, \mathbf{d})(s_{n-2}, _)(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})(s_1, _)$.

Last (nth) pass: p appears as itself at location s_n . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})$, so the pair (s_{n-1}, \mathbf{t}) is rotated by rule $\mathbf{w} \mathbf{p} \to \mathbf{t}$. The next context is $(s_n, \mathbf{p})(s_1, \mathbf{w})$, so the pair (s_n, \mathbf{w}) is rotated. The last pass concludes with the string $(s_1, \mathbf{w})...(s_{n-4}, \mathbf{w})(s_{n-3}, \mathbf{d})(s_{n-2}, \mathbf{w})(s_{n-1}, \mathbf{t})(s_n, \mathbf{w})$, establishing Claim 2.

Proof of Claim 3: Consider the cycles in order i=2,...,n-3. The *i*th cycle will render the following behaviour. At the beginning of the *i*th cycle the memory string will be $(s_1, _)...(s_{n-i-1}, \mathbf{d})(s_{n-i}, _)...(s_{n-1}, \mathbf{t})(s_n, _)$.

First pass: p appears implicitly as t at location s_{n-1} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-2}, _)(s_{n-1}, t)$, so the pair (s_{n-2}, p) is rotated, then the next context is $(s_{n-1}, t)(s_n, _)$, so the pair (s_{n-1}, w) is rotated. The first pass concludes with the string $(s_n, _)(s_1, _)...(s_{n-i-1}, d)(s_{n-i}, _)...(s_{n-2}, p)(s_{n-1}, w)$.

Next i-2 passes: p appears as itself followed by w or a blank, so after each pass p moves one position counterclockwise. The token w at s_{n-1} and d at s_{n-i-1} are also followed by blanks and thus stay at the same location. Therefore, at the end of the next i-2 passes the memory string will be $(s_{n-i+2}, _)...(s_{n-1}, w)(s_n, _)(s_1, _)...(s_{n-i-1}, d)(s_{n-i}, p)(s_{n-i+1}, _)$.

Next (ith) pass: p appears as itself at location s_{n-i} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-i-1}, \mathbf{d})(s_{n-i}, \mathbf{p})$, so the pair (s_{n-i-1}, \mathbf{g}) is rotated by rule $\mathbf{d} \mathbf{p} \to \mathbf{g}$. The next context is $(s_{n-i}, \mathbf{p})(s_n, \mathbf{p})$, so the pair (s_{n-i}, \mathbf{p}) is rotated. The *i*th pass concludes with the string $(s_{n-i+1}, \mathbf{p})...(s_{n-1}, \mathbf{w})(s_n, \mathbf{p})(s_1, \mathbf{p})...(s_{n-i-1}, \mathbf{g})(s_{n-i}, \mathbf{p})$.

Next ((i+1)st) pass: p appears implicitly as g at location s_{n-i-1} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-i-2}, _)(s_{n-i-1}, g)$, so the pair (s_{n-i-2}, z) is rotated. The next context is $(s_{n-i-1}, g)(s_{n-i}, _)$, so the pair $(s_{n-i-1}, _)$ is rotated by rule $g _ \to _$. Therefore, at the end of the (i+1)st pass the memory string will be $(s_{n-i}, _)...(s_{n-1}, w)(s_n, _)(s_1, _)...(s_{n-i-2}, z)(s_{n-i-1}, _)$.

Next ((i+2)nd) pass: p appears implicitly as z at location s_{n-i-2} . The first n-3 iterations rotate the first pair unaltered. The next context is $(s_{n-i-3}, \bot)(s_{n-i-2}, z)$, so (s_{n-i-3}, p) is

rotated. The next context is $(s_{n-i-2}, \mathbf{z})(s_{n-i-1}, \square)$, so (s_{n-i-2}, \mathbf{d}) is rotated. The (i+2)nd pass concludes with the memory string $(s_{n-i-1}, \square)...(s_{n-1}, \mathbf{w})(s_n, \square)(s_1, \square)...(s_{n-i-3}, \mathbf{p})(s_{n-i-2}, \mathbf{d})$.

Next n-i-3 passes: p appears as itself followed by d or a blank, so after each pass p moves one position counterclockwise. The token w at s_{n-1} and d at s_{n-i-2} are also followed by blanks and thus stay at the same location. Therefore, at the end of the next n-i-3 passes the memory string will be $(s_2, _)...(s_{n-i-2}, d)...(s_{n-1}, w)(s_n, p)(s_1, _)$.

Last (nth) pass: The same argument as the last pass of Claim 2 can be used to show the resulting string is $(s_1, _)...(s_{n-i-2}, d)...(s_{n-1}, t)(s_n, _)$.

Therefore, each of the cycles i = 2, ..., n-3 retains t at s_{n-1} while shifting d one position clockwise, establishing Claim 3.

Proof of Claim 4: Immediate from the outcome of Claim 3.

Proof of Claim 5: The cycle starts with the memory string $(s_1, d)(s_2, \bot)...(s_{n-1}, t)(s_n, \bot)$ obtained at the end of Claim 4.

First n-1 passes: Following the same proof as Claim 3 with i=n-2 for the first i+1=n-1 passes, the resulting memory string will be $(s_2, _)...(s_{n-1}, \mathbf{w})(s_n, \mathbf{z})(s_1, _)$.

Last (nth) pass: The first n-2 iterations rotate the first pair unaltered. The next context is $(s_n, \mathbf{z})(s_1, \mathbf{z})$, so the pair (s_n, \mathbf{d}) is rotated, yielding the final string $(s_1, \mathbf{z})...(s_{n-1}, \mathbf{w})(s_n, \mathbf{d})$, establishing Claim 5.

Proof of Claim 6: First pass: The first n-2 iterations rotate the first pair unaltered. The next context is $(s_{n-1}, \mathbf{w})(s_n, \mathbf{d})$, so the pair (s_{n-1}, \mathbf{v}) is rotated by $\mathbf{w} \mathbf{d} \to \mathbf{v}$, resulting in the memory string $(s_n, \mathbf{d})(s_1, \bot)...(s_{n-1}, \mathbf{v})$.

Next n-2 passes: Observe that the symbol v will behave identically to p, in that for any pass where it is followed by a blank (or d), v shifts one position counterclockwise according to the rules $v \to v$, $v \to v$ and $v \to v$. Moreover, v stays stationary whenever it is followed by a blank. Therefore, for the next v passes, v will remain stationary while v shifts one position counterclockwise, resulting in the memory string v of v and v derivatively.

Last (nth) pass: The first n-2 iterations rotate the first pair unaltered. The next context is $(s_n, \mathbf{d})(s_1, \mathbf{v})$, so the pair (s_n, \mathbf{r}) is rotated by the rule $\mathbf{d} \mathbf{v} \to \mathbf{r}$, resulting in the string $(s_1, \mathbf{v})(s_2, \mathbf{u})...(s_n, \mathbf{r})$.

One final iteration renders the string asserted in Claim 6, thereby establishing the claim and the lemma.

Overall, we have demonstrated in this section that a Lag system can exert bidirectional position control using only Algorithm 2, where a rule set R_{left} allows a memory access location to be moved one position counterclockwise in O(n) iterations (Proposition 3), and a rule set R_{right} allows memory access location to be moved one position clockwise in $O(n^3)$ iterations (Lemma 6). From these two capabilities, it becomes clear that a Turing machine can be simulated by a Lag system.

5 Turing machines

Formally, a Turing machine T consists of a tuple $T = (Q, \Gamma, b, q_0, H, f)$, where Q is a finite set of states, Γ is a finite set of tape symbols, $b \in \Gamma$ is the unique "blank" symbol, $q_0 \in Q$ is

the unique start state, $H \subseteq Q \times \Gamma$ is the set of halting (state, symbol) pairs, and $f: Q \times \Gamma \to \Gamma \times Q \times \{-1, +1\}$ is a finite set of transition rules that specify the operation of the machine in each compute cycle. The machine has access to a memory tape that is uni-directionally unbounded, so memory locations can be indexed by a natural number $i \in \mathbb{N}$, i > 0, such that there is a leftmost memory location at i = 1 but no rightmost memory location.

The execution of a Turing machine is defined as follows. The tape is initialized with an input that is expressed by a finite number of non-blank symbols, with all other locations blank, T starts in state q_0 , and the tape head starts at a specified location i_0 (default $i_0 = 1$). At the start of each compute cycle, T is in some state $q \in Q$, the tape head is at some location i > 0, and a symbol $\gamma \in \Gamma$ is currently being read from the tape. The combination (q, γ) determines the update $f(q, \gamma) \mapsto (\gamma', q', D)$, specifying that the symbol γ' is written at the current memory location i, the machine state q is updated to q', and the tape head is moved to i + D (i.e., one step left or right depending on the sign of D). It is assumed the machine never moves off the left end of the tape. The compute cycle repeats until the machine encounters a configuration $(q, \gamma) \in H$. Non-halting computations are possible.

To facilitate subsequent proofs, it will be helpful to understand how the computation of a Turing machine can be simulated using only finite memory. A standard simulation strategy is depicted by Algorithm 3, where a new delimiter symbol $\# \notin \Gamma$ is used to mark off the end of the visited memory, which allows additional space to be allocated when necessary. This enables a potentially unbounded memory to be simulated without ever having to allocate infinite storage.⁵

```
Algorithm 3 Finite memory simulation of a Turing machine T = (Q, \Gamma, b, q_0, H, f)
```

```
Input: \gamma_1...\gamma_{n-1}; i_0
Require: 1 < n, 0 < i_0 < n
  m \leftarrow \gamma_1...\gamma_{n-1} \#
                                                                    ▷ Copy input and append delimiter #
  n \leftarrow \text{length}(m)
                                                                                             ▷ Track array size
  i \leftarrow i_0
                                                             \triangleright Initial tape head location (default i_0 = 1)
                                                                                                  \triangleright Initial state
  q \leftarrow q_0
  for k=0,1,... do
       if (q, m_i) \in H then
                                                                            ▶ Halt if encounter halting pair
       halt
       (\gamma', q', D) \leftarrow f(q, m_i)
                                                             ▷ Get update from matching transition rule
       m_i \leftarrow \gamma'
                                                                     ▷ Overwrite current memory location
       q \leftarrow q'
                                                                                                 ▶ Update state
       i \leftarrow i + D
                                                                                             if m_i = \# (hence i = n) then
           m \leftarrow m_1...m_{n-1}b\#
                                                              ▷ Insert new blank symbol before delimiter
           n \leftarrow n + 1
                                                                                           ▶ Update array size
```

⁵Note that Algorithm 3 requires the input to consist of at least one symbol γ_1 . To handle an empty input, one can simply set $\gamma_1 = b$ (blank). Also, Algorithm 3 does not need to explicitly track the array length n, but it will be helpful to refer to in some proofs.

6 Simulating a Turing machine with a Lag system

Our first main result is to show that any Turing machine can be simulated by a restricted (2, 2)-Lag system. The proof will also imply that any linear bounded automaton [Myhill, 1960] can be simulated by a restricted (2, 1)-Lag system. Lag systems were already shown to be computationally universal in [Wang, 1963], however the original proof relies on a reduction from a lesser known form of register machine [Shepherdson and Sturgis, 1963], which proves inconvenient for our purposes. Instead, we develop a direct reduction of Turing machines to Lag systems that allows us to exploit the existence of small universal Turing machines in the subsequent argument.

Given a Turing machine $T = (Q, \Gamma, b, q_0, H, f)$, we construct a corresponding Lag system as follows. The Lag system will use an alphabet $\Sigma = (\Gamma \cup \{\#\}) \times (Q \cup \{\bot\}) \times (\Sigma_{\text{left}} \cup \Sigma_{\text{right}})$, where $\# \notin \Gamma$ is a delimiter symbol, Q is the finite set of states from T (such that $\bot \notin Q$), and Σ_{left} and Σ_{right} are the position control alphabets from Section 4. That is, each symbol in the Lag system is a triple, consisting of a memory symbol from $\Gamma_\# = \Gamma \cup \{\#\}$, a state symbol from $Q \cup \{\bot\}$, and a position control symbol from one of the alphabets developed in Section 4.

We design rules for the Lag system so that its memory string tracks the state of the local variables in the Turing machine simulation, Algorithm 3. In particular, at the start of each iteration $k \in \mathbb{N}$, Algorithm 3 maintains a set of local variables, m, n, q and i, where $m = m_1...m_{n-1}\#$ is an array representing the current tape contents, n is the current length of m, q is the current state of T's controller, and i is the current location of the tape head. To mirror the values of these local variables, the Lag system will maintain a memory string $s = (m_1, \ldots, \ldots)...(m_i, q, \ldots)...(m_{n-1}, \ldots, \ldots)(\#, \ldots, \ldots)$, such that the sequence $m_1...m_{n-1}\#$ corresponds to m, the length of s is n, q corresponds to the same controller state, and the location of the Turing machine's tape head, i, is represented by the location of the only non-blank state symbol q in the second position of a triple. Specifically, for a given Turing machine T, we define the corresponding Lag system L by the rule set determined by Algorithm 4.6

The first main result in this paper is to establish that Algorithm 2, operating on the Lag system L produced by Algorithm 4, simulates the execution of Algorithm 3 for a given Turing machine T on any input $\gamma_1...\gamma_{n-1}$.

Theorem 7 Given a Turing machine $T = (Q, \Gamma, b, q_0, H, f)$, an input string $\gamma_1...\gamma_{n-1}$, and a start location i, such that $n \geq 3$ and 0 < i < n, let the initial string for the Lag system be $(\gamma_1, \ldots, \ldots)...(\gamma_{i-1}, \ldots, \ldots)(\gamma_i, q_0, \ldots)(\gamma_{i+1}, \ldots, \ldots)...(\gamma_{n-1}, \ldots, \ldots)(\#, \ldots, \ldots)$. Then Algorithm 2 with the rule set L determined by Algorithm 4 simulates the execution of Algorithm 3 on T with input

⁶Additional notes about Algorithm 4: The rules in L_5 and L_6 can ignore the case when both $\gamma_1 = \#$ and $\gamma_2 = \#$, since the delimiter symbol can only appear once within a string. The rules in L_6 are the only ones that emit a pair of triples rather than a single triple. If L_6 is omitted, the remaining rule set can never increase the size of the initial string. The conditions defining the rule set L_5 omit the rule ----, but this is explicitly included as L_1 .

⁷Note that this theorem requires the input to consist of at least two symbols $\gamma_1\gamma_2$. To handle shorter input strings, one can simply pad the input with blank memory symbols (b) as necessary.

 $\gamma_1...\gamma_{n-1}$, in the sense that, for every iteration $k \in \mathbb{N}$ of Algorithm 3—which results in local variables $m = m_1...m_{n-1}\#$, n, q and i satisfying 0 < i < n—there is an iteration $t(k) \in \mathbb{N}$ of Algorithm 2 (t(k) monotonically increasing in k) such that the memory of the Lag system is $s = (m_1, \underline{\ }, \underline{\ })...(m_i, q, \underline{\ })...(m_{n-1}, \underline{\ }, \underline{\ })(\#, \underline{\ }, \underline{\ })$, thus maintaining the correspondence with the local variables of Algorithm 3.

Proof: The proof is by induction on the iteration $k \in \mathbb{N}$ of Algorithm 3. The base case, when k = 0, holds by the construction of the initialization.

Assume that at the start of iteration k the correspondence holds between the local variables of Algorithm 3 and the memory string of Algorithm 2. That is, if the local variables in Algorithm 3 are $m^{(k)} = m_1...m_{n^{(k)}}\#$, $n^{(k)}$, $q^{(k)}$ and $i^{(k)}$, with $0 < i^{(k)} < n^{(k)}$, then the memory string of Algorithm 2 will be

$$s^{t(k)} = (m_1, \square, \square) \dots (m_{i^{(k)}}, q^{(k)}, \square) \dots (m_{n^{(k)}}, \square, \square) (\#, \square, \square).$$

$$(9)$$

We show that this correspondence will continue to hold at the start of the next iteration k+1 of Algorithm 3 after some finite number of additional iterations of Algorithm 2.

Given the current state of the local variables, the update $(\gamma', q', D) \leftarrow f(q^{(k)}, m_{i^{(k)}})$ of Algorithm 3 is uniquely determined. We consider three cases depending on whether in the update the tape head of the Turing machine T moves left (i.e., D = -1 and $1 < i^{(k)} < n^{(k)} - 1$), the tape head moves right but not onto the delimiter (i.e., D = +1 and $0 < i^{(k)} < n^{(k)} - 1$), or the tape head moves right and onto the delimiter (i.e., D = +1 and $i^{(k)} = n^{(k)} - 1$).

Case 1: the tape head moves left: In this case we assume D=-1 and $1 < i^{(k)} < n^{(k)}-1$. (Recall that the Turing machine is not allowed to move off the left end of the tape, so $i^{(k)} > 1$.) The new values for the local variables in Algorithm 3 after the (k+1)st iteration will become $m^{(k+1)} = m_1...m_{i-1}\gamma' m_{i+1}...m_{n^{(k)}}\#$, $n^{(k+1)} = n^{(k)}$, $i^{(k+1)} = i^{(k)} - 1$ and $q^{(k+1)} = q'$.

By the induction hypothesis, the memory string for the Lag system will start out as (9). Consider how Algorithm 2 will update this memory string. To de-clutter the notation, drop the superscript (k).

The initial i-2 triples will be rotated unmodified, by the rule L_1 . The next context will be $(m_{i-1}, \underline{\ }, \underline{\ })(m_i, q, \underline{\ })$, so by L_2 the triple $(m_{i-1}, \underline{\ }, \underline{\ })$ will be rotated unmodified. The next context will be $(m_i, q, \underline{\ })(m_{i+1}, \underline{\ }, \underline{\ })$, so by the first rule in L_3 the first triple will be rotated as (γ', q', L) . Then by L_1 each of the next n-2 triples will be rotated unmodified, yielding the string

$$(m_{i-1}, \bot, \bot)(\gamma', q', L)(m_{i+1}, \bot, \bot)...(m_{n-1}, \bot, \bot)(\#, \bot, \bot)(m_1, \bot, \bot)...(m_{i-2}, \bot, \bot).$$
(10)

Since the next context is $(m_{i-1}, \underline{\ }, \underline{\ })(\gamma', q', \underline{\ })$, the triple $(m_{i-1}, q', \underline{\ })$ will be rotated to the end according to the second rule in L_3 . Finally, given the next context $(\gamma', q', \underline{\ })(m_{i+1}, \underline{\ }, \underline{\ })$, the triple $(\gamma', \underline{\ }, \underline{\ })$ will be rotated by the third rule in L_3 , which will yield the memory string

$$(m_{i+1}, \underline{\ }, \underline{\ }, \underline{\ })...(m_{n-1}, \underline{\ }, \underline{\ })(\#, \underline{\ }, \underline{\ })(m_1, \underline{\ }, \underline{\ })...(m_{i-1}, q', \underline{\ })(\gamma', \underline{\ }, \underline{\ }).$$

$$(11)$$

Algorithm 4 Reducing a Turing machine to a Lag system

Input: Turing machine $T = (Q, \Gamma, b, q_0, H, f)$

Output: Restricted (2,2)-Lag system $L: \Sigma^2 \to \Sigma \cup \Sigma^2$

$$L_1 \leftarrow \left\{ (\gamma_1, \square, \square)(\gamma_2, \square, \square) \rightarrow (\gamma_1, \square, \square) \ \forall \gamma_1, \gamma_2 \in \Gamma_\# \right\}$$
 $\triangleright (L_1)$

$$L_2 \leftarrow \left\{ (\gamma_1, \bot, \bot)(\gamma_2, q, \bot) \rightarrow (\gamma_1, \bot, \bot) \ \forall \gamma_1, \gamma_2 \in \Gamma_\#, \ \forall q \in Q \right\}$$
 $\triangleright (L_2)$

 $L \leftarrow L_1 \cup L_2$

for each transition rule $(\gamma', q', D) \leftarrow f(q, \gamma)$ in T do

if
$$D = -1$$
 then

$$L \leftarrow L \cup \text{LeftRules}(q, \gamma, q', \gamma')$$

else (D=1)

 $L \leftarrow L \cup \text{RIGHTRULES}(q, \gamma, q', \gamma')$

return L

function LeftRules (q, γ, q', γ')

$$L_{3} \leftarrow \left\{ \begin{array}{ccc} (\gamma, q, \square)(\gamma_{1}, \square, \square) & \rightarrow & (\gamma', q', L) & \forall \gamma_{1} \in \Gamma \\ (\gamma_{1}, \square, \square)(\gamma', q', L) & \rightarrow & (\gamma_{1}, q', \square) & \forall \gamma_{1} \in \Gamma_{\#} \\ (\gamma', q', L)(\gamma_{1}, \square, \square) & \rightarrow & (\gamma', \square, \square) & \forall \gamma_{1} \in \Gamma_{\#} \end{array} \right\}$$

$$\mathbf{return} \ L_{3}$$

function RIGHTRULES (q, γ, q', γ')

$$L_4 \leftarrow \left\{ (\gamma, q, \square)(\gamma_1, \square, \square) \rightarrow (\gamma', q', \mathbb{R}) \ \forall \gamma_1 \in \Gamma_\# \right\}$$
 $\triangleright (L_4)$

 $\mathbf{for}\ \mathtt{ab} \to \mathtt{c} \in \mathit{R}_{right}\ \mathbf{do}$

$$L_{\mathrm{abc}} \leftarrow \left\{ \begin{array}{ll} (\gamma_{1},q',\mathtt{a})(\gamma_{2},q',\mathtt{b}) & \rightarrow & (\gamma_{1},q',\mathtt{c}) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} \neq \, _, \ \mathtt{b} \neq \, _, \ \mathtt{c} \neq \, _ \\ (\gamma_{1},_,_)(\gamma_{2},q',\mathtt{b}) & \rightarrow & (\gamma_{1},q',\mathtt{c}) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} = \, _, \ \mathtt{b} \neq \, _, \ \mathtt{c} \neq \, _ \\ (\gamma_{1},q',\mathtt{a})(\gamma_{2},_,_) & \rightarrow & (\gamma_{1},q',\mathtt{c}) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} \neq \, _, \ \mathtt{b} = \, _, \ \mathtt{c} \neq \, _ \\ (\gamma_{1},q',\mathtt{a})(\gamma_{2},q',\mathtt{b}) & \rightarrow & (\gamma_{1},_,_) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} \neq \, _, \ \mathtt{b} \neq \, _, \ \mathtt{c} = \, _ \\ (\gamma_{1},_,_)(\gamma_{2},q',\mathtt{b}) & \rightarrow & (\gamma_{1},_,_) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} = \, _, \ \mathtt{b} \neq \, _, \ \mathtt{c} = \, _ \\ (\gamma_{1},q',\mathtt{a})(\gamma_{2},_,_) & \rightarrow & (\gamma_{1},_,_) & \forall \gamma_{1},\gamma_{2} \in \Gamma_{\#} & \mathrm{if} \ \mathtt{a} \neq \, _, \ \mathtt{b} = \, _, \ \mathtt{c} = \, _ \end{array} \right)$$

$$L_6 \leftarrow \left\{ (\gamma_1, q', \mathbf{r})(\gamma_2, \underline{\ }, \underline{\ }) \rightarrow (\gamma_1, q', \underline{\ }) \ \forall \gamma_1, \gamma_2 \in \Gamma_\# \right\}$$
 $\triangleright (L_6)$

$$L_7 \leftarrow \left\{ (\#, q', \square)(\gamma_1, \square, \square) \rightarrow (b, q', \square)(\#, \square, \square) \ \forall \gamma_1 \in \Gamma \right\}$$
 $\triangleright (L_7)$

return
$$L_4 \cup \left(\bigcup_{\mathtt{ab} \to \mathtt{c} \in R_{\mathrm{right}}} L_{\mathtt{abc}}\right) \cup L_6 \cup L_7$$

After another n-i rotations using L_1 , Algorithm 2 will obtain

$$(m_1, \underline{\ }, \underline{\ })...(m_{i-1}, q', \underline{\ })(\gamma', \underline{\ }, \underline{\ })(m_{i+1}, \underline{\ }, \underline{\ })...(m_{n-1}, \underline{\ }, \underline{\ })(\#, \underline{\ }, \underline{\ }).$$
 (12)

Thus, the correspondence with the next state of the local variables in Algorithm 3's simulation has been reestablished, after a total of i - 1 + 1 + n - 2 + 1 + 1 + n - i = 2n iterations of Algorithm 2.

Case 2: the tape head moves right but not onto the delimiter: In this case we assume D=1 and $0 < i^{(k)} < n^{(k)} - 1$. The new values for the local variables in Algorithm 3 will become $m^{(k+1)} = m_1...m_{i-1}\gamma' m_{i+1}...m_{n-1}\#$, $n^{(k+1)} = n^{(k)}$, $i^{(k+1)} = i^{(k)} + 1$ and $q^{(k+1)} = q'$ at the start of the (k+1)st iteration.

By the induction hypothesis, the memory string for the Lag system will start out as (9). Consider how Algorithm 2 updates this memory string. Once again, drop the superscript $^{(k)}$ to de-clutter the notation.

The initial i-1 triples will be rotated unmodified to the end, by the rules L_1 and L_2 . The next context will be $(m_i, q, _)(m_{i+1}, _, _)$, so by L_4 the first triple will be rotated as (γ', q', R) . The next triple will be rotated unmodified, by L_1 , resulting in the string

$$(m_{i+2}, \underline{\ }, \underline{\ }, \underline{\ })...(m_n, \underline{\ }, \underline{\ })(\#, \underline{\ }, \underline{\ })(m_1, \underline{\ }, \underline{\ })...(m_{i-1}, \underline{\ }, \underline{\ }, \underline{\ })(\gamma', q', \mathbb{R})(m_{i+1}, \underline{\ }, \underline{\ }, \underline{\ }).$$
 (13)

At this point, we make three key observations. First: each triple in (13) has a blank second position if and only if the third position is also blank. As long as this property holds, the only rules that can match a context are in $L_1 \cup L_5 \cup L_6$.⁸ Second: q' is the only non-blank symbol that can appear in the second position of any triple for the remainder of this case, since it is the only non-blank symbol in the second position of any triple in (13) and the only non-blank symbol that can be generated in the second position by any rule in $L_1 \cup L_5 \cup L_6 \cup L_7$. Third: the set of rules $L_{\text{right}} = L_1 \cup L_5$ corresponds to the position control rules in R_{right} , and every rule in L_{right} either generates a blank in both the second and third position, or a non-blank in both the second (q') and third position, thus maintaining the first observation.

From these properties we conclude that, until the position symbol \mathbf{r} is generated and appears in a context, the only applicable rules are in L_{right} . Therefore, the third position of any triple generated by Algorithm 2 will behave identically to Lemma 6. This implies that after another $n(n-1)^2 + n + 1$ iterations the memory string will have a position control sequence that matches Lemma 6's conclusion; namely,

$$(m_{i+3}, \square, \square)...(m_n, \square, \square)(\#, \square, \square)(m_1, \square, \square)...(m_{i-1}, \square, \square)(\gamma', \square, \square)(m_{i+1}, q', \mathbf{r})(m_{i+2}, \square, \square).$$
 (14)

The next n-2 iterations will rotate the first triple unaltered. The subsequent context will be $(m_{i+1}, q', \mathbf{r})(m_{i+2}, \ldots, \ldots)$, so the triple (m_{i+1}, q', \ldots) will be rotated on the next iteration, by L_6 . Then for the next n-i-1 iterations the first triple will be rotated unaltered, yielding

$$(m_1, \square, \square)...(m_{i-1}, \square, \square)(\gamma', \square, \square)(m_{i+1}, q', \square)...(m_n, \square, \square)(\#, \square, \square).$$
 (15)

⁸Although the second and third rules in L_3 also satisfy this property, these rules can be ignored in this case, since the position control symbol L cannot be generated from the string (13) before the case concludes.

Thus, the correspondence with the next state of the local variables in Algorithm 3's simulation has been reestablished, after a total of $i-1+1+1+n(n-1)^2+n+1+n-2+1+n-i-1=n(n-1)^2+3n$ iterations.

Case 3: the tape head moves right and onto the delimiter: In this case we assume D=1 and $i^{(k)}=n^{(k)}-1$. The new values for the local variables in Algorithm 3 will therefore become $m^{(k+1)}=m_1...m_{n-2}\gamma'b\#$, $n^{(k+1)}=n^{(k)}+1$, $i^{(k+1)}=i^{(k)}+1$ and $q^{(k+1)}=q'$ at the start of the (k+1)st iteration; that is, Algorithm 3 inserts a new blank symbol b at memory location $n^{(k)}$ and expands the length of its memory array by one.

By the induction hypothesis, the memory string for the Lag system will start out as

$$s^{t(k)} = (m_1, \underline{\ }, \underline{\ })...(m_{n(k)-2}, \underline{\ }, \underline{\ })(m_{n(k)-1}, q, \underline{\ })(\#, \underline{\ }, \underline{\ }).$$

$$(16)$$

Consider how Algorithm 2 will update this string. Once again, drop the superscript $^{(k)}$ to de-clutter the notation.

Initially this case proceeds isomorphically to the previous case, until the position control symbol \mathbf{r} appears and is subsequently replaced, meaning that after $n(n-1)^2 + 3n$ iterations Algorithm 2 will produce the memory string

$$(m_1, \bot, \bot)...(m_{n-2}, \bot, \bot)(\gamma', \bot, \bot)(\#, q', \bot).$$
 (17)

Now, in this situation, the state symbol q' has become associated with the delimiter #, so we continue iterating. From this string, another n-1 iterations will rotate the first triple unmodified to the end, resulting in the context $(\#, q', _)(m_1, _, _)$. At this point, L_7 is applied to expand the size of Algorithm 2's memory string by one, since after deleting the first triple $(\#, q', _)$, the pair of triples $(b, q', _)(\#, _, _)$ is appended to the end of the string, resulting in

$$(m_1, \bot, \bot)...(m_{n-2}, \bot, \bot)(\gamma', \bot, \bot)(b, q', \bot)(\#, \bot, \bot).$$
 (18)

Thus, the correspondence with the next state of the local variables in Algorithm 3's simulation has been reestablished, after a total of $n(n-1)^2 + 3n + n - 1 + 1 = n(n-1)^2 + 4n$ iterations.

Since the induction step holds in each case, the theorem has been proved.

There are a few interesting corollaries that follow from this result. First, Theorem 7 shows that any Turing machine can be simulated by a restricted (2, 2)-Lag system, since all of the rules in Algorithm 4 use a context of size 2 and emit either one or two output symbols (here each symbol is a triple).

Corollary 8 Any linear bounded automaton can be simulated, in the same sense as Theorem 7, by a restricted (2,1)-Lag system.

Proof: A linear bounded automaton is simply a Turing machine T that never moves off the left or right of its input [Myhill, 1960, Sipser, 2013]. For such a T, given an input string $\gamma_1...\gamma_{n-1}$, Algorithm 3 always maintains the index variable i between 0 < i < n. From the proof of Theorem 7, one can see that the corresponding Lag system L produced by

Algorithm 4 will maintain an exact simulation of T without ever applying L_7 , which is the only rule that emits more than one output triple. Hence, L_7 can be dropped from the simulation, and the remaining rule set defines a restricted (2, 1)-Lag system.

Note that a restricted (N, 1)-Lag system can be simulated by generalized autoregressive (greedy) decoding of an (N+1)-gram model, which implies that a 3-gram model can also simulate any linear bounded automaton. However, these models are insufficient for simulating an arbitrary Turing machine.

Corollary 9 Any restricted (N,1)-Lag system is not Turing universal, hence, generalized autoregressive (greedy) decoding of any K-gram model also cannot be Turing universal.

Proof: This follows directly from the fact that a restricted (N, 1)-Lag system can never change the length of its memory string after initialization, since every iteration deletes and appends exactly one symbol, hence its computation can be simulated by a linear bounded automaton. Linear bounded automata cannot simulate every Turing machine, since every language recognizable by a linear bounded automaton is context sensitive [Landweber, 1963], yet there are recursively enumerable languages that are not context sensitive [Chomsky, 1959, Salomaa, 1973].

Corollary 10 Any Turing machine can be simulated by extended autoregressive (greedy) decoding of a K-gram model with $K \geq 4$.

Proof: Follows directly from the proof of Theorem 7, since every rule produced by Algorithm 4 is expressible by a 3-gram (L_1-L_6) or a 4-gram (L_7) , where L_7 is necessary for Turing completeness and requires extended autoregressive decoding to realize. Note that simulating the application of L_7 requires an extended autoregressive decoding strategy (as discussed in Section 2) to generate an output bigram given a bigram context.

7 A universal Lag system

The primary goal of this paper is to demonstrate that current language models are computationally universal under extended autoregressive decoding. For any given language model, the most direct method for establishing such a result is to identify a known computationally universal system that the model can simulate.

Ultimately, any assertion of computational universality relies on the Church-Turing thesis, that all computational mechanisms are expressible by a Turing machine [Sipser, 2013, Moore and Mertens, 2011]. The concept of a universal Turing machine—a Turing machine U that can simulate the execution of any other Turing machine U on any input—was developed by Alan Turing to solve the Entscheidungsproblem [Turing, 1937]. Proving computational universality of a system therefore reduces to establishing that the system can simulate the operation of a universal Turing machine.

Given the similarity between autoregressive decoding of a language model and the updates of a Lag system, as established in Proposition 1, it is natural to seek a universal Lag system to provide the foundation for a proof of universality. The results of the previous section (Theorem 7) now make the construction of a universal Lag system straightforward.

Let L(T) denote the Lag system that is produced by applying Algorithm 4 to a Turing machine T.

Theorem 11 For any universal Turing machine U, the Lag system L(U) obtained by applying Algorithm 4 to U is also universal, in the sense that Algorithm 2 operating on L(U), like U, is able to simulate the execution of any Turing machine T on any input $\gamma = \gamma_1...\gamma_{n-1}$; i₀.

Proof: The universality of U implies that for any Turing machine T there exist two computable functions—an encoding function c_T and a location function ℓ_T —such that, for any input γ and initial location i_0 , the execution of U on $c_T(\gamma)$ starting from tape location $\ell_T(i_0)$ simulates the execution of T on γ starting from i_0 . By Theorem 7 it then immediately follows that, using L(U), Algorithm 2 is able to simulate the execution of U on $c_T(\gamma)$ starting from location $\ell_T(i_0)$, hence it is also able to simulate the execution of T on γ starting from i_0 .

The concluding argument below will be based on formulating then simulating a universal Lag system L(U) with extended autoregressive decoding of a language model. A key challenge will be to identify a compact universal Lag system, since the entire prompt design for a large language model needs to fit within the model's bounded context window. To that end, we consider a universal Lag system derived from a small universal Turing machine.

There has been a long running effort to identify the smallest universal Turing machines in terms of the number of states and tape symbols used, starting with [Shannon, 1956]. A gap remains between the known upper and lower bounds on the state and symbol counts for a universal Turing machine [Neary, 2008, Neary and Woods, 2009, Neary and Woods, 2012], but progressively smaller universal Turing machines have been identified. In this paper, we will consider one such machine, $U_{15,2}$, that uses only 15 states and 2 tape symbols [Neary and Woods, 2009], which is Pareto optimal in terms of the smallest known universal Turing machines [Neary, 2008]. Formally, the Turing machine $U_{15,2}$ can be defined by a tuple $U_{15,2} = (Q, \Gamma, b, q_0, H, f)$, where $Q = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O\}$, $\Gamma = \{0, 1\}, b = 0, q_0 = A, H = \{(J, 1)\}$, and the transition function f is defined in Table 1. The proof of universality of $U_{15,2}$ also requires the initial position of the tape head, i_0 , to depend on the target system being simulated [Neary and Woods, 2009].

Given its compact size, we therefore adopt the universal Lag system $L(U_{15,2})$ as the target system to be simulated for the remainder of this paper.

Corollary 12 Algorithm 2 operating on $L(U_{15,2})$ is able to simulate the execution of any Turing machine T on any input $\gamma = \gamma_1...\gamma_{n-1}$; i_0 .

Proof: Immediate from the universality of $U_{15,2}$ [Neary and Woods, 2009] and by Theorem 11.

		A														
0	0, -	+, B	1, +	-, C	0, –	$\overline{-,G}$	0, –	\overline{F}	1,+	, A	1, -	,D	0, -	,H	1, -	$\overline{\cdot,I}$
1	1, -	+, B +, A	1, +	-, A	0, -	-, E	1, -	\cdot, E	1, -	, D	1, -	, D	1, -	\cdot, G	1, -	,G
				J												
	0	0, +, A		1, -, K		0, +, L		0, +, M		0, -, B		0, -, C		0, +, N		
	1	1, -	, J	ha	lt	1, +	\cdot, N	$1, \dashv$	\vdash , L	1, -	+, L	$0, \dashv$	-, O	1, +	-, N	

Table 1: Transition table for the universal Turing machine $U_{15,2}$. Rows are indexed by the read symbol γ , columns are indexed by the state q, and each table entry (γ', D, q') specifies the write symbol γ' , the tape head move $D \in \{-1, +1\}$, and the next state q'. The initial state is A.

The universal Lag system $L(U_{15,2})$ consists of 2027 rules defined over 262 "symbols", where each symbol is a triple, as explained in Section 6. Of the 2027 rules, only 16 produce a pair of output symbols for the matching context (i.e., L_7), while the remaining 2011 only output a single symbol for the matching context (i.e., L_1 – L_6).

8 Simulating a universal Lag system with a language model

Our final task is to demonstrate that an existing large language model can simulate the execution of the universal Lag system $L(U_{15,2})$ on any input string. This will be achieved by developing a specific prompt for the language model that drives extended autoregressive (greedy) decoding to mimic the behaviour of $L(U_{15,2})$.

As a preliminary step, note that the 262 symbols used by $L(U_{15,2})$ are not in a convenient format for this purpose, as each symbol is a triple that requires auxiliary characters to describe (such as parentheses and commas). Rather than have the language model generate a bunch of extraneous syntactic detail, we streamline the task by introducing a simple invertible mapping between the 262 triples and 262 token pairs. In particular, we leverage a simple bijection where each triple is assigned a unique token pair, implementable by two dictionaries: an encoding dictionary that maps triples to token pairs, and an inverse decoding dictionary that maps token pairs to triples. The computation then proceeds by encoding the Turing machine simulation as a sequence of token pairs, rather than a sequence of triples. (Note that realizing this simulation requires extended autoregressive decoding, as introduced in Section 2, since the language model must generate 2 or 4 tokens for each given context.) The resulting formulation simulates the execution of the universal Turing machine by incorporating this bijection in the computable function $c_{U_{15,2}}$ used in the proof of Theorem 11.

We develop a prompting strategy that consists of two components: a "system prompt" that provides the full rule set (expressed over the token pair encoding) to the language model, and a "sliding window prompt" where the next symbol pair (4 tokens) from the input sequence is appended. (This prompt structure is reminiscent of the recent streaming

Figure 3: Prompting strategy for simulating a universal Lag system. The $system_prompt$ consists of copies of the entire rule set, which is prepended to the prompt used in each call to the language model. A sliding context window of size 2 is moved through the symbol sequence, emitting 1 or 2 symbols that are appended to the end of the sequence in each iteration. (Note that each symbol is a pair of tokens.) The sliding context window advances 1 position per iteration. For example, in the first row, the prompt given to the language model on the initial iteration is $system_prompt s_1s_2$; for the next iteration, the prompt given to the language model is $system_prompt s_2s_3$, then $system_prompt s_3s_4$ in the subsequent iteration, and so on.

model proposed by [Xiao et al., 2024].) On each iteration, the next symbol pair (4 tokens) from the sequence is appended to the system prompt and given to the language model as input; the output of the language model (2 or 4 tokens) is then appended to the end of the sequence, as shown in Figure 3. To ensure a deterministic system, we use a temperature of zero and fix all the random seeds that define the language model's behaviour, as discussed in Section 2. To allow the language model to emit a variable number of tokens for each context window, we employ extended autoregressive decoding as explained in Section 2 where an implicit latent halt token h outside the base alphabet of 262 token pairs is used.

Finally, we choose a specific large language model to verify whether extended autoregressive (greedy) decoding is indeed able to replicate the behaviour of $L(U_{15,2})$. For this purpose, we use gemini-1.5-pro-001, which is a publicly released large language model that can be accessed at https://ai.google.dev/gemini-api/docs/models/gemini. After some experimentation, we developed a single system prompt that drives this model to correctly execute each of the 2027 rules. We refer to this system prompt as S_{gemini} . Based on these developments, we reach the final claim.

Theorem 13 For any string $\gamma_1...\gamma_{n-1}$, gemini-1.5-pro-001 with extended autoregressive (greedy) decoding, using the system prompt string S_{gemini} and a sliding window of length 2, is able to simulate the execution of the universal Lag system $L(U_{15,2})$ on $\gamma_1...\gamma_{n-1}\#$.

Proof: The proof consists of an enumeration of the 2027 cases, where, for each rule $x_1x_2 \to y$ in $L(U_{15,2})$, the prompt $S_{\text{gemini}}x_1x_2$ is provided to the language model and the model's response is verified to correspond to the rule's output y.

From this theorem, by the Church-Turing thesis, we conclude that <code>gemini-1.5-pro-001</code> under extended autoregressive (greedy) decoding is a general purpose computer. Notably, no additional computational mechanisms beyond extended autoregressive decoding are required to achieve this result.

9 Related work

It has already been intuited in the literature that large language models are able to express arbitrary computation via autoregressive decoding [Jojic et al., 2023]—an informal conclusion reached through a series of prompt designs that elicit increasingly sophisticated computational behaviour. However, a proof has not been previously demonstrated. By establishing such a result in this paper, we observe more precisely that generalized autoregressive decoding (Section 2) is only sufficient for realizing linear bounded automata (Corollary 8), whereas extended autoregressive decoding is required for simulating general Turing machines (Theorem 7 and Corollary 9). Previous work [Schuurmans, 2023] has demonstrated that augmenting a language model with an external random access memory allows universal computational behaviour to be elicited via prompting, but such a result is weakened by the need to introduce external regular expression parsers for managing the interaction between the language model and the memory. Here we do not consider any such augmentations of the model, and instead only consider extended autoregressive decoding introduced in Section 2. That is, even though variety of other extensions to transformer-based language models continue to be investigated [Mialon et al., 2023], these are unnecessary for achieving computational universality (although they undoubtedly have practical benefits).

A significant amount of prior work has investigated the hypothetical expressiveness of different architectures for sequential neural networks, asking whether a given architecture is able, in principle, to express a given model of computation (e.g., Turing machine, formal language class, or circuit family). Such works typically argue for the existence of model parameters that realize targeted behaviour, but do not account for the learnability of such parameters from data. Moreover, such results generally do not apply directly to pre-trained models where the parameters have been frozen.

Regarding Turing machine simulation, an extensive literature has investigated the ability of various sequential neural network architectures to express universal computation, focusing, for example, on recurrent neural networks [Siegelmann and Sontag, 1992, Chen et al., 2018], transformers [Pérez et al., 2019, Bhattamishra et al., 2020], attention mechanisms in particular [Pérez et al., 2021], transformers with recurrence (informally) [Dehghani et al., 2019], and neural Turing machines [Pérez et al., 2019]. A related line of research has investigated novel computational architectures tailored to transformers that retain universal computational properties [Giannou et al., 2023]. A common shortcoming of these studies is that, by focusing on representational capacity, they exploit high precision weights to encode data structures, such as multiple stacks, without considering whether these specific weight configurations are learnable from data. More recently, there has been some attempt to control the precision of the weights and still approximate general computational behaviour with reasonable statistical complexity [Wei et al., 2022a]. Another interesting result is that Turing completeness can still be achieved by a recurrent neural network with bounded precision neurons, provided the hidden state can grow dynamically [Chung and Siegelmann, 2021]. These prior results nevertheless do not apply to existing large language models without altering their weights (as far as anyone currently knows).

The literature on neural Turing machines, although directly inspired by theoretical mod-

els of computation, has remained largely empirical [Graves et al., 2014, Graves et al., 2016, Kaiser and Sutskever, 2016, Kurach et al., 2016]. Some of this work has shown that augmenting recurrent neural networks with differentiable memory in the form of stack or queue does allow rich language structures to be learned from data [Grefenstette et al., 2015]. The computational universality of such models has only been demonstrated theoretically by [Pérez et al., 2019], with the same shortcomings as above.

Another line of work has considered practical restrictions on neural network parameterizations, focusing on the effects of limited precision representation. Again, these works focus on hypothetical expressiveness rather than learnability or applicability to pre-trained models. Useful insights have been obtained regarding the ability of alternative architectures to realize restricted forms of computation with finite precision parameterizations, including the expressiveness of recurrent neural networks (including LSTMs and GRUs) [Weiss et al., 2018, Korsky and Berwick, 2019], attention mechanisms with depth and width bounds [Hahn, 2020], linear transformers [Irie et al., 2023], transformers [Weiss et al., 2021, Feng et al., 2023, Merrill and Sabharwal, 2024, Li et al., 2024], and state space models [Merrill et al., 2024]. None of these results apply to existing, pre-trained language models, nor do they establish Turing completeness, but they do identify clear limitations of myopic next token or bounded chain of thought processing with finite precision representations.

In this paper, we have primarily focused on the ability of language models to simulate general Turing machines or linear bounded automata. There is an extensive literature that investigates the relationship between various models of computation based on automata and alternative grammar classes [Chomsky, 1959, Chomsky and Schützenberger, 1963]. The relationship between linear bounded automata [Myhill, 1960] and context sensitive grammars [Chomsky, 1959] is particularly interesting. Initially it was established by [Landweber, 1963] that non-deterministic linear bounded automata can only recognize context sensitive languages [Chomsky, 1959], with their full equivalence subsequently established by [Kuroda, 1964] Remarkably, the question of whether deterministic versus non-deterministic linear bounded automata exhibit the same language recognition ability (and thus equivalent to context sensitive grammars) has remained open for 60 years [Kutrib et al., 2018], even though it is easy to show a corresponding equivalence for Turing machines [Sipser, 2013]. The results for Lag systems developed in Section 4 were inspired by the finding that any context sensitive grammar can be converted to a purely one-sided or direction-independent context sensitive grammar [Penttonen, 1974, Kleijn et al., 1984]. The earlier Tag systems studied by [Post, 1943] also inspired a long sequence of works that establish Turing completeness of nearby variants [Wang, 1963, Cocke and Minsky, 1964], ultimately leading to the key findings that one dimensional cellular automata are Turing complete [Cook, 2004], and later to the identification of the smallest known universal Turing machines [Neary and Woods, 2009, Neary and Woods, 2012. We have directly exploited the results of the latter work in this paper, but it remains open whether early variants of Tag systems might lead to more compact reductions and simpler proofs of universality.

10 Conclusion

We have shown that a particular large language model, gemini-1.5-pro-001, is Turing complete under extended autoregressive (greedy) decoding. This result is not hypothetical: the proof establishes Turing completeness of the current model in its publicly released state. That is, gemini-1.5-pro-001 is already a general purpose computer, not merely a hypothetical computer. We expect that the same is true of almost all currently released large language models, both closed and open source, but we have not yet conducted the necessary verifications on other alternatives. Performing such a verification likely requires re-engineering of the system prompt and possibly a fair amount of experimentation in each case, but the proof strategy remains the same: verify that each of the 2027 production rules in the universal Lag system $L(U_{15,2})$ are correctly simulated. We do not know whether a single system prompt can be successful for a large set of language models, but that too is a possibility worth considering.

It remains future work to consider alternative reductions from a universal model of computation (universal Turing machine, Lag system, or some other alternative) that leads to a more compact verification. The 2027 rules in the current proof is a significant reduction from our earlier attempts, and it seems like even this rule set can likely be compressed further, leading to easier verifications for other language models.

There are some interesting implications from these findings. Essentially, we have established that a current large language model is already able to simulate any algorithm or computational mechanism, by the Church-Turing thesis. However, the key benefit of a large language model over a classical (formal) computer is that, while a language model sacrifices nothing in terms of ultimate computational power, it offers a far superior interface for human users to express their computational intent. In principle, a human user can bypass full formalization their problem, and can even avoid providing precise step by step instructions for how to solve infinitely many variations of their particular problem instance (i.e., design an algorithm or write a program), yet the user can still hope to elicit useful computational behaviour toward generating a solution. The consequences of providing a more natural interface for producing desired computational behaviour have already become widely apparent.

Also, as is the case with classical computers, long computational sequences are necessary for solving arbitrary tasks. Although it is not yet the most common mode of operation for language models, processing very large inputs and emitting extremely long output sequences is inevitable for tackling serious use cases.

Finally, we conclude with an (admittedly extreme) conjecture: training to perform next token prediction on human generated text does not "teach" a large language model to reason or compute; rather, with probability nearly 1 a randomly initialized language model is already a universal computer, and next token prediction is merely a mechanism for making its input-output behaviour understandable to humans. This hypothesis is falsifiable in principle, but we have not yet been able to prove it either way.

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