Fast Fluid Dynamics for Modelling Flow Through a City

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1 Finite Difference Implementation

To solve the necessary PDEs we implement a staggered grid finite difference scheme. In this scheme, we split the domain into a number of cells. On each cell we will have 7 total degrees of freedom, 2 for each velocity component and 1 for the pressure. The velocity degrees of freedom will be located at the middle of the edge normal to the velocity direction, and the pressure degree of freedom will be located at the centre of the cell. This is illustrated in figure 1.

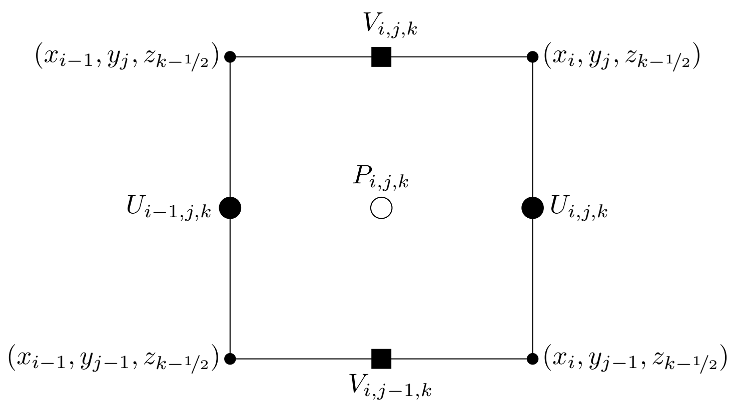


Figure 1: View of edge of cell i, j, k, with k constant. $W_{i,j,k}$ and $W_{i,j,k-1}$ are defined above and below $P_{i,j,k}$ respectively.

1.1 Boundary Conditions

Since $U_{i-1,j,k}$ lies on the boundary already, it can be set directly to $\mathcal{U}(x_{i-1}, y_{j-1/2}, z_k)$. The nodes for V and W do not lie on the boundary however, so we must do an averaging of the nodes adjacent to the boundary to get the correct boundary values:

$$\begin{aligned}
\frac{V_{i-1,j,k} + V_{i,j,k}}{2} &= \mathcal{V}(x_{i-1}, y_j, z_k) \\
\frac{V_{i-1,j-1,k} + V_{i,j-1,k}}{2} &= \mathcal{V}(x_{i-1}, y_{j-1}, z_k) \\
\frac{W_{i-1,j,k} + W_{i,j,k}}{2} &= \mathcal{W}(x_{i-1}, y_j, z_k) \\
\frac{W_{i-1,j,k-1} + W_{i,j,k-1}}{2} &= \mathcal{W}(x_{i-1}, y_j, z_{k-1})
\end{aligned}$$

To enforce homogeneous Neumann boundary conditions on the pressure, we must have that P does not change in the direction normal to boundary, this means:

$$P_{i,j,k} = P_{i-1,j,k}$$

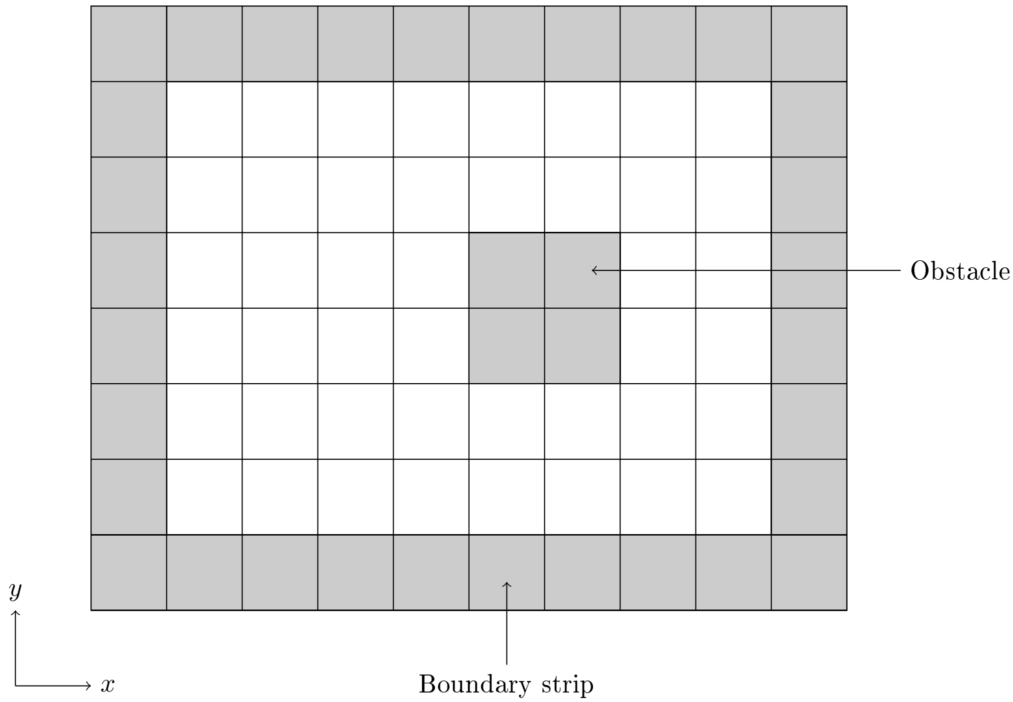


Figure 2: Cross section of a domain discretization. We must have an extra layer of cells around our domain to accommodate the boundary conditions.

1.1.1 Edge Cases

For edge cases, the only case to consider is the velocity which is not normal to either edge. As seen in figure 4 if a cell is normal to both the x and y axis we need to average 4 W nodes to prescribe the correct velocity at the edge. That is:

$$\frac{W_{i-1,j+1,k} + W_{i,j+1,k} + W_{i,j,k} + W_{i-1,j,k}}{4} = \mathcal{W}_{x_{i-1},y_{j+1},z_k}$$

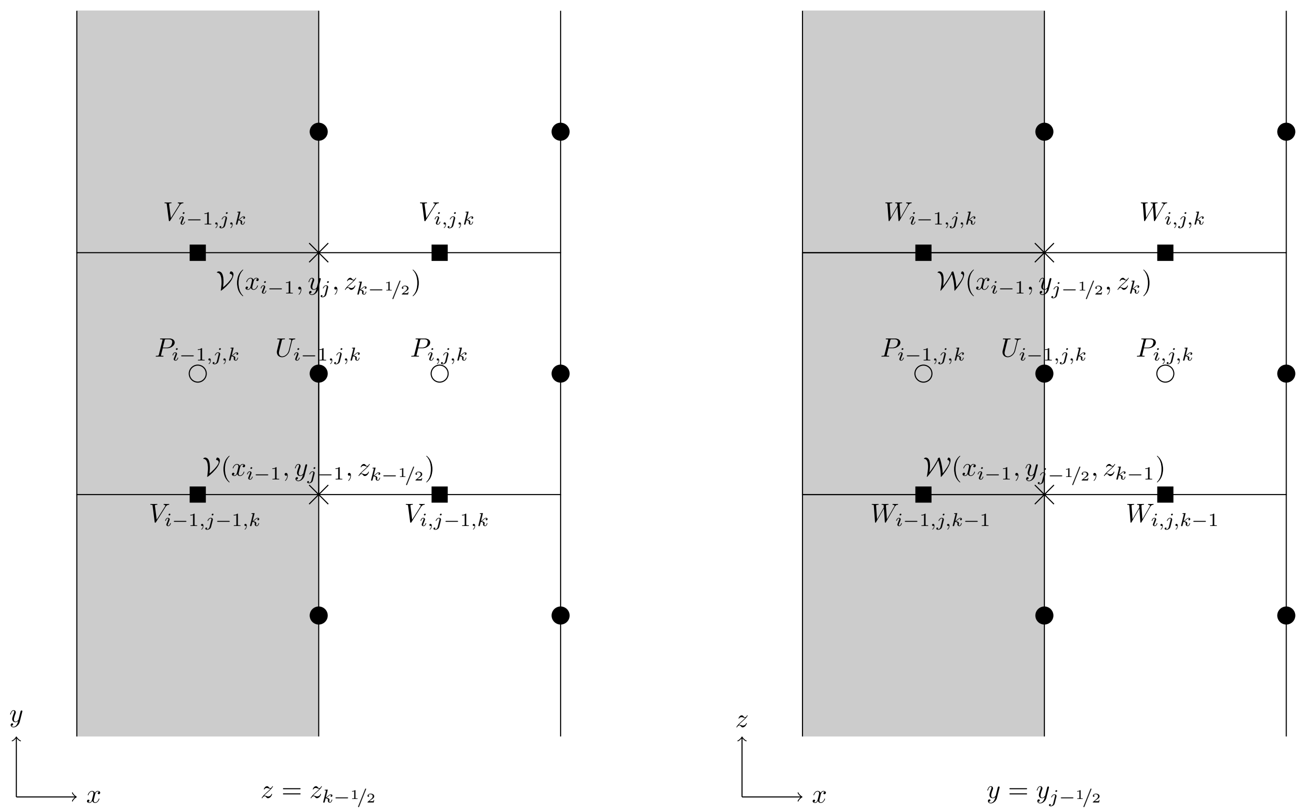


Figure 3: Cross sections of boundary nodes at a boundary normal to $-\hat{\mathbf{i}}$.

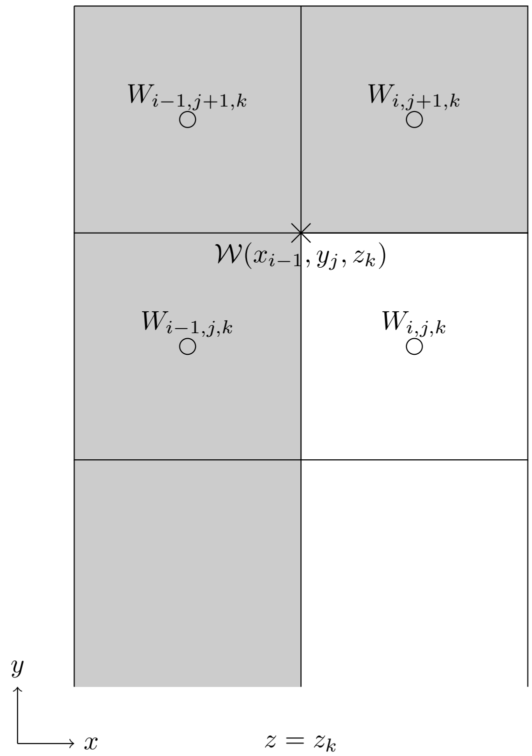


Figure 4: Cross sections of boundary nodes at a boundary normal to both $-\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

2 User Guide

To use the FFD library, the following information must be provided:

- a flag for each cell indicating if it is an obstacle cell, or a ghost cell outside the domain (`obstacle_cells`)
- flags for each cell indicating if it borders an obstacle or ghost cell, and if so the direction which it borders (`boundary_normal_x`, `boundary_normal_y`, `boundary_normal_z`),
- a flag for each cell indicating if it is part of an outflow boundary (`outflow_cells`)
- boundary conditions for u, v and w (`boundary_u`, `boundary_v`, `boundary_w`)

2.1 Cell Information

To create a domain of N cells in each direction actually requires $N + 2$ cells in each direction to account for the ghost cells that enclose the domain. We will call these cells "obstacles" eventhough they do not need to be physical obstacles. Inside the domain, certain cells can be denoted as obstacles. These are physical obstacles and typically come with no-slip boundary conditions. Figures 5 and 6 show cross sections of obstacle cells for a domain with a step inside the domain.

The flags indicating boundary cells can be either -1, 1 or 0. If a fluid cell (i, j, k) does not border an obstacle cell, then all boundary flags are 0. If it borders a boundary cell in the $+x$ direction, (i.e. to its right) then the flag indicating `boundary_normal_x[i,j,k]` should be set to 1. If it borders in the $-x$ direction, then `boundary_normal_x[i,j,k]` should be set to -1. Likewise for the other coordinate directions. Figures 7 - 10 show some cross sections of boundary flag indicators for the centre step problem.

The last flag that needs to be given to each cell is one indicating outflow cells. These are fluid cells bordering an obstacle (typically ghost cells) at the outflow. Figure 11 shows a cross section of outlflow cell indicators.

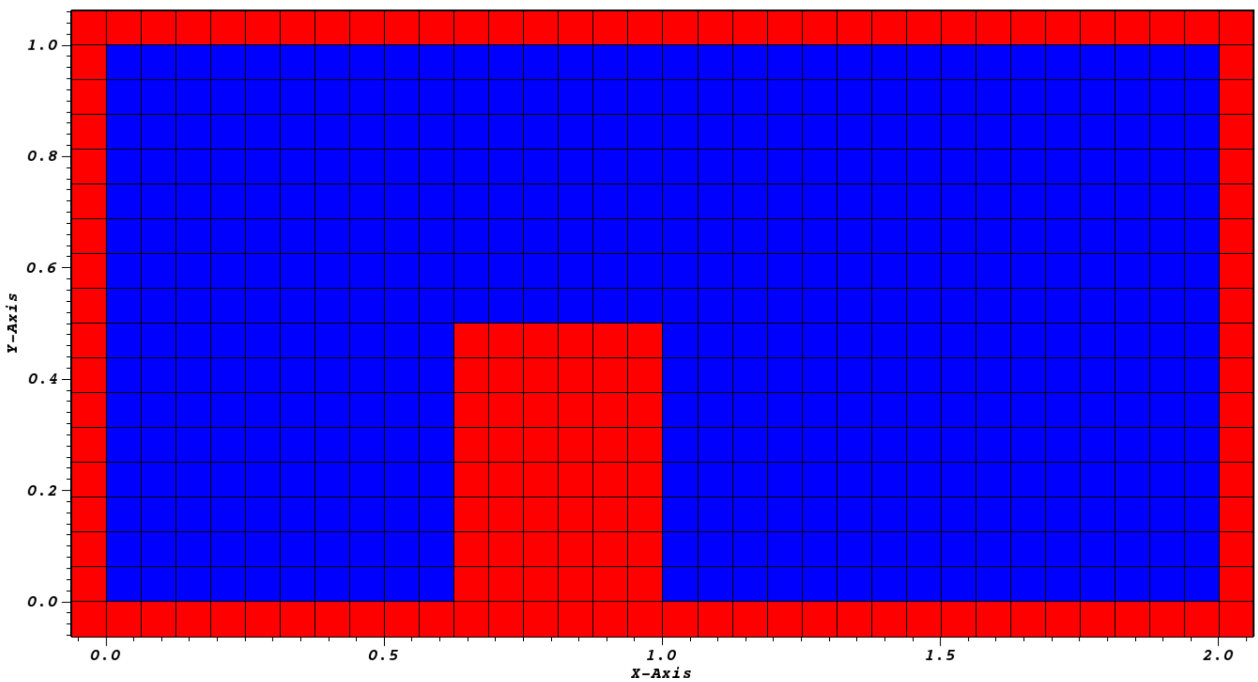


Figure 5: $x - y$ cross section of a centre step domain at $z = 0.5$. Red cells are obstacle cells, blue cells are fluid cells.

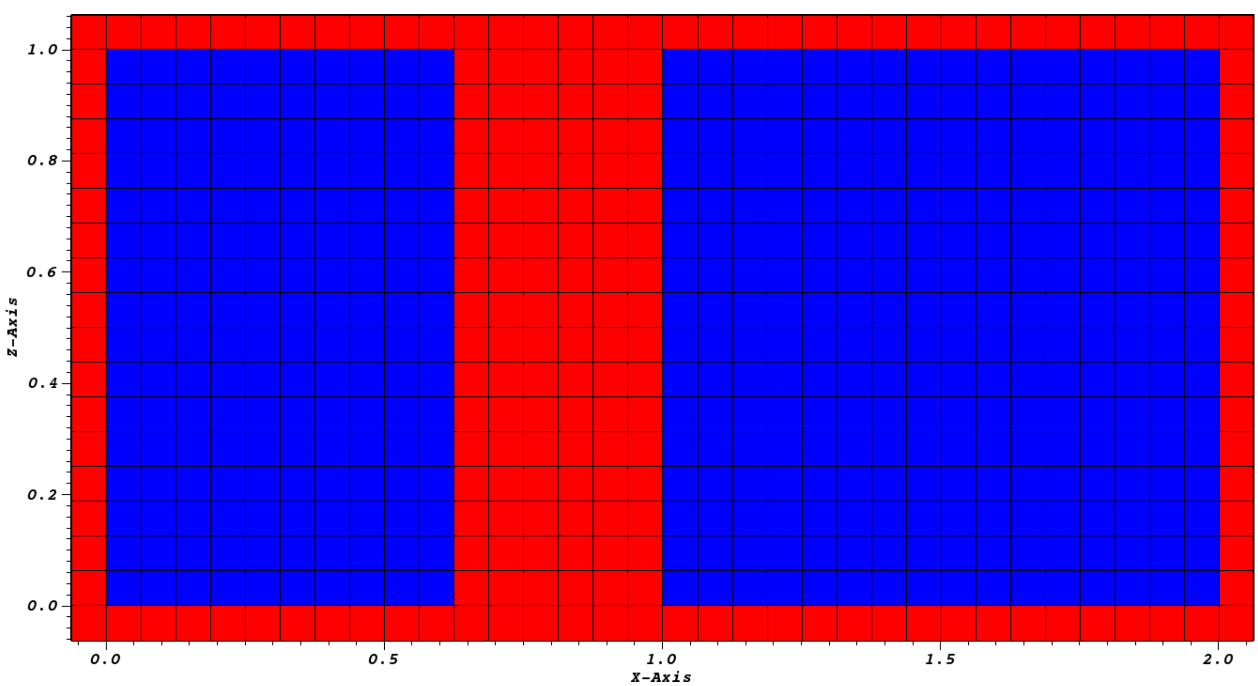


Figure 6: $x - z$ cross section of a centre step domain at $y = 0.5$. Red cells are obstacle cells, blue cells are fluid cells.

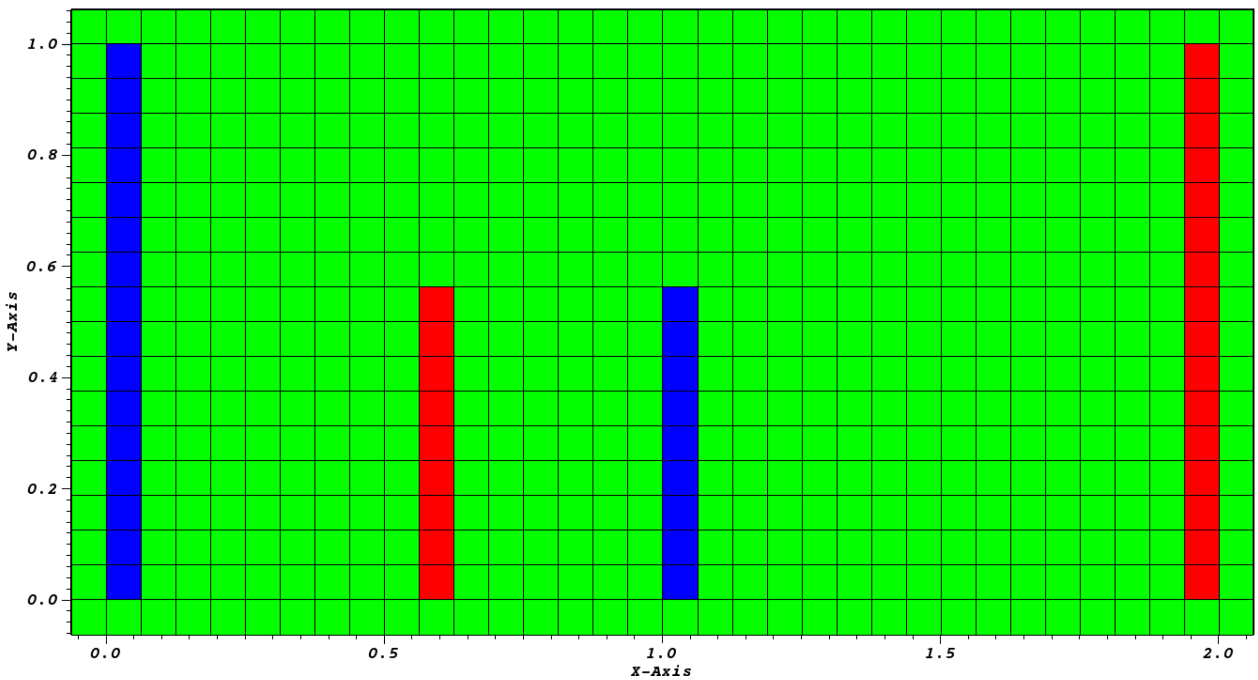


Figure 7: $x - y$ cross section of a centre step domain at $z = 0.5$. Red cells are `boundary_normal_x` $= 1$, blue cells are `boundary_normal_x` $=$ -1 and green cells are `boundary_normal_x` $= 0$.

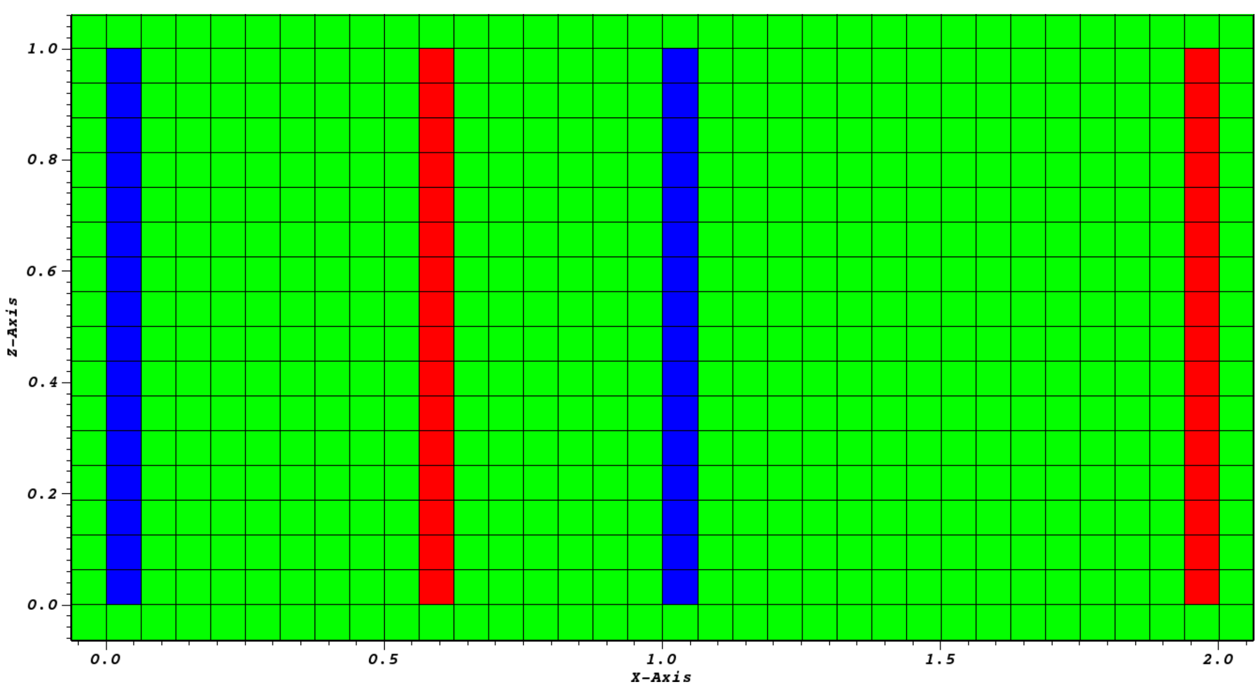


Figure 8: $x - z$ cross section of a centre step domain at $y = 0.5$. Red cells are `boundary_normal_x` $= 1$, blue cells are `boundary_normal_x` $=$ -1 and green cells are `boundary_normal_x` $= 0$.

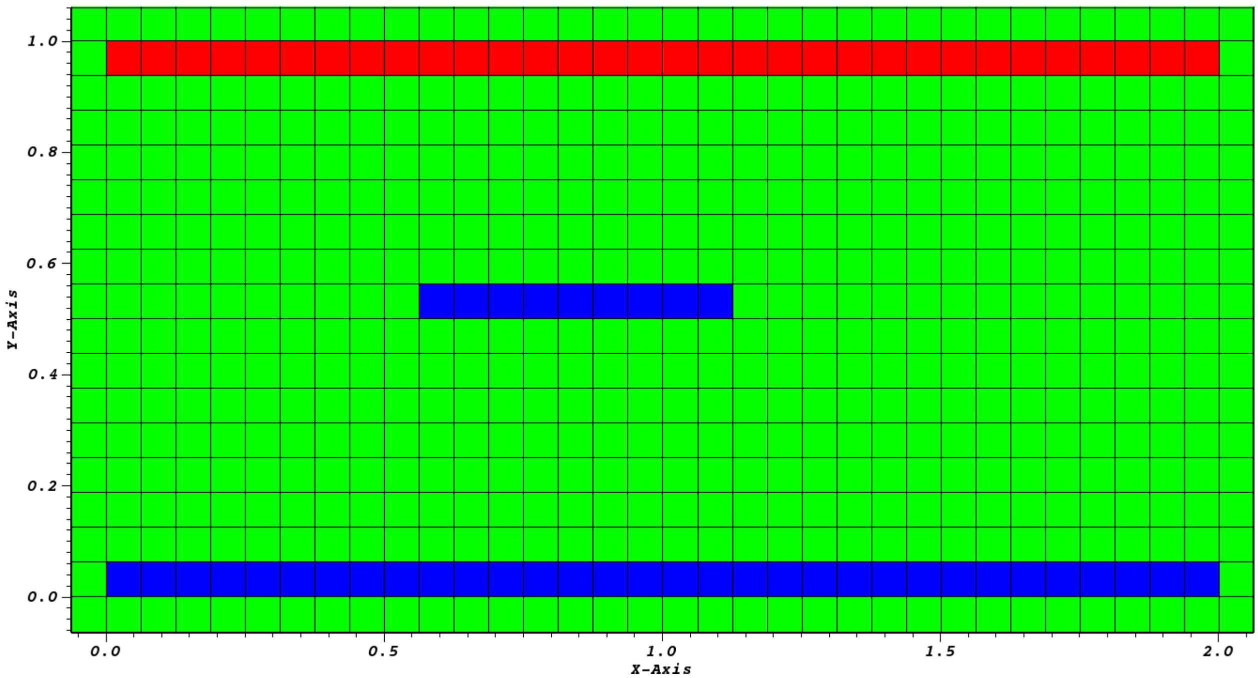


Figure 9: $x - y$ cross section of a centre step domain at $z = 0.5$. Red cells are `boundary_normal_y` $= 1$, blue cells are `boundary_normal_y` $=$ -1 and green cells are `boundary_normal_y` $= 0$.

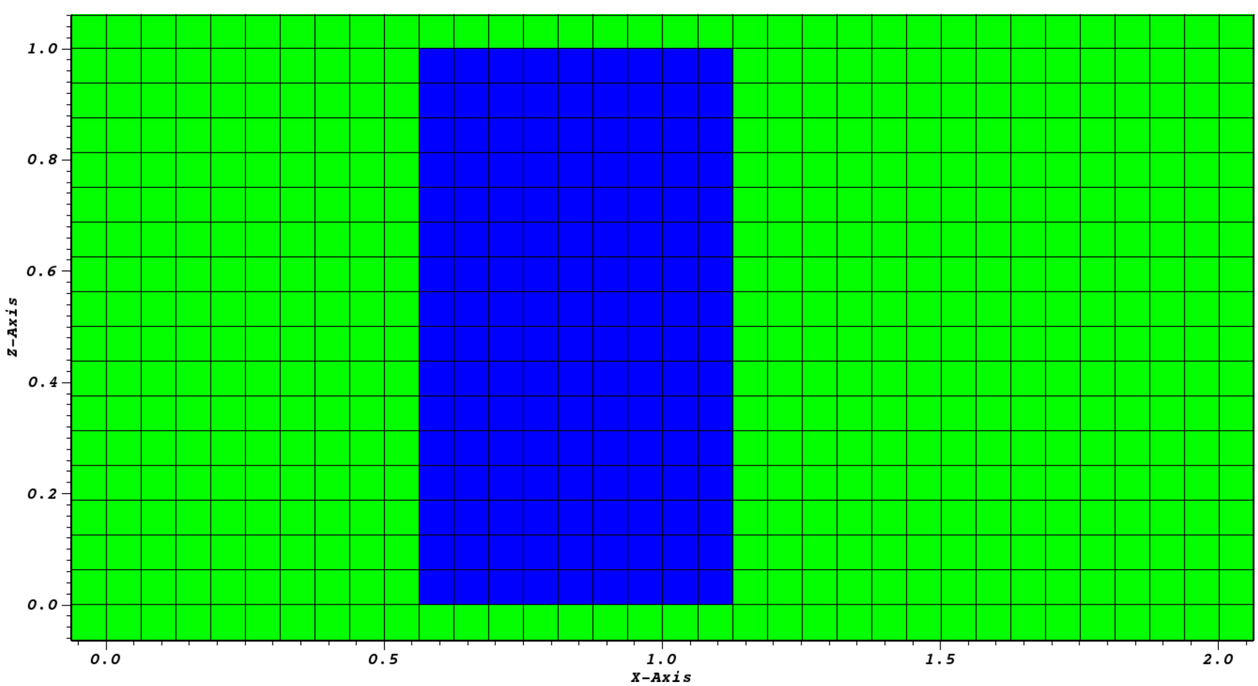


Figure 10: $x - y$ cross section of a centre step domain at $y = 0.5$. Red cells are `boundary_normal_y` $= 1$, blue cells are `boundary_normal_y` $=$ -1 and green cells are `boundary_normal_y` $= 0$.

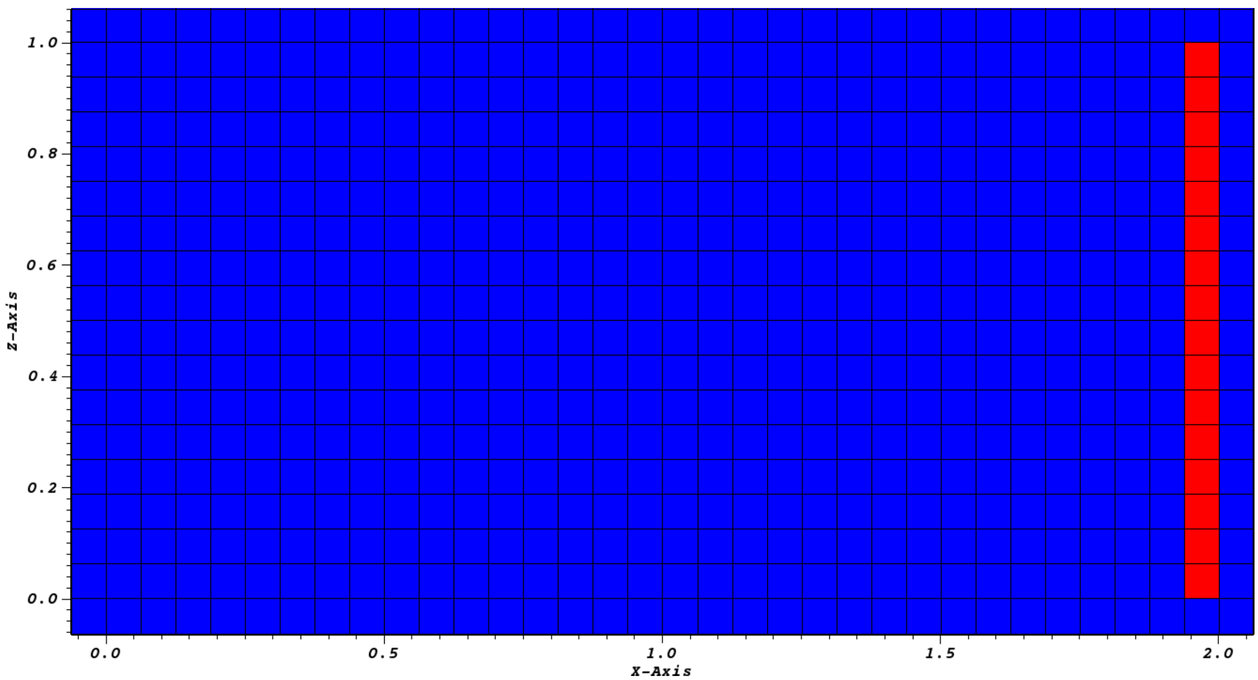
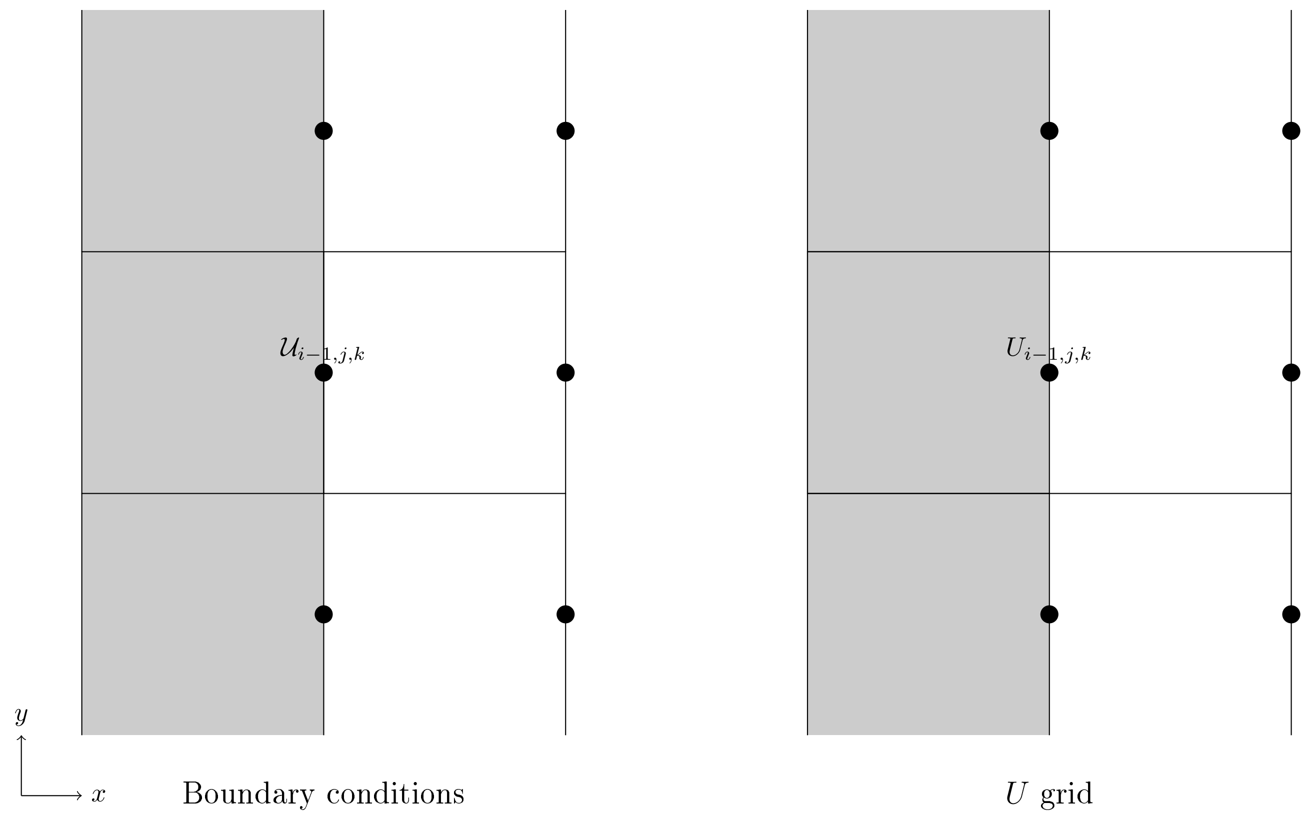


Figure 11: $x - z$ cross section of a centre step domain at $y = 0.5$. Red cells are outflow cells.

2.2 Boundary Conditions

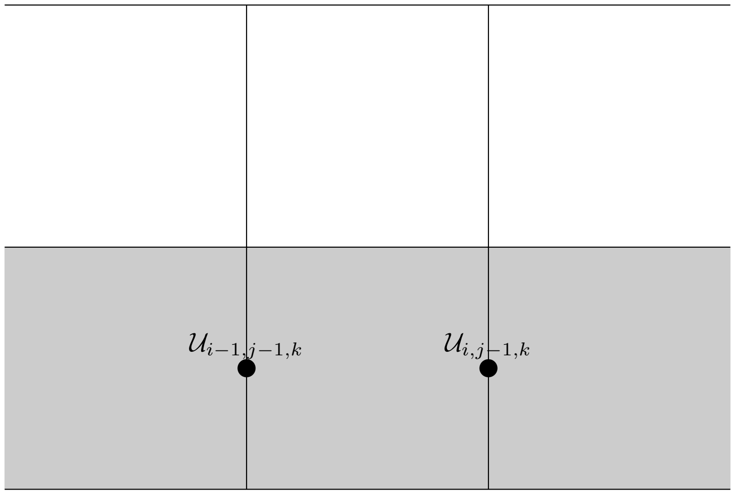
The boundary conditions for u, v and w are given on grids corresponding to that variable. Consider the variable u, all other variables are treated the same way. On boundaries orthogonal to the x axis, we can simply prescribe the value of u directly:



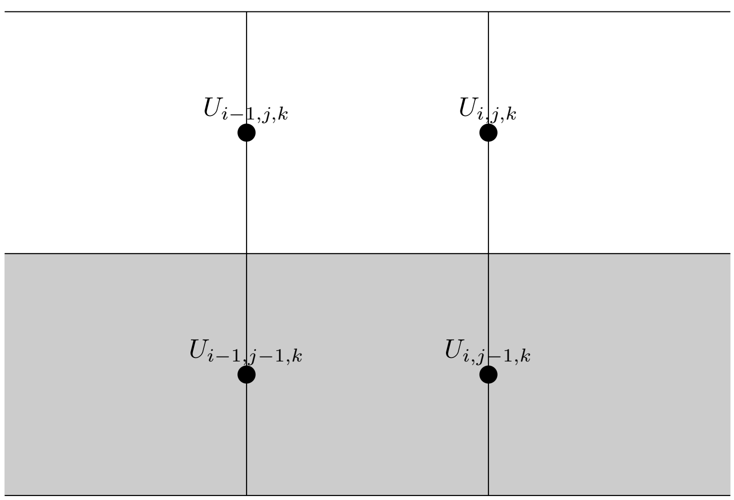
Boundary conditions

U grid

i.e. $U_{i-1,j,k} = \mathcal{U}_{i-1,j,k}$. For other boundaries, we must average the values of a point inside and a point outside the domain. In this case, the boundary condition must be given at a ghost node and must be equal to the desired value on the boundary:



Boundary conditions



U grid

In other words:

$$\frac{U_{i,j-1,k} - U_{i,j,k}}{2} = \mathcal{U}_{i,j-1,k}.$$