# Kalkulus (1230012)

## Bab V Turunan

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# Sub Pokok Bahasan

- Turunan Fungsi Eksponen
- Turunan Fungsi Logaritma
- Turunan Fungsi Implisit
- Turunan Fungsi secara Logaritmis

# Kompetensi Khusus

Mahasiswa mampu menyelesaikan berbagai turunan fungsi

# Bilangan Alam

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^k \quad . \qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \text{kombinasi}$$

$$= a^n + n.a^{n-1} \cdot b + \frac{n.(n-1)}{2!} a^{n-2} \cdot b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} \cdot b^3 + \cdots + b^n$$

$$\left(1+\frac{1}{n}\right)^n \Rightarrow a = 1 \text{ dan } b = \frac{1}{n}$$

$$= 1^n + \cancel{n}.1^{n-1} \cdot \frac{1}{\cancel{n}} + \frac{n.(n-1)}{2!}.1^{n-2}\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n.n} + \frac{1}{3!}\frac{(n)}{n}\frac{(n-1)}{n}\frac{(n-2)}{n} + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 2 + \frac{1}{2!}.1.\left(1-\frac{1}{n}\right) + \frac{1}{3!}1.\left(1-\frac{1}{n}\right).\left(1-\frac{2}{n}\right) + \cdots + \left(\frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$= 2{,}718281828459 \cdots$$

$$= e$$

$$\therefore \boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}$$

$$u = \frac{1}{x} \Rightarrow x = \frac{1}{u}.$$

$$x \to \infty \Rightarrow u \to 0.$$

$$\boxed{\lim_{u \to 0} (1 + u)^{1/u} = e}$$

$$\lim_{x\to 0} \frac{a^x - 1}{x}$$

$$u = a^x - 1 \longrightarrow a^x = u+1$$

$$x\to 0 \Rightarrow u \to 0 \qquad \log a^x = \log(u+1)$$

$$x \log a = \log(u+1)$$

$$x = {}^a\log(u+1)$$

$$= \lim_{u\to 0} \frac{u}{{}^a\log(u+1)}$$

$$= \lim_{u\to 0}\frac{1}{\frac{1}{u}\cdot {}^a\log(u+1)} = \frac{1}{\lim\limits_{u\to 0} {}^a\log(u+1)^{1/u}}$$

$$= \frac{1}{{}^a\log \lim\limits_{u\to 0}(u+1)^{1/u}} = \frac{1}{{}^a\log e} = \frac{\log a}{\log e} = {}^e\log a = \ln a$$

$$\therefore \boxed{\lim_{x\to 0}\frac{a^x-1}{x} = \ln a}$$

# TURUNAN FUNGSI EKSPONEN

$$f(x) = a^x$$

$$y' = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} = a^x \cdot \lim_{h \to 0} \frac{(a^h - 1)}{h}$$

$$= a^x \cdot \ln . a$$

(*) $$\boxed{f(x) = e^x \Rightarrow \frac{dy}{dx} = e^x \cdot \ln e = e^x}$$

# TURUNAN FUNGSI LOGARITMA

TURUNAN FUNGSI LOGARITMA

$$f(x) = y = {}^{a}\log x \rightarrow x = a^{y} \quad \text{(invers)}$$

$$\frac{dx}{dy} = a^{y} . \ln a = x \ln a$$

$$\frac{dy}{dx} = \frac{1}{x . \ln a}$$

$$\boxed{f(x) = y = \ln x = {}^{e}\log x \rightarrow \frac{dy}{dx} = \frac{1}{x . \ln e} = \frac{1}{x}}$$

# TURUNAN FUNGSI EKSPONEN & LOGARITMA

- $y = f(x) = e^x \rightarrow f'(x) = dy/dx = e^x$
- $y = f(x) = \ln x \rightarrow f'(x) = dy/dx = 1/x$

# ATURAN RANTAI BERSUSUN

Jika y = f (u) dan u = g (v), v = h (x)

Maka : $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dx}$

**Contoh :**

1)Carilah dy/dx dari y = $\sin^3(4x)$

Jawab : $y = u^3$ dan $u = \sin v$, $v = 4x$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dx}$$

$= 3u^2 . \cos v . 4$

$= 3.\sin^2(4x).\cos(4x).4$

$= 12 \sin^2(4x).\cos 4x$

# Aturan Rantai Bersusun (Langsung):

- Turunkan dari fungsi terluar sampai fungsi terdalam→ dikalikan

# Soal

Tentukan dy/dx dari:

1. $y = sin^6(2x^4 - 7x^2)$
2. $y = \sin(2x^4 - 7x^2)^6$
3. $y = cos^6(2x^4 - 7x^2)$
4. $y = \tan(2x^4 - 7x^2)^6$
5. $y = sec^8\left(\frac{3x+2}{4x-3}\right)$
6. $y = \ln(x^3 + 4x)^{15}$
7. $y = ln^{15}(x^3 + 4x)$
8. $y = e^{\csc(x^2-6x^3)}$
9. $y = e^{\arcsin(x^2+1)}$
10. $y = ln^8\left(\frac{x^2+2}{x^2-3}\right)$

# TURUNAN FUNGSI IMPLISIT

Fungsi eksplisit : $y = f(x)$

Fungsi implisit : $y - f(x) = 0 \; atau \; F(x, y) = 0$

Kunci Penyelesaian :

1. Turunan x seperti biasa, tetapi semua turunan y selalu diikuti/dikalikan dengan dy/dx
2. Mengikuti aturan turunan

## Contoh :

1)Tentukan $\frac{dy}{dx}$ dari persamaan $y^3 + 7y = x^3$

Jawab :

$$\frac{d}{dx}(y^3 + 7y) = \frac{d}{dx}(x^3)$$

$$3y^2\frac{dy}{dx} + 7.\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx}\left(3y^2 + 7\right) = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 + 7}$$

2)Cari $\frac{dy}{dx}$ jika : $x^2 + 5y^3 = x + 9$

Jawab : $$\frac{d}{dx}\left(x^2 + 5y^3\right) = \frac{d}{dx}(x+9)$$

$$2x + 15y^2\frac{dy}{dx} = 1$$

$$15y^2\frac{dy}{dx} = 1 - 2x$$

$$\frac{dy}{dx} = \frac{1-2x}{15y^2}$$

3)Cari $\dfrac{dy}{dt}$ jika $t^3 + t^2 y - 10y^4 = 0$

**Jawab :**

$$\frac{d}{dt}(t^3 + t^2 y - 10y^4) = \frac{d}{dt}(0)$$

$$3t^2 + t^2.\frac{dy}{dx} + 2t.y - 40y^3.\frac{dy}{dt} = 0$$

$$\frac{dy}{dt}\left(t^2 - 40y^3\right) = -3t^2 - 2ty$$

$$\frac{dy}{dt} = \frac{3t^2 + 2ty}{40y^3 - t^2}$$

4) Cari $\frac{dy}{dx}$ jika $x^3 + y^3 = 6xy$

Jawab : $\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(6xy)$

$$3x^2 + 3y^2 . \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

# TURUNAN FUNGSI SECARA LOGARITMIS

Diketahui : $y = [f(x)]^{g(x)}$ , tentukan $\dfrac{dy}{dx}$ !

$$\ln y = ln[f(x)]^{g(x)}$$

$$= g(x).\ln[f(x)]$$

Kedua sisi diturunkan ; $\dfrac{1}{y}.\dfrac{dy}{dx} = g'(x).\ln[f(x)] + g(x).\dfrac{f'(x)}{f(x)}$

$$\frac{dy}{dx} = y[g'(x).lnf(x) + g(x).\frac{f'(x)}{f(x)}]$$

$$= [f(x)]^{g(x)}.[g'(x).lnf(x) + g(x).\frac{f'(x)}{f(x)}]$$

Contoh :

1) $y = (x+1)^x$

$$\frac{dy}{dx} = (x+1)^x[1.\ln(x+1) + x.\frac{1}{x+1}]$$

$$= (x+1)^x[\ln(x+1) + \frac{x}{x+1}]$$

2) $y = (tg\ x)^{\sin x} \rightarrow$

$$\frac{dy}{dx} = (tg\ x)^{\sin x}.[\cos x.\ln(tg\ x) + \sin x.\frac{\sec^2 x}{tg\ x}]$$

$$= (tg\ x)^{\sin x}[\cos x.\ln(tg\ x) + \sec x]$$

# Referensi


- Purcell, Varberg, *Kalkulus dan Geometri Analitis*, Penerbit Erlangga, 1993
- Frank Ayres, *Calculus*, Mc.Graw Hill, New York, 1972
- J.Salas and Hill, *Calculus One and Several Variables*, John Willey& Sons, NewYork, 1982