CS203B, Assignment 2

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1. Cyclic Groups
   (a) Prove that every cyclic group of order n is isomorphic to $\mathbb{Z}_n$.
   (b) Prove that every subgroup of a cyclic group is cyclic.
   (c) For any cyclic group of order n; prove that given any divisor d of n, there exists a unique subgroup of order d.
   (d) In a cyclic group of order n, how many generators are there?
   (e) Prove the following equality:
   $$\sum_{d|n} \phi(d) = n, \text{ where } \phi \text{ is euler's totient function.}$$
   (f) Hence or otherwise prove that $\mathbb{Z}_p^*$ is cyclic, where p is a prime.
2. A few nice results:
   (a) We talked about $\mathbb{Z}_n^*$ in the last assignment. Now, prove Euler's theorem.
   $$a^{\phi(n)} \equiv 1 \ (mod\ n)\ \forall a \in \mathbb{Z}$$
   Realise that Fermat's little theorem is a corollary of the above.
   (b) Prove Wilson's theorem:
   $$(p-1)! \equiv -1 \ (mod\ p), \text{ where } p \text{ is a prime.}$$
3. Conjugacy class of an element $a \in G$ is defined as :
   $$Cl(a) = \{gag^{-1} | g \in G\}$$
   (a) Now if G is abelian, what can you say about conjugacy classes?
   (b) Prove that conjugacy classes form an equivalence relation.
   (c) Prove that two elements in same conjugacy class have same order.
   (d) Define
   $$\begin{gathered} f : G \to Aut(G) \\ g \to \Phi_g \\ \Phi_g(h) = ghg^{-1}\ \forall h \in G \end{gathered}$$

Prove f is a group homomorphism and Φ_g is a group automorphism.

(e) Find the kernel of f.

(f) Using results from above parts or otherwise prove that if $Aut(G)$ is cyclic then G is abelian. (You will have to get deeper insights for the way the quotient group $\frac{G}{Z(G)}$ looks!)

4. Find all the conjugacy classes of D_{2n} the dihedral group. (Look up the definition of dihedral group)

5. Determine all isomorphic classes of abelian groups of order p^k, where p is a prime.