ACA Summer School

Data Structures and Algorithms
Lecture 3

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Outline

Graph Definitions

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- If vertex u is connected to v by an edge, then v is called a neighbour of u, and vice-versa.
- **Degree** of a vertex if the number of neigbours of the vertex.
- Maximum number of edges $|E| < |V|^2$.

Graph Terminologies

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Connected graph

A graph is called **connected** if every pair of vertices in the graph are connected.

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- A **directed graph** is a graph with orientations in the edges.
- An **undirected graph** is a graph without any orientations in the edges.
- A **regular graph** is a graph in which every vertex has the same degree.

Uses of Graphs

Many problems of practical interest can be represented by graphs. For example

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- If the vertices represent cities, and edges represent the roads between cities, then a graph can be used to model the road transport network of an area.
- If the vertices represent brain regions, and edges represent the series of neurons connecting the brain regions, then a graph can be used to model the brain neural network.

Uses of Graphs

Minimum distance between two cities: Suppose you have a road structure which connects various cities by road. You need to find the minimum distance road-path between two places.

This problem can be framed as a weighted graph. The places represents the vertices and edges are the connected roads between two places.
The distance between two cities is the weight given to the particular edge.

Outline

Graph Terminologies

Adjacency Matrices and Lists

Graph Traversal

Shortest Path Algorithms

Adjacency Matrix

Given a undirected graph G with n vertices and m edges, the adjacency matrix of G is an $n \times n$ matrix such that

$$\begin{aligned} A[i,j] &= 1, \text{ if } i \text{ and } j \text{ are connected,} \\ &= 0, \text{ otherwise} \end{aligned}$$

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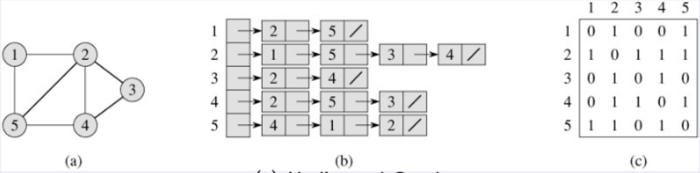
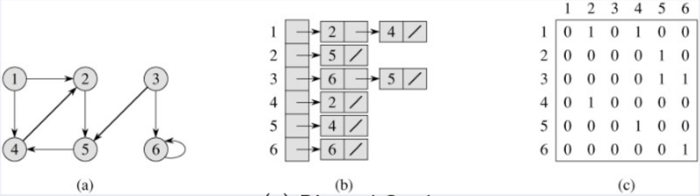
Adjacency Lists

Given a undirected graph G with n vertices and m edges, the adjacency list of a vertex u contains all the vertices that are neighbours of u in G. The array of these adjacency lists can be used to represent G.

Adjacency Matrices and Lists



(t) Undirected Graph



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- Adjacency matrix is more useful for dense graphs

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  - Time taken to determine if u is neighbour of $v = \Theta(1)$

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Adjacency Matrices and Lists

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Shortest Path Algorithms

Breadth First Search

Given a graph $G = (V, E)$ and a source vertex s.

BFS

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- Then for each of those nearest nodes, it explores their unexplored neighbor nodes.
- It continues step 2, until it explores the entire graph.

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- A vertex whose all the adjacent vertices have been discovered is colored black.

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- A queue Q to store the next elements to be visited.

Breadth First Search

```
for each u in V - {s}                    \\ for each vertex except s.
   do color[u] = WHITE
       d[u] = infinity
       P[u] = NIL
color[s] = GRAY                           \\ Source vertex discovered
d[s] = 0                                  \\ initialize
P[s] = NIL                                \\ initialize
Q = {}                                    \\ Clear queue Q
ENQUEUE(Q, s)
while Q is non-empty
do u = DEQUEUE(Q)                            \\ That is, u = head[Q]
     for each v adjacent to u
          do if color[v] = WHITE
             then  color[v] = GRAY
                   d[v] = d[u] + 1
                   P[v] = u
                   ENQUEUE(Q, v)
     color[u] = BLACK
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- Running time of BFS $= O(m + n)$.

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- DFS progresses by expanding the first node that appears.
- It goes deeper and deeper from one node to its child until it hits the node that has no children.
- The search backtracks, returning to the most recent node it hasn't finished exploring.

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- Blackened when its adjacency list has been examined completely.

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- d[u] : Stores the timestamp when u is first discovered.

Depth First Search

```
DFS (V, E)
      for each vertex u in V[G]
         do color[u] = WHITE
                 P[u] = NIL
      time = 0
      for each vertex u in V[G]
         do if color[u] = WHITE
                 then DFS-Visit(u)

DFS-Visit(u)
      color[u] = GRAY
      time = time + 1
      d[u] = time
      for each vertex v adjacent to u
         do if color[v] = WHITE
                 then P[v] = u
                         DFS-Visit(v)
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- Time taken by the function DFS-VISIT is $\Theta(m)$.
- Total time taken in DFS is $\Theta(m + n)$.

Problems on DFS and BFS

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- Finding a cycle in the graph
- Checking if the graph is connected
- Finding the diameter of a tree
- Finding minimum distance for all pairs in an unweighted graph

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Finding the Shortest Path

- To find the shortest path from vertex s to v in an unweighted graph, we can use BFS.

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- To find the shortest path from vertex s to v in an unweighted graph, we can use BFS.
- Dijkstra's algorithm is a well-known algorithm used to find the shortest path between two vertices in a weighted graph.

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- Visit all the unvisited neighbours of the current node, and update their tentative distance.

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- If the target node has been visited, then report the distance to that node.

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- When we are done considering all of the neighbors of the current node, mark the current node as visited and remove it from the unvisited set.
- If the target node has been visited, then report the distance to that node.
- Otherwise, choose any node in the univisted set and mark it as the current noode and repeat Step 4.

Dijkstra's Algorithm

```
function Dijkstra(G, s):

    Create empty set Q

    for each vertex v :
        dist[v] = INFINITY
        add v to Q

    dist[s] = 0
    while Q is not empty:
        u = vertex in Q with min dist[u]
        remove u from Q

        for each neighbor v of u:
            alt = dist[u] + length(u, v)
            if alt < dist[v]:
                dist[v] = alt

    return dist[]
```

Dijkstra's Algorithm

- Using Fibonacci heaps or self-balancing binary search trees, the above algorithm can run in $O(m + n \log n)$ time.

Reading Assignment

- All-Pair Shortest Path Algorithms

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- Minimum Spanning Tree

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- Directed Graph Traversal

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- All-Pair Shortest Path Algorithms
- Minimum Spanning Tree
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- Topological Sorting