Worksheet - 1

1) $A = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$

a) $x \neq y \neq z$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - x^2 R_1]{R_2 \to R_2 - x R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y^2-x^2 & z^2-x^2 \end{pmatrix}$$

$R_3 \to R_3 - (y+x) R_2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & 0 & (z^2-x^2) - (z-x)(y+x) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & 0 & (z-x)(z-y) \end{pmatrix}$$

$(z-x)(z+\cancel{x}-y-\cancel{x})$

$\underline{\underline{(z-x)(z-y)}}$

$\neq 0.$

Since $y-x$; $z-x$ and $z-y$ are non - zero the $\rho(A) = \underline{\underline{3}}$

8. b) $x = y \neq z$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & x & z \\ x^2 & x^2 & z^2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - x^2 R_1]{R_2 \to R_2 - x R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & z-x \\ 0 & 0 & z^2-x^2 \end{pmatrix}$$

$R_3 \to R_3 - (z+x) R_2$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & z-x \\ 0 & 0 & 0 \end{pmatrix}$ $\rho(A) = 2$

c) $x = y = z$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x & x & x \\ x^2 & x^2 & x^2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - x^2 R_1]{R_2 \to R_2 - x R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\rho(A) = 1}}$$

2. a) $\rho(A+B) = \rho(A) + \rho(B)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix}$$

$\det(A) \neq 0 \Rightarrow \rho(A) = 2$

$\det(B) \neq 0 \Rightarrow \rho(B) = 2$

$$A + B = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A+B) \neq \rho(A) + \rho(B)$

$\rho(A+B) = 1$

b) $$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$\rho(A) = 1$ $\qquad$ $\rho(B) = 1$

$$A+B = \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}$$

$\cancel{\rho(A+B) = 1}$

~~$\therefore \rho(A+B)$ is not less than~~

$\rho(A+B) = 2$

$\therefore \rho(A+B) = \rho(A) + \rho(B)$

not less than $\rho(A) + \rho(B)$.

The correct inequality is,

$$\underline{\rho(A+B) \leq \rho(A) + \rho(B)}$$

3. $A = [a_{ij}]_{5\times5}$ $\quad a_{ij} = i^2 - j^2$; $1 \le i \le 5$, $1 \le j \le 5$.

$$A = \begin{bmatrix} 0 & -3 & -8 & -15 & -24 \\ 3 & 0 & -5 & -12 & -21 \\ 8 & 5 & 0 & -7 & -16 \\ 15 & 12 & 7 & 0 & -9 \\ 24 & 21 & 16 & 9 & 0 \end{bmatrix}$$

$0+9$ (9)

$25 + 120 - 8 \cdot 15$

$145 - 120$
$= 25$

$a_{ij} = -a_{ji}$

~~As~~ skew-symmetric of odd order

$a_{ij} = -a_{ji}$

det

$$\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$$

det $(A) = 0$.

$\therefore r(A) \neq 5$.

$3 \times 40 - 8 \cdot 15$
$120 - 120$
$= 0$

4. $v = [a \quad b \quad c]_{1\times3}$ Non-zero column vector.

$$v^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times1}$$

a) $v v^T = [a^2 + b^2 + c^2]$ $\quad a^2+b^2+c^2 \neq 0$ as atleast one of a, b, c is non-zero.

$\Rightarrow r(v v^T) = 1$

b) $v^T v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [a \quad b \quad c]$

$$= \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a \begin{vmatrix} a & ab & ac \\ b & b^2 & bc \\ c & bc & c^2 \end{vmatrix} = abc \begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$$

$= 0$ //

$\therefore \rho(A) \neq 3$

Atleast one a, b, c is non-zero.

$\Rightarrow \rho(A)$ is either 1 or 2.

Echelon form

$$\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$bc - \frac{ac \cdot b}{a}$

$R_2 \to R_2 - \frac{b}{a} R_1$

$R_3 \to R_3 - \frac{c}{a} R_1$

$$\begin{bmatrix} a^2 & ab & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$bc - \frac{ab \cdot c}{a}$

$c^2 - \frac{ac \cdot c}{a}$

$\Rightarrow \rho(A) = 1$.

$\rho(v^T v) = 1$

5. $P_1 : x - y - \lambda^2 z = 0$

$P_2 : x - y + z = 0$

$P_3 : x + y - z = 0$

$$A = \begin{bmatrix} 1 & -1 & -\lambda^2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

If $\rho(A) = 3$, then only one solution

$\rho(A) < 3$ then infinite no.of solution.

So, there exists a line of intercection L only if these system has infinite no.of solutions ($\rho(A) < 3$)

echelon form

~~Solⁿ~~ $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & -\lambda^2 \\ 0 & 0 & 1+\lambda^2 \\ 0 & 2 & -1+\lambda^2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & -\lambda^2 \\ 0 & 2 & -1+\lambda^2 \\ 0 & 0 & 1+\lambda^2 \end{bmatrix}$$

$\therefore \rho(A) = 3$, there exists a unique solution,

$x = 0, y = 0, z = 0$

(For ~~R~~ Homogeneous system, $x = 0$ is the unique solution).

If $1 + \lambda^2 = 0$, then ~~it~~ $\rho(A) = 2$, it will have infinite no. of solutions.

$1 + \lambda^2 = 0 \Rightarrow$ ~~[illegible]~~ $\lambda = i \notin \mathbb{R}$.

$\therefore$ we can't find such λ.

$P(0, 0, 0)$.

6. $2x + y + z = 1$

$3x - y + \lambda z = 2$

$x + \mu y + z = 3$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & \lambda \\ 1 & \mu & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If $\rho(A) = \rho(A|B) \neq 3$, we will get infinite no. of solutions

$$A|B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & -1 & \lambda & 2 \\ 1 & \mu & 1 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & \mu & 1 & 3 \\ 3 & -1 & \lambda & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & \mu & 1 & 3 \\ 0 & -1-3\mu & \lambda-3 & -7 \\ 0 & 1-2\mu & -1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - \frac{(1-2\mu)}{\cancel{8}(1-3\mu)}$

$$\begin{bmatrix} 1 & \mu & 1 & 3 \\ 0 & -1-3\mu & \lambda-3 & -7 \\ 0 & 0 & -1 - \frac{(\lambda-3)(1-2\mu)}{-1-3\mu} & -5 + \frac{7(1-2\mu)}{-1-3\mu} \end{bmatrix}$$

$-1 - \frac{(\lambda-3)(1-2\mu)}{}$

For infinite solutions,

$$\frac{-5 + 7(1-2\mu)}{-1-3\mu} = 0.$$

$$-5(-1-3\mu) + 7(1-2\mu) = 0.$$

$$\Rightarrow 5 + 15\mu + 7 - 14\mu = 0$$

$$\Rightarrow \mu = -12$$

and $$-1 - \frac{(\lambda-3)(1-2\mu)}{-1-3\mu} = 0$$

$$\Rightarrow -1 - \frac{(\lambda-3)\,25}{35} = 0.$$

$$\Rightarrow -35 - 25\lambda + 75 = 0$$

$$\Rightarrow 40 = 25\lambda$$

$$\Rightarrow \frac{8}{5} = \lambda$$

$$(A|B) = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & -1 & 8/5 & 2 \\ 1 & -12 & 1 & 3 \end{bmatrix} \sim$$

for μ

Echelon form will be

$$\begin{bmatrix} 1 & -12 & 1 & 3 \\ 0 & 35 & -7/3 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - 12y + z = 3 \quad \text{—(1)}$$

$$35y - \frac{7}{3}z = -7 \quad \text{—(2)}$$

(1) $\times \frac{7}{3}$

$$\frac{7}{3}x - 28y + \frac{7}{3}z = 7$$

$$35y - \frac{7}{3}z = -7$$

$$\frac{7}{3}x + 7y = 0.$$

$$\frac{1}{3}x = -y$$

$$x = -3y$$

$$35y - 7 = \frac{7}{3}z$$

$$\Rightarrow \frac{z}{3} = 5y - 1$$

$$z = 15y - 3$$

$\Rightarrow y = t \quad x = -3t \quad z = 15t - 3.$

$$X = \begin{bmatrix} -3t \\ t \\ 15t-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 15 \end{bmatrix}$$

$$(x, y, z) = (-3t, t, 15t - 3)$$

7. $a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$$A = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array}\right]$$

Parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$R_2 \to R_2 - \frac{a_2}{a_1} R_1$

Not parallel

$$A \sim \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 - \frac{b_1 a_2}{a_1} & c_2 - \frac{c_1 a_2}{a_1} & d_2 - \frac{d_1 a_2}{a_1} \end{bmatrix}$$

$$\frac{b_2}{b_1} \neq \frac{a_2}{a_1}$$

or $\frac{c_2}{c_1} \neq \frac{a_2}{a_1}$

$R(A) = 2 = R(A|B)$ and

$R(A) = R(A|B) \neq$ no. of variables $= 3$.

$\Rightarrow$ The system has infinite no. of solns.

$d_2 - d_1 \frac{a_2}{a_1} = 0$

$d_2 = d_1 \frac{a_2}{a_1}$

$\frac{d_2}{d_1} = \frac{a_2}{a_1}$.

$\therefore$ the planes intersect at a line L.

8. $P_1 \equiv -2x - 3y + z = 0$

$P_2 \equiv 6z - 5 = 0$

$P_3 \equiv ay + 2z = 5$

a) P_3 intersects L at a unique pt means, P_1, P_2 & P_3 intersect at that pt. ~~That mean~~

$\therefore$ we need a unique soln for the system P_1, P_2, and P_3.

$\therefore \rho(A) = \rho(A|b) = 3$ if

$$A = \begin{bmatrix} -2 & -3 & 1 \\ 0 & 0 & b \\ 0 & a & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$$

$$(A|B) = \left[\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 0 & 0 & b & 5 \\ 0 & a & 2 & 5 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & a & 2 & 5 \\ 0 & 0 & b & 5 \end{bmatrix}$$

If $b \neq 0$ and $a \neq 0$

then $\rho(A) = \rho(A|B) = 3$

$\Rightarrow$ unique solution.

b) Never intersects

$\Rightarrow$ no solution.

$\rho(A) \neq \rho(A|B)$

$\Rightarrow$ If $a = 0$, ~~then~~

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & b & 5 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - \frac{b}{2} R_2$

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 - \frac{5b}{2} \end{bmatrix}$$

If $5 - \frac{5b}{2} \neq 0$.

$\Rightarrow b \neq 2$; then

no solution.

because $\left[\rho(A) = 2, \ \rho(A|B) = 3\right]$

If $b = 0$

$\Rightarrow \rho(A) = 2, \ \rho(A|B) = 3$

no solution.

c) Contains L,

~~unique soln~~ infinite soln.

$a = 0$

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & b & 5 \end{bmatrix} \quad a = 0.$$

$R_3 \longrightarrow R_3 - \frac{b}{2} R_2$

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 - \frac{5b}{2} \end{bmatrix}$$

If $5 - \frac{5b}{2} = 0$

$b = 2$

$\Rightarrow \rho(A) = \rho(A|B) = 2 < 3$

$\Rightarrow$ infinite soln.