Indian Institute of Technology Indore
MA204 Numerical Methods
(Spring Semester 2022)
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Tutorial Sheet 3

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1. (a) A $n\times n$ matrix A is said to be **positive definite** if it is symmetric and if $\boldsymbol{x}^\top A\boldsymbol{x} > 0$ for every n-dimensional vector $\boldsymbol{x} \neq \mathbf{0}$; here the symbol '$\top$' denotes the transpose. Using this definition, show that the 4×4 symmetric Pascal matrix [pascal(4)] computed in tutorial sheet 2 is positive definite.

   [Hint: Take $\boldsymbol{x}$ as a general vector, say $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^\top$; compute $\boldsymbol{x}^\top A\boldsymbol{x}$; and try to express the result as a sum of squares.]

   (b) Being positive definite, the matrix pascal(4) has a Cholesky factorization (of the form $A = LL^\top$). Determine the Cholesky factorization for the matrix pascal(4). Observe that the Cholesky factorization for the matrix pascal(4) turns out to be exactly the same as its LU-factorization determined in tutorial sheet 2. [Note that these factorizations for a general symmetric positive definite matrix may not be the same.]

2. Find a rotation matrix P with the property that PA has a zero entry in the second row and first column, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

   Also write down PA.

3. The system

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6, \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

   has a unique (exact) solution $\boldsymbol{x} = [1, 2, -1, 1]^\top$. Starting with an initial guess $\boldsymbol{x}^{(0)} = [0, 0, 0, 0]^\top$, solve the system using the (a) Jacobi and (b) Gauss–Seidel methods to find an approximate solution of the system until the relative error

$$\frac{\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\|_\infty}{\|\boldsymbol{x}\|_\infty} < 10^{-6}.$$

   Write down the approximate solutions at each iteration in the form of a table that also includes the relative error at every iteration, and write down all the numbers till 4 digits after decimal using rounding-off.

4. The Gauss–Seidel iteration for the i^{th} component of the vector $\boldsymbol{x}$, while solving $A\boldsymbol{x} = \boldsymbol{b}$, is given by

$$x_i^{(\text{new})} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(\text{new})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(\text{old})} \right].$$

   If $x_i^{(\text{new})} = x_i^{(\text{old})}$ for all $i = 1, 2, \ldots, n$, how does this show that the solution $\boldsymbol{x}$ is correct?

5. Compute the condition numbers of the following matrices relative to $\|\cdot\|_\infty$.

$$(a)\ A = \begin{bmatrix} 1 & 2 \\ 1.00001 & 2 \end{bmatrix}, \qquad (b)\ B = \begin{bmatrix} 58.9 & 0.03 \\ -6.10 & 5.31 \end{bmatrix},$$

$$(c)\ C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad (d)\ D = \begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}.$$

6. Apply the power method to compute (a) the largest and (b) the smallest eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 6 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(c) Hence, compute all the eigenvalues of the matrix A.

7. (a) Draw the Geršgorin discs (row-wise) for the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

(b) What can you conclude about the diagonalization of the matrix A by looking at the Geršgorin discs for it? Give the corresponding general statement for any square matrix.

(c) If A is diagonalizable, represent it in the diagonalized form.

(d) Now draw the Geršgorin discs (column-wise) for the matrix A. Notice that, similarly to the row-wise Geršgorin discs, the column-wise Geršgorin discs for A are also disjoint. However, this need not to be true in general.

Remarks:

(i) If a Geršgorin disc of radius zero is disjoint from all other Geršgorin discs of a matrix (which is the case in this example; notice this in parts (a) and (d) both), the center of the disc (the one having zero radius and disjoint from the others) is an eigenvalue of the matrix.

(ii) The converse of the statement made in part (b) is not true in general.

8. Compute all the eigenvalues of the matrix

$$A = \begin{bmatrix} 6 & 4 & 1 & 1 \\ 4 & 6 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$$

by performing five iterations of the QR-factorization method.