FINITE ELEMENT ANALYSIS OF ONE-DIMENSIONAL BAR

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Abstract

In this project, we develop a MATLAB code for One-dimensional Finite Element Analysis of a Bar fixed at right end and free at left end. We perform two types of analysis- Modal Analysis and Dynamic Analysis. The code takes user input for the type of analysis to perform and displays the result.In modal analysis, natural frequencies are obtained using lumped mass matrix formulation and are compared with values obtained from exact method. Mode shapes for first 10 natural frequencies are shown in this report. In dynamic analysis, we use Heaviside function as boundary condition to obtain displacement,velocity and acceleration. Displacement and stress are plotted for different time intervals.

1 Introduction

We begin the coding process with modal analysis and end with dynamic analysis. The process is implemented for a reason that the later analysis requires the values obtained from the preceding analysis. User inputs a wrapper file containing all the required input data and parameters. Several functions are created in-order to simplify the coding process. Inside these functions, calculations are performed that takes the values from wrapper file and output value is generated in return. A driver file is created that calls the functions that are created and will display the required results and plots.

Consider a bar is fixed at x = L and free at x = 0. The governing differential equation for this is

$$\frac{\partial}{\partial x}(EA\frac{\partial u}{\partial x}) + q(x,t) = \rho A\frac{\partial^2 u}{\partial t^2} \tag{1}$$

For free vibrations, q=0. Assume that E and A are constant.

$$E\frac{\partial^2 u}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \text{ or,} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C_b^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

$$C_b = \sqrt{\frac{E}{\rho}} \tag{4}$$

2 Modal Analysis

The aim of modal analysis in finite element method is to determine natural frequencies and mode shapes of a body during free vibration. It is basically performed to get the frequency at which the body will resonate. It is also used in Noise, Vibration and Harness(NVH) Analysis. If the frequency of input load matches with the resonant frequency of the component, the body produces deformation and might damage the component. Modal Analysis can be done using Exact Method or Approximate Method.

2.1 Exact Method

Let us consider solutions of the form

$$u(x,t) = A(x)\sin\omega t \tag{5}$$

Substituting the above in the governing differential equation, we find that

$$A_{xx}(x) - \lambda^2 A(x) = 0,\ where\ \lambda = \frac{\omega}{C_b} \tag{6}$$

The general solution to the above is given by

$$A(x) = c_1\cos\lambda x + c_2\sin\lambda x \tag{7}$$

where $c_1 and c_2$ are constants. The boundary require that

$$u(L,t) = 0\ and\ u_x(0,t) = 0 \tag{8}$$

which translate to

$$A_{,x}(x=0) = 0\ and\ A(x=L) = 0 \tag{9}$$

These conditions lead to

$$c_2 = 0\ and\ \lambda_n = \frac{(2n+1)\pi}{2L}, n = 0,1,2,3... \tag{10}$$

The values λ_n are the modes of free vibrations of the bar and the mode shapes are given by

$$A_n(x) = \cos\lambda_n x, n = 0,1,2,3... \tag{11}$$

2.2 Finite Element Method

The governing equation of Modal Analysis in Finite Element Method is

$$\mathrm{M}\ddot{\mathbf{d}} + K\mathbf{d} = 0 \tag{12}$$

where M is Global Mass Matrix, K is Global Stiffness Matrix

$$\mathrm{Let, d(t)} = \mathbf{U}\sin(\omega t) \tag{13}$$

where U is Mode Shape Substituting (2) in (1),

$$(\mathrm{K} - \omega^2 M)\mathrm{U} = 0 \tag{14}$$

The above equation is also called Generalized Eigenvalue Problem.
The characteristic equation for non-trivial U is

$$\det(\mathrm{K} - \omega^2 M) = 0 \tag{15}$$

This equation is used to find natural frequencies.

There are two types of Mass Matrix Formulation - Consistent Mass Matrix Formulation and Lumped Mass Matrix Formulation. For a large number of elements, the mass matrix size increases. This increases memory space, calculation time and overall cost. The problem demands to analyze using Lumped Mass Matrix formulation. One of the advantage of using Lumped Mass Matrix formulation is that it reduces the memory space occupied by the file in the computer. Lumped Mass Matrix being a diagonal matrix becomes easy to inverse. Thus, it reduces calculation time and becomes cost-efficient.

2.3 Gauss Quadrature Rule

We use Gauss Quadrature Rule to find Element Stiffness Matrix and Element Lumped Mass Matrix. For any arbitrary function $f(\xi)$, the Gaussian quadrature rule is given by,

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=0}^{n} w_i f(\xi_i) \tag{16}$$

For $n = 1$, ξ takes value

$$\xi_0 = -1/\sqrt{3} \tag{17}$$
$$\xi_1 = 1/\sqrt{3} \tag{18}$$

and w_i takes value

$$w_0 = 1 \tag{19}$$
$$w_1 = 1 \tag{20}$$

2.4 Stiffness and Mass Matrix Formulation

For linear elements, components of global stiffness matrix is given by,

$$K_{AB} = \int_0^L N_{A,x} E(x) A(x) N_{B,x} dx \tag{21}$$
$$K_{AB} = \mathop{\mathrm{A}}_{e=1}^{n_{el}} k^e$$

Element stiffness matrix is given by,

$$\begin{aligned} k^e : k^e_{ab} &= \int_{\Omega_e} N_a(x)' E(x) A(x) N_b(x)' dx \\ &= \int_{-1}^{1} N_{a,\xi}(\xi) E(\xi) A(\xi) N_{b,\xi}(\xi) (x_{,\xi})^{-1} d\xi \\ \text{where, } E &= \sum_{c=1}^{n_{int}} E^e_c N_c \ and\ A = \sum_{c=1}^{n_{int}} A^e_c N_c \end{aligned} \tag{22}$$

For linear elements $a, b = 1, 2$

$$x_{,\xi} = \frac{h_e}{2} \tag{23}$$
$$N_a(\xi) = \frac{1 + \xi_a \xi}{2} \tag{24}$$

Using Gauss Quadrature Rule,

$$k^e = \frac{w_i E(\xi) A(\xi)}{2h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{25}$$

Components of global mass matrix is given by,

$$M_{AB} = \int_0^L N_A\ \rho(x)A(x)N_B dx \tag{26}$$

$$M_{AB} = \mathop{\mathrm{A}}_{e=1}^{n_{el}} m^e$$

Element mass matrix is given by,

$$\begin{aligned} m^e : m^e_{ab} &= \int_{\Omega_e} N_a(x)\rho(x)A(x)N_b(x)dx \\ &= \int_{-1}^{1} N_a(\xi)\rho(\xi)A(\xi)N_b(\xi)x_{,\xi}\, d\xi \\ \text{where, } \rho &= \sum_{c=1}^{n_{int}} \rho^e_c N_c \ and \ A = \sum_{c=1}^{n_{int}} A^e_c N_c \end{aligned} \tag{27}$$

For linear elements a,b =1,2

$$x_{,\xi} = \frac{h_e}{2} \tag{28}$$

$$N_a(\xi) = \frac{1+\xi_a\xi}{2} \tag{29}$$

Using Gauss Quadrature Rule,

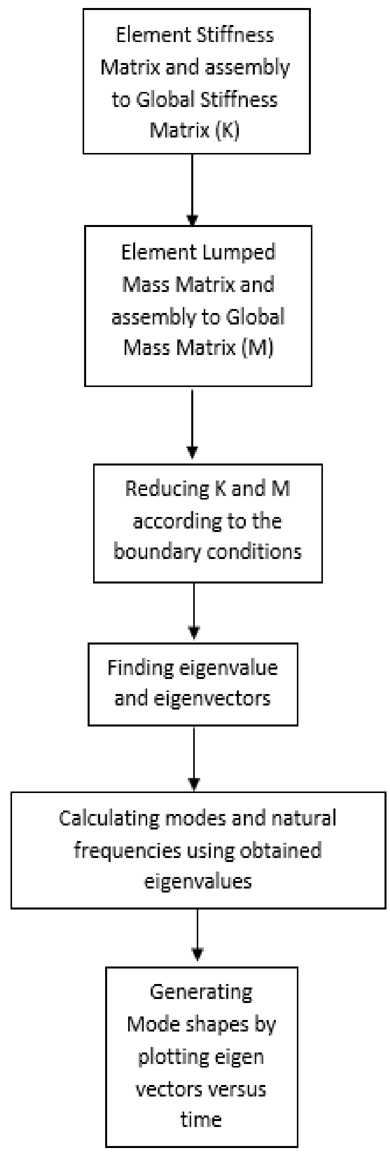
$$m^e = m^e_c = \frac{w_0\rho(\xi)A(\xi)N_a(\xi)N_b(\xi)h_e + w_1\rho(\xi)A(\xi)N_a(\xi)N_b(\xi)h_e}{12}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \tag{30}$$

The element mass matrix is also called consistent mass matrix. Element lumped mass matrix is given by,

$$m^e_l = \frac{3(w_0\rho(\xi)A(\xi)N_a(\xi)N_b(\xi)h_e + w_1\rho(\xi)A(\xi)N_a(\xi)N_b(\xi)h_e)}{4}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{31}$$

2.5 Coding Process for Modal Analysis

The coding process of modal analysis is



The code is structured to take both constant as well as variable values for the parameters. Any change in input parameters does not affect the structure of the code. To run for variable values, comment from line "19-26" and to run for constant values, comment from line '9-15" in 'wrapper_modal_analysis' file.

The user inputs following data in the wrapper file,

Parameter	Value
E	$200e9$ Pa
A	$6e{-}4m^2$
ρ	$7800Kg/m^3$
n_{el}	25
L	1.2 m
bias	1.0

Table 1: Input Parameters and Values

Several functions are created which are called in 'onedfem_driver_modal' file. The following function were used:

Parameter	Variable	Function
Nodal values of displacement	xc	getx
Nodal values of distributed load	qc	getq
Nodal values of Young's modulus	Ec	getE
Nodal values of cross sectional area	Ac	getA
Nodal values of Density	rhoc	getrho
Gauss Quadrature Points and weights	Ng, w	gauss
Global Lumped Mass Marix	M	getM
Global Stiffness Matrix	K	getK
Global Force Vector	F	getF
Mode	Mode	getMode
Natural Frequency	Natural Frequency	getMode
Mode Shape	ModeShape	getmodeshape

Table 2: Table showing Parameters, Variables and Functions that are used in the code

3 Results of Modal Analysis

We generate a table of natural frequencies for different mode number and compare them with values obtained from exact method.

Mode Number	Lumped Mass Matrix Formulation Method(Hz)	Exact Method(Hz)
1	0.1986×10^5	0.1988×10^5
2	0.3301×10^5	0.3314×10^5
3	0.4603×10^5	0.4639×10^5
4	0.5886×10^5	0.5965×10^5
5	0.7147×10^5	0.7291×10^5
6	0.8379×10^5	0.8617×10^5
7	0.9579×10^5	0.9942×10^5
8	1.0740×10^5	1.1268×10^5
9	1.1859×10^5	1.2594×10^5
10	1.2932×10^5	1.3920×10^5

Table 3: Table showing Mode number and comparison between values of Natural Frequency obtained using Lumped Mass Matrix Formulation Method and Exact Method

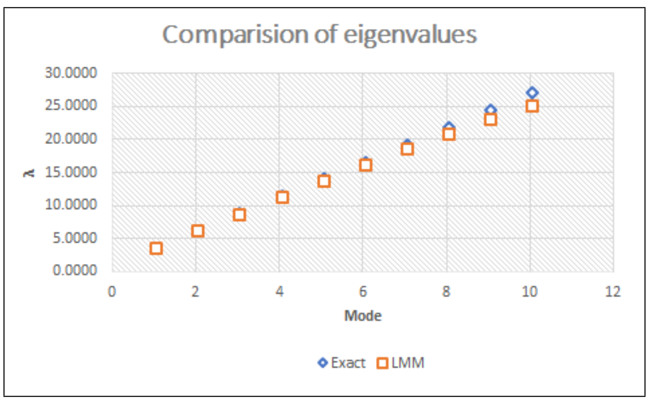
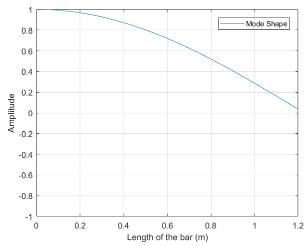


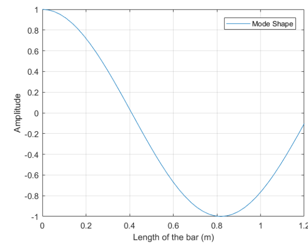
Figure 1: Comparison of eigenvalues with exact method and values obtained using lumped mass matrix formulation

In comparison with natural frequency obtained by Exact Method, as the mode number increases, the natural frequency obtained by lumped mass matrix formulation gradually decreases in accuracy. This can be due to the fact in the discretizations, insufficient number of modes were used to solve the problem.

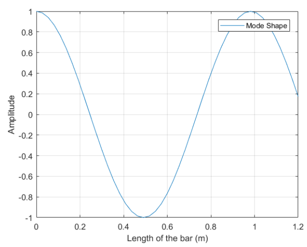
Mode Shape signifies the pattern of vibrations at a particular frequency. For 1st Mode, refer Figure 2a, the amplitude of vibration begins from 1 and decreases nearly to 0. This also signifies that for 1st Mode, the bending moment acting on the bar would be maximum. Similarly, as mode number increases, the frequency increases, the pattern converts into a wave form.



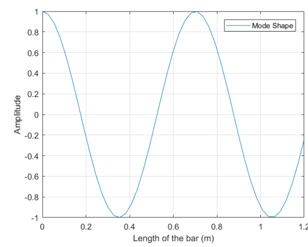
(a) For $\lambda = 3.921$



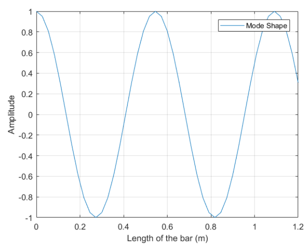
(b) For $\lambda = 6.518$



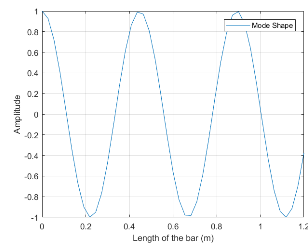
(c) For $\lambda = 9.089$



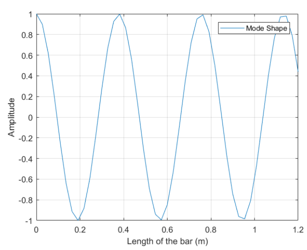
(d) For $\lambda = 11.625$



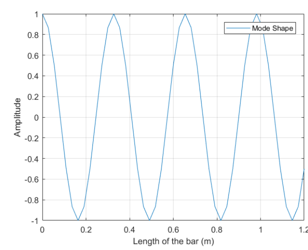
(e) For $\lambda = 14.114$



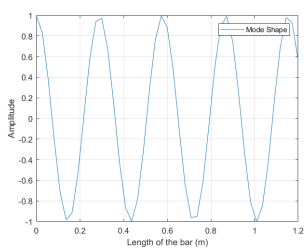
(f) For $\lambda = 16.548$



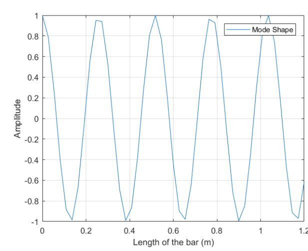
(g) For $\lambda = 18.916$



(h) For $\lambda = 21.21$



(i) For $\lambda = 23.42$



(j) For $\lambda = 25.538$

Figure 2: Mode Shapes for first 10 natural frequencies

4 Dynamic Analysis

Dynamic analysis is used to evaluate the impact of transient loads or to design out potential noise and vibration problems. It involves calculation of displacement and stress values as a function of time.

4.1 Time Discretization

Let [0,T] be the time interval of interest. Partition [0,T] into intervals,

$$[t_0, t_1], [t_1, t_2], [t_2, t_3], ...[t_{M-1}, t_M]$$

We try to find d at $t_0, t_1, t_2, t_3.....t_M$. So let d_i denote $d(t_i)$, $i = 0, 1, 2, ...M$

4.2 Explicit Method

Explicit method is used to calculate displacement of the bar.
The explicit method uses

$$\mathrm{M}\ddot{\mathbf{d}} + \mathrm{K}\mathbf{d} = \mathrm{F(t)} \tag{32}$$

Here $\mathbf{d}$ is displacement and $\ddot{d}$ is acceleration. We break it up in time domain.
Suppose we know the solution $d_i = d(t_i)$, $v_i = v(t_i)$, and $a_i = a(t_i)$.

$$Ma_{i+1} + Kd_{i+1} = F_{i+1} \tag{33}$$

$$d_{i+1} = d_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i \tag{34}$$

$$v_{i+1} = v_i + \frac{(\Delta t)^2}{2}[a_i + a_{i+1}] \tag{35}$$

$$a_{i+1} = M^{-1}[F_{i+1} - Kd_{i+1}] \tag{36}$$

Δt cannot be arbitrary. Therefore, the method is said to be conditionally stable.

4.3 Stability Criterion

$$\omega^h_{max}\Delta t \leq 2 \tag{37}$$

where ω^h_{max} is the maximum frequency of the system which we get using eigenvalue problem.

$$\omega^h_{max} \leq \omega^\wedge_{max} \tag{38}$$

$$\omega^\wedge_{max} = max^{n_{el}}_{e=1}(\omega^e_{max}) \tag{39}$$

where ω^e_{max} is maximum natural frequency of Ω_e obtained from $\det(k^e - \omega^2 m^e) = 0$
For linear elements,

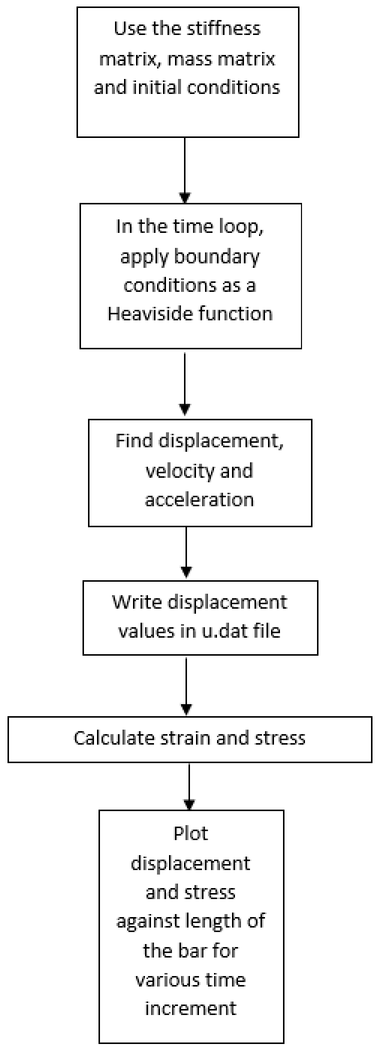
$$\omega^e_{max} = \sqrt{\frac{12E}{\rho h_e^2}} = \sqrt{\frac{12C_b^2}{h_e^2}} = \sqrt{12}\frac{C_b}{h_e} \tag{40}$$

when E and A are constant.
For lumped mass matrix formulation, stability criterion is

$$\frac{C_b\Delta t}{h_e} \leq 1 \tag{41}$$

4.4 Coding Process for Dynamic Analysis

The coding process includes:



The code is structured to take both constant as well as variable values for the parameters. Any change in input parameters does not affect the structure of the code. To run for variable values, comment from line "18-26" and to run for constant values, comment from line '8-15" in 'wrapper_dynamic_analysis' file.

The user inputs following data in the wrapper file,

Parameter	Value
E	$200e9$ Pa
A	$6e{-}4m^2$
ρ	$7800Kg/m^3$
L	1.2 m
bias	1.0
$u(x,0)$	0
$\dot{u}(x,0)$	0
$q(t)$	$\sum[H(t) - H(t - t_c)]$

Table 4: Input Parameters and Values

where $\sum = 100$ MPa, H(t) is the Heaviside function and $t_c = 0.1\tau$, where $\tau = \frac{L}{C_b}$
For Heaviside function,

$$H(t - t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t \leq t_0 \end{cases}$$

Several functions are created which are called in 'onedfem_driver_dynamic' file. The following function were used:

Description	Variable	Function
Nodal values of displacement	xc	getx
Nodal values of distributed load	qc	getq
Nodal values of Young's modulus	Ec	getE
Nodal values of cross sectional area	Ac	getA
Gauss Quadrature Points and weights	Ng, w	gauss
Global Lumped Mass Marix	M	getM
Global Stiffness Matrix	Km	getKm
Global Force Vector	F	getf
Displacement	d_dyn	getdispexp
Velocity	v_dyn	getdispexp
Acceleration	a_dyn	getdispexp
Runs displacement plot		visualize
Plot displacement		plotdispexp
Calculate and runs stress plot		visualize
Plot stress		plotstress

Table 5: Table showing Parameters, Variables and Functions that are used in the code

5 Results of Dynamic Analysis

We generate plots for displacement and stress by the values calculated in the time loop. The displacement values were written in u.dat file. Displacement and stress are plotted against length of the bar for different time increments. Time increments are :

Sr.No.	Time
1	0.05τ
2	0.12τ
3	0.5τ
4	1.05τ
5	1.2τ

Table 6: Time increments

The figure 3 shows displacement plot for the time increments in table 6.

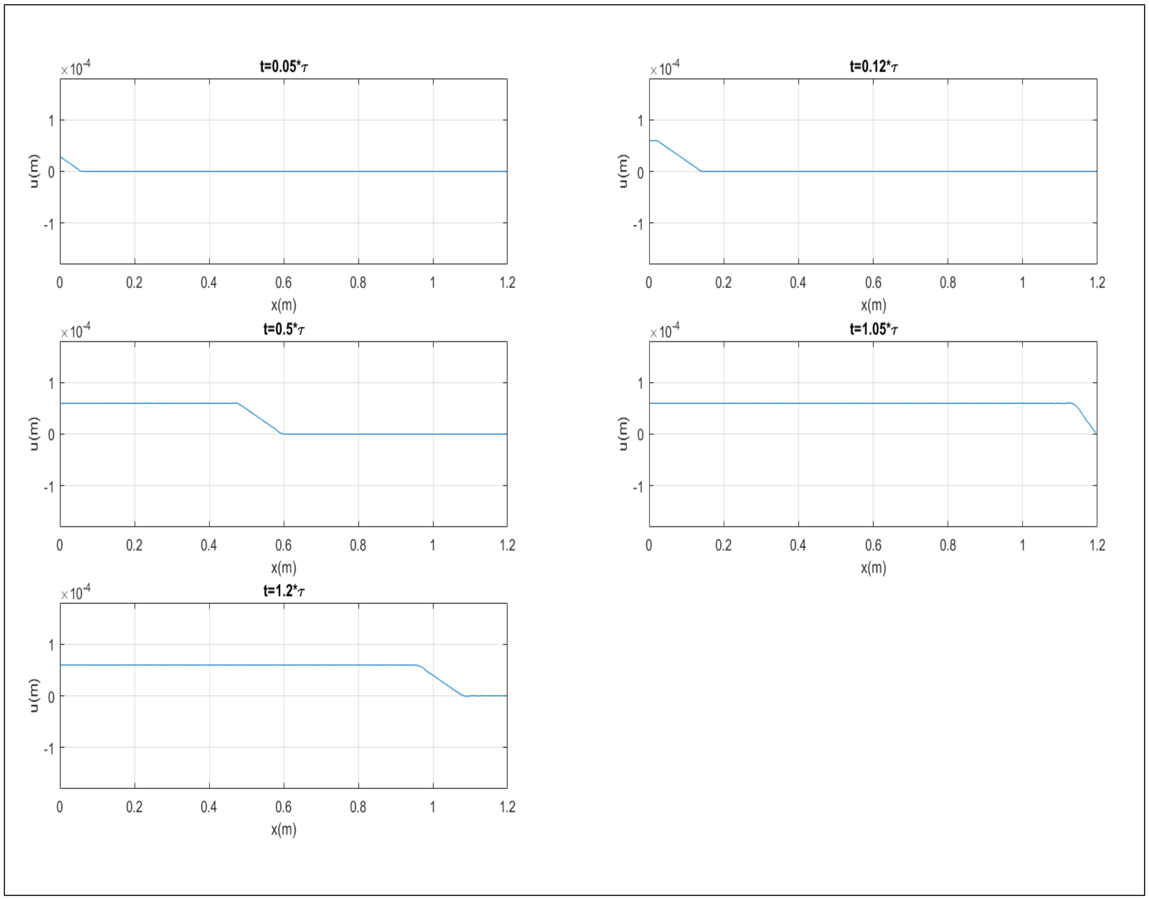


Figure 3: Displacement Plot

From the free end, as time increases from $t = 0.0\tau$ to $t = 1.05\tau$, the displacement plot travels like a wave. The magnitude remains constant at all time increments. When $t = 1.05\tau$, it reaches the end of the bar(fixed end). The wave reflects after colliding the end. It returns with same magnitude. When $t = 2.0\tau$, the wave gets inverted that is, it changes its direction (positive to negative) and magnitude gets doubled.

The figure 4 shows displacement plot for the time increments in table 6.

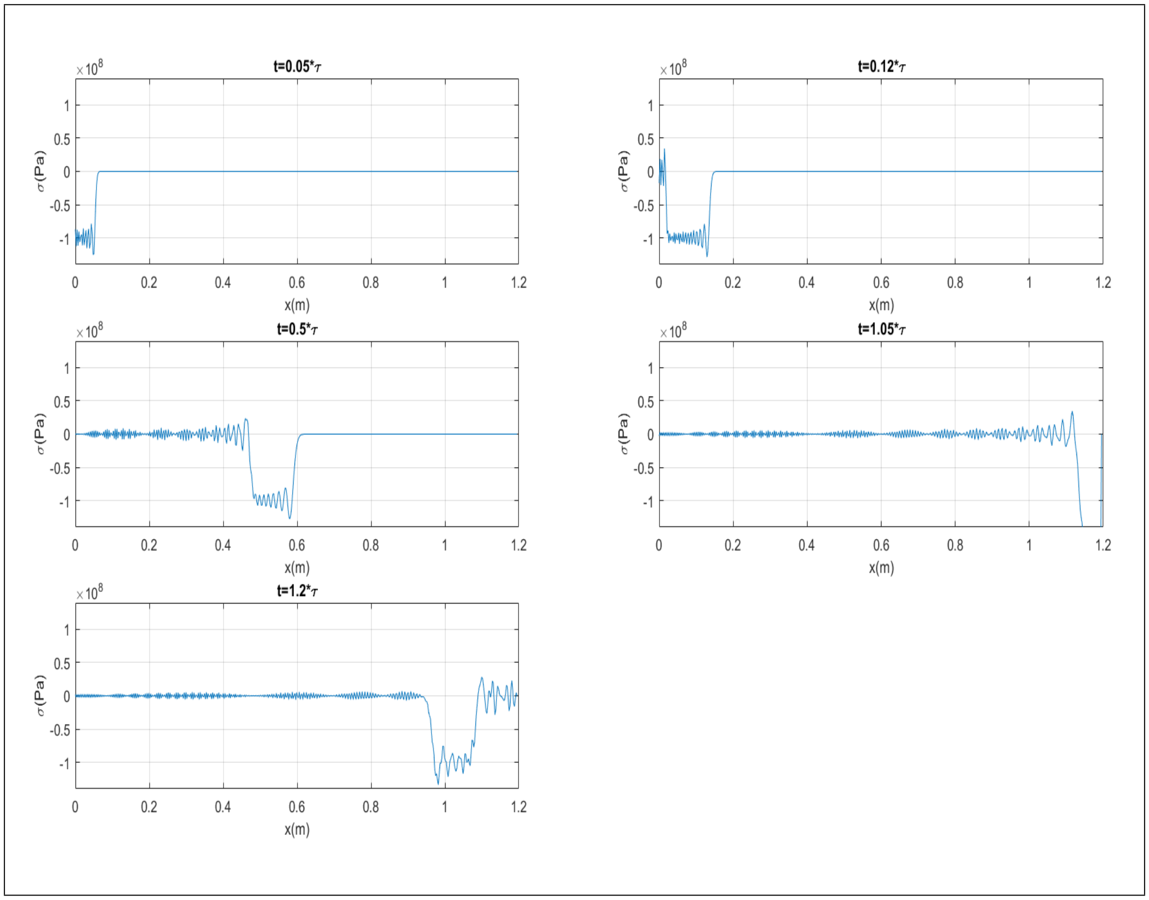


Figure 4: Stress Plot

As time increases from $t = 0.0\tau$ to $t = 1.05\tau$, the stress plot also travels like a wave. The magnitude remains constant at all time increments. When $t = 1.05\tau$, it reaches the end of the bar(fixed end). The wave reflects after colliding the end. It returns with same magnitude. When $t = 2.0\tau$, even the stress wave gets inverted that is, it changes its direction. The waviness seen in the plots is due to the numerical error.

6 Conclusion

Modal and Dynamic analysis were carried for the bar. Modes and natural frequencies were obtained, compared and plotted. A plot shows the variation in values of mode using exact and finite element method. For dynamic analysis, the time was discretized and explicit method was used to calculate displacement of the bar. Heaviside function was used for boundary conditions. Strain and stress were calculated. Plots for displacement and stress show motion of respective wave travel with respect to time.

7 Reference

[1] Handout- Modal Analysis of a bar by Dr. Harish Cherukuri
[2] Handout- Numerical Integration in Finite Element Methods by Dr. Harish Cherukuri