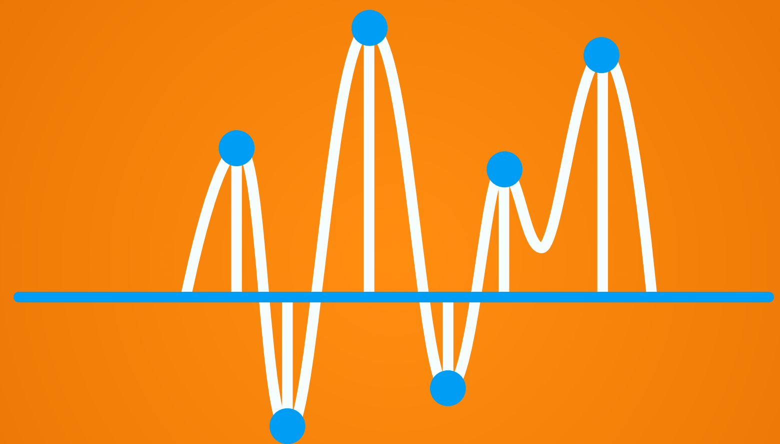
Chapter 1

Digital Signal Processing

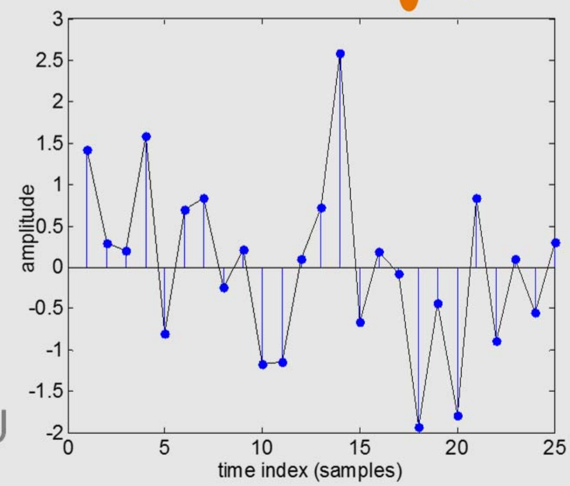


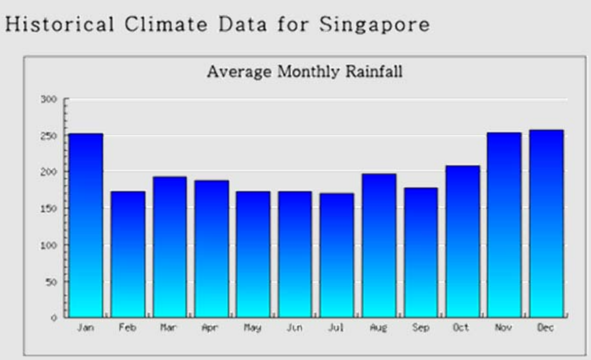
Dr. Andy W. H. Khong

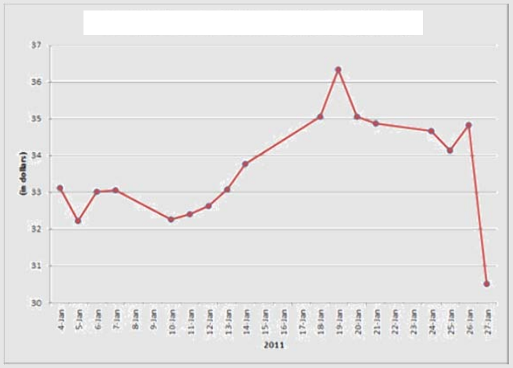
NANYANG TECHNOLOGICAL UNIVERSITY

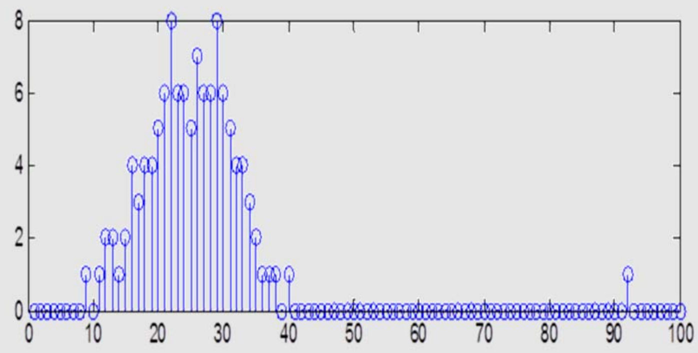
Digital signals

- discrete-time sequence
- sampling data at regular intervals
- *Examples:*
  a) rainfall record for a given period
  b) stock prices for a particular firm
  c) grade distribution for a particular course in NTU









1.1 Definition

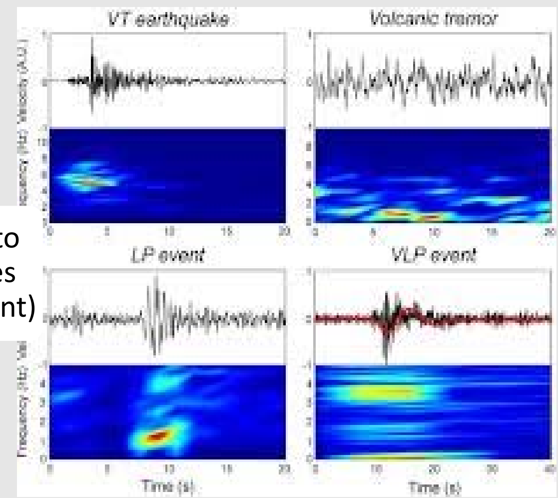
Processing

- Representation
- Transformation
- Manipulation

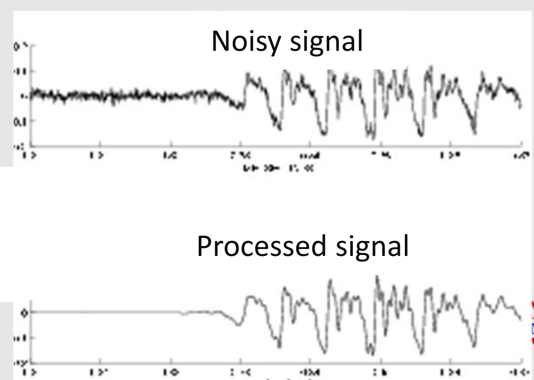
Analog Signals → Digital Signals → Representation / Transformation / Manipulation

Representation: Data compression

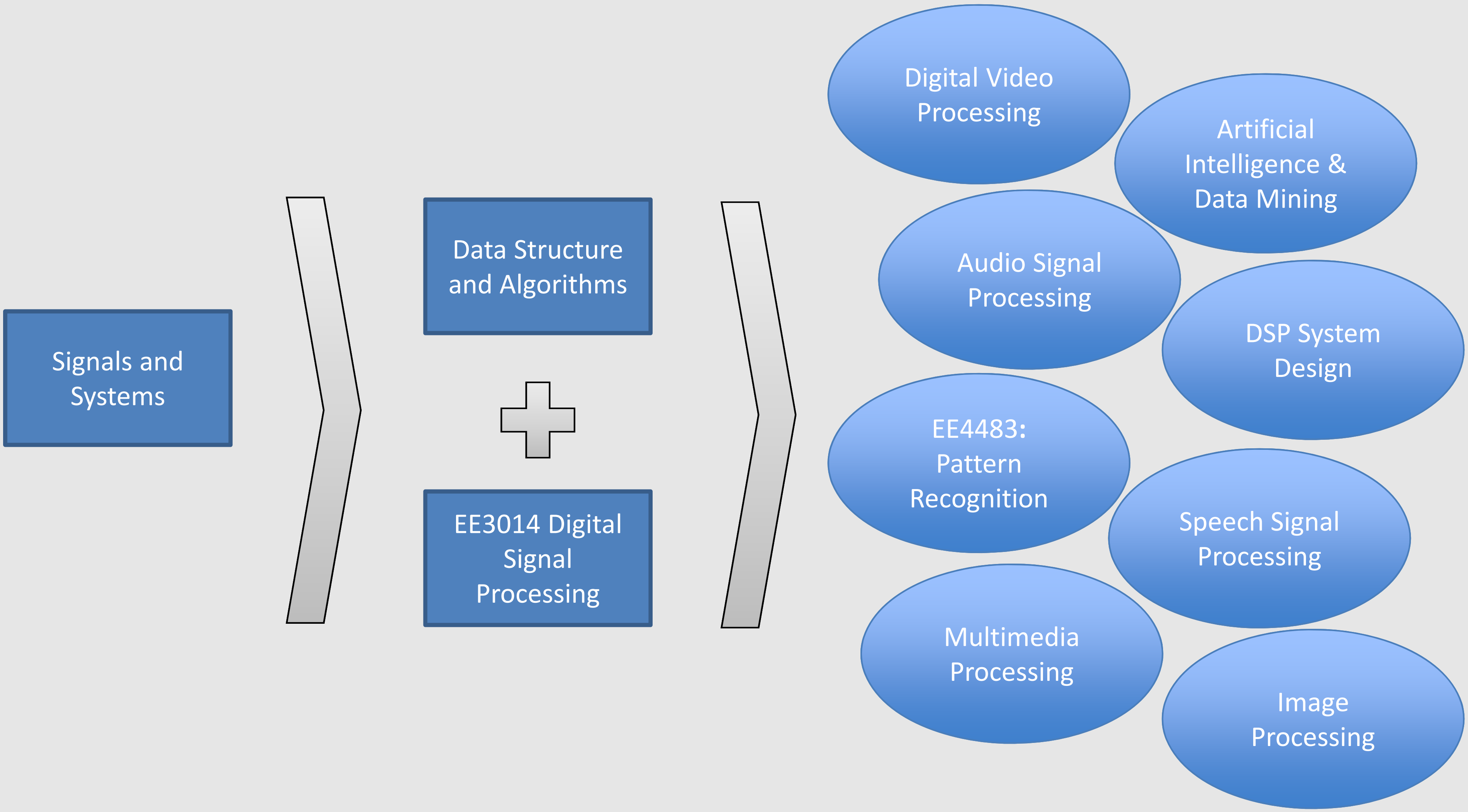
Transformation: Gain insights into signal properties (frequency content)



Manipulation: Reduce noise, sharpen image

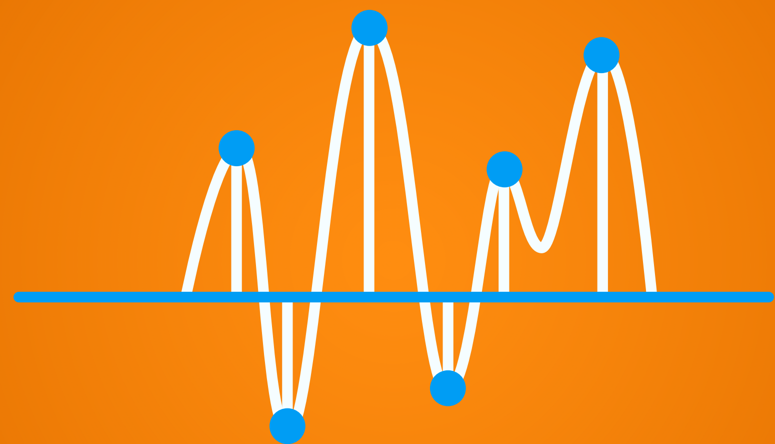


1.3 Related Courses



Chapter 2

Discrete-Time Signals



Dr. Andy W. H. Khong

Chapter Aims

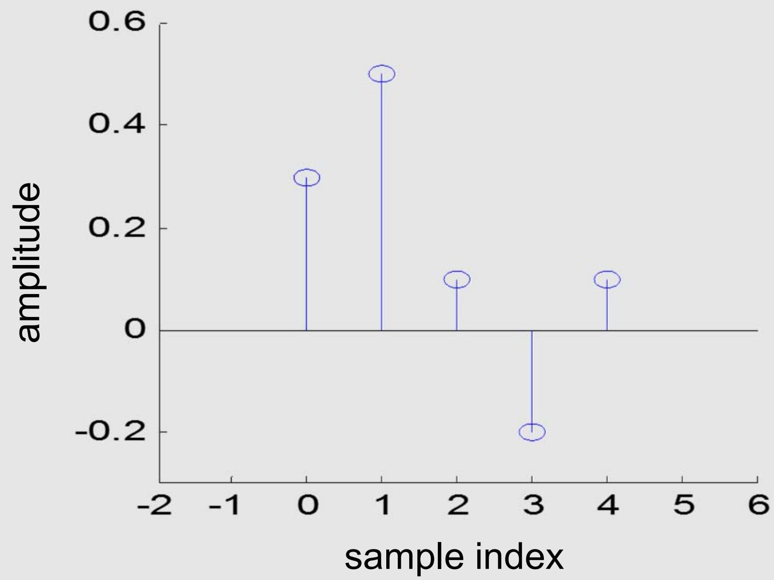
The aims of this chapter are to:

1. construct and compare the basic types of discrete signals
2. differentiate between different types of signal operations
3. formulate the process of sampling an analog signal
4. analyze and interpret properties of discrete signals

Discrete-time signals

- sequence of numbers
- normally represented in a vector form notation, e.g., $\mathbf{x}[n]$
- n is known as the sample index
- sometimes an arrow denotes the value when $n = 0$
- if there is no arrow, the first element is taken $n = 0$

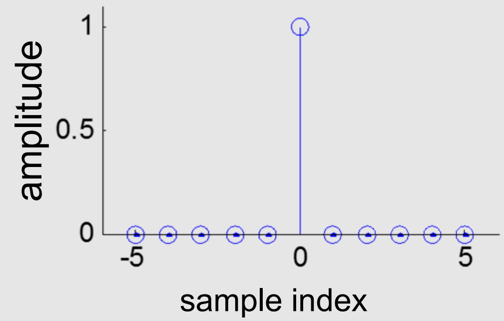
$$\mathbf{x}[n] = [\underset{\uparrow}{0.3} \quad 0.5 \quad 0.1 \quad -0.2 \quad 0.1]$$



Some basic signals

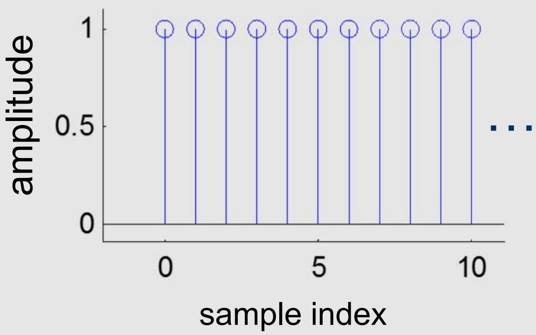
- Impulse

$$\delta[n] = \begin{cases} 0, & n \neq 0; \\ 1, & n = 0 \end{cases}$$



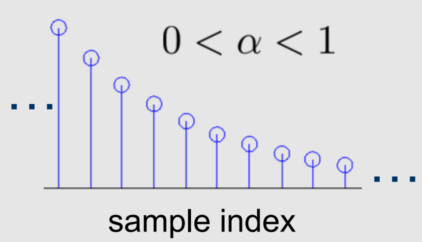
- Unit step

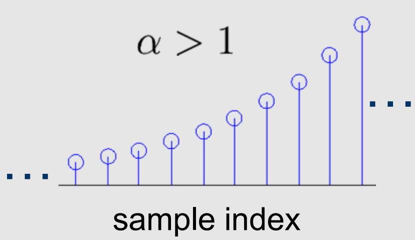
$$u[n] = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0 \end{cases}$$



- Exponential

$$x[n] = A\alpha^n$$

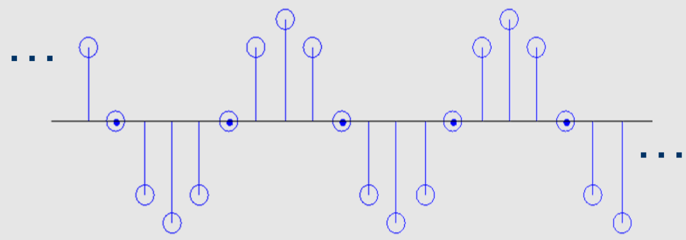




2.2 Basic Signals

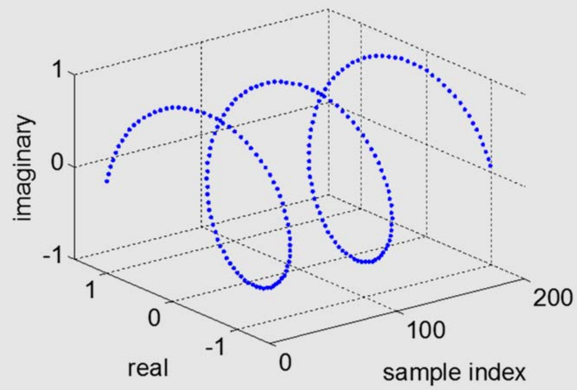
- Sinusoid

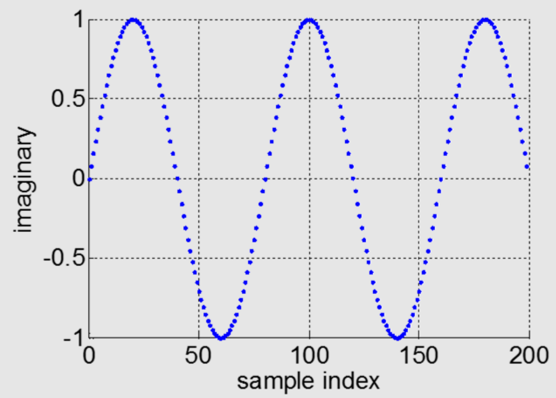
$$
\begin{aligned}
x[n] &= A\cos(\omega_0 n + \phi) \\
A &: \text{amplitude} \\
\omega_0 &: \text{angular frequency (radian/sample)} \\
\phi &: \text{phase}
\end{aligned}
$$

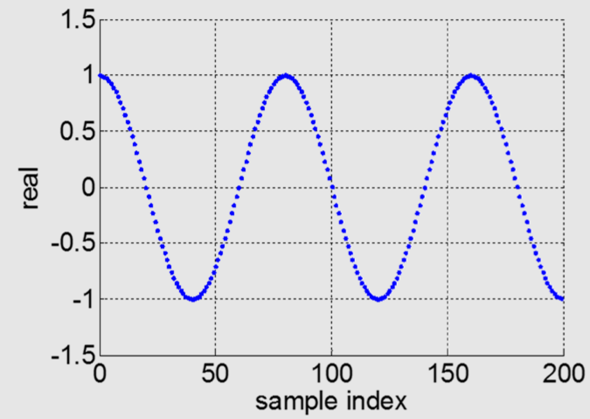


- Complex sinusoid

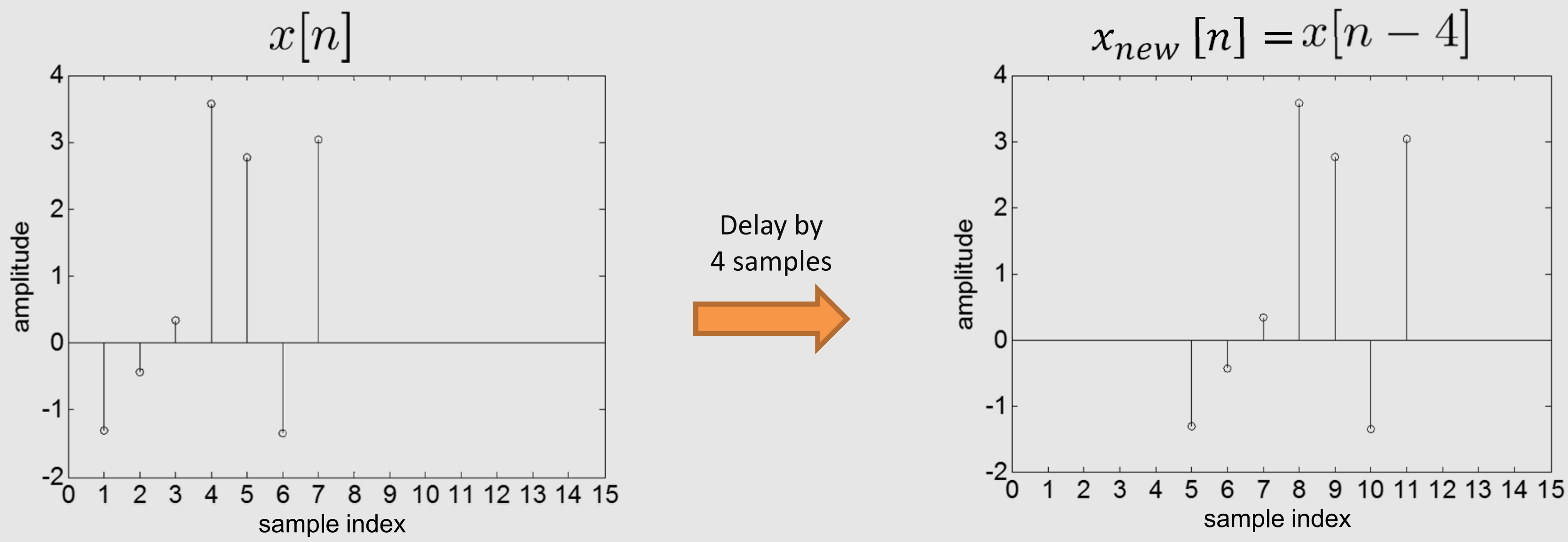
$$
\begin{aligned}
x[n] &= Ae^{\jmath(\omega_0 n + \phi)} \\
&= A\cos(\omega_0 n + \phi) + \jmath A\sin(\omega_0 n + \phi)
\end{aligned}
$$





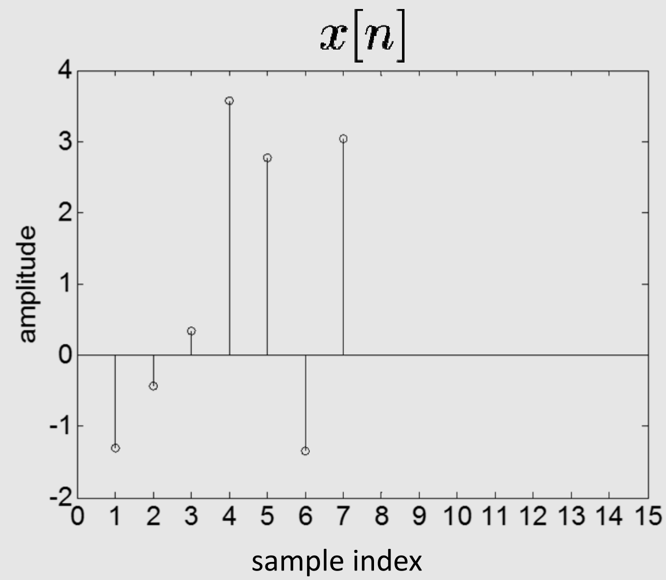


- Signal shift (delay)



2.3 Basic Operations of Signals

- Expressing a signal using the impulse function



We can express $x[n]$ by

$$x[n] = -1.3\delta[n-1] - 0.4\delta[n-2] + 0.3\delta[n-3] + \ldots + 3\delta[n-7]$$

More compactly, we can express a given signal as

$$x[n] = \sum_{k=0}^{\infty} A_k \delta[n-k]$$

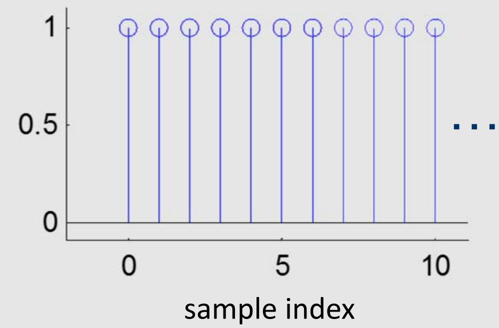
$$A_k \quad : \quad \text{coefficent of time index } k$$

2.3 Basic Operations of Signals

Examples:

a) a unit step sequence can be expressed as

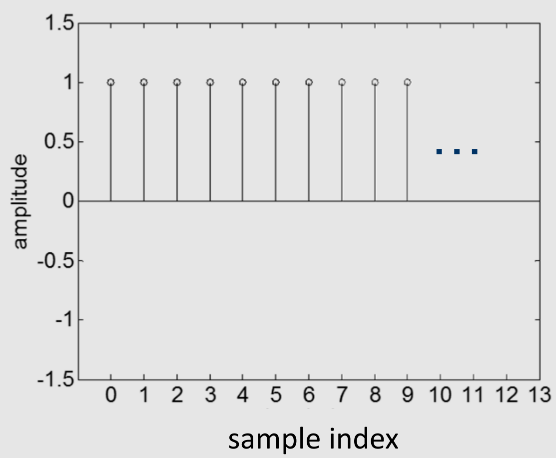
$$x[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



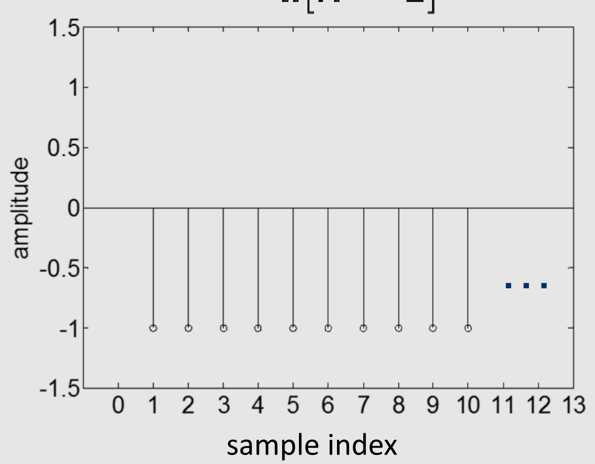
b) a unit impulse can be expressed as

$$\begin{aligned}\delta[n] &= u[n] - u[n-1] \\ &= \sum_{k=0}^{\infty} \delta[n-k] - \sum_{k=1}^{\infty} \delta[n-k]\end{aligned}$$

$u[n]$

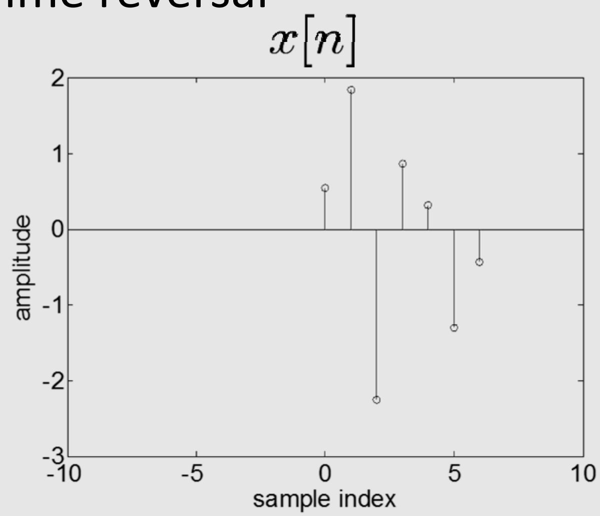


$-u[n-1]$

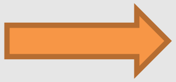


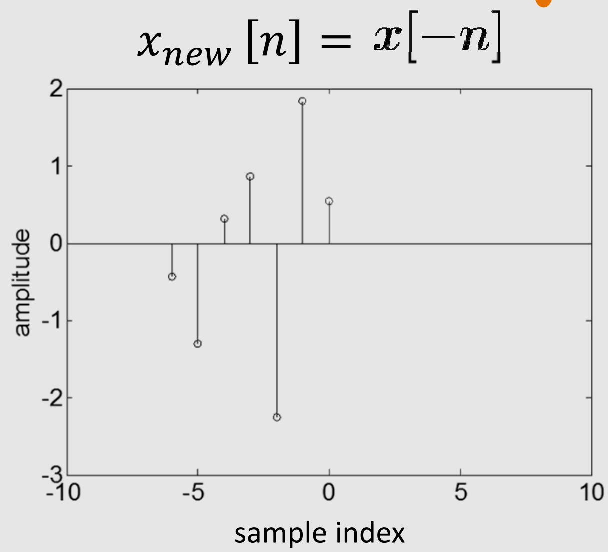
2.3 Basic Operations of Signals

- Time reversal

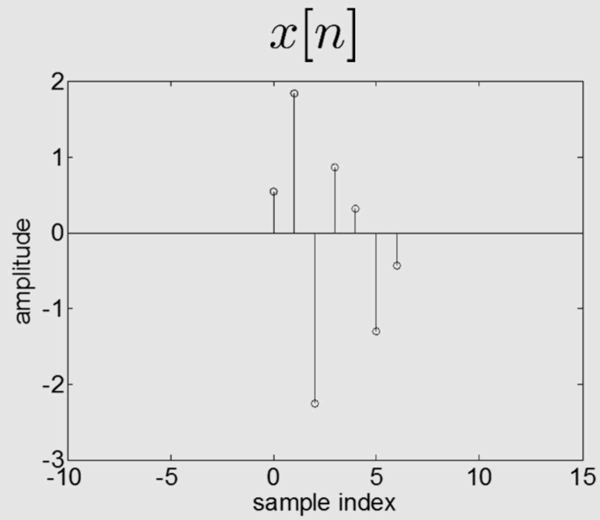


Time reversal

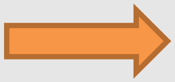


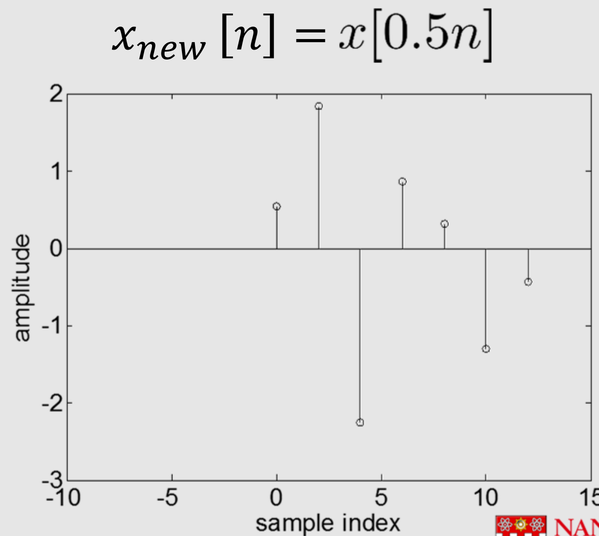


- Time scaling



Time scaling by a factor of 0.5





This page is intentionally left blank.

2.4 Discrete-Time Signal from Continuous Time Signal

A) Definition

- An analog sinusoid is in the form of

$$\begin{aligned} x(t) &= A\cos(2\pi f t) \\ &= A\cos(\omega t) \end{aligned}$$

Therefore, by definition,

$$\begin{aligned} f &: \text{analog frequency in cycles/sec (Hz)} \\ \omega &: \text{angular (analog) frequency in rad/sec} \end{aligned}$$

- A digital sinusoid is in the form of

$$\begin{aligned} x[n] &= A\cos(2\pi f_0 n) \\ &= A\cos(\omega_0 n) \end{aligned}$$

Therefore, by definition,

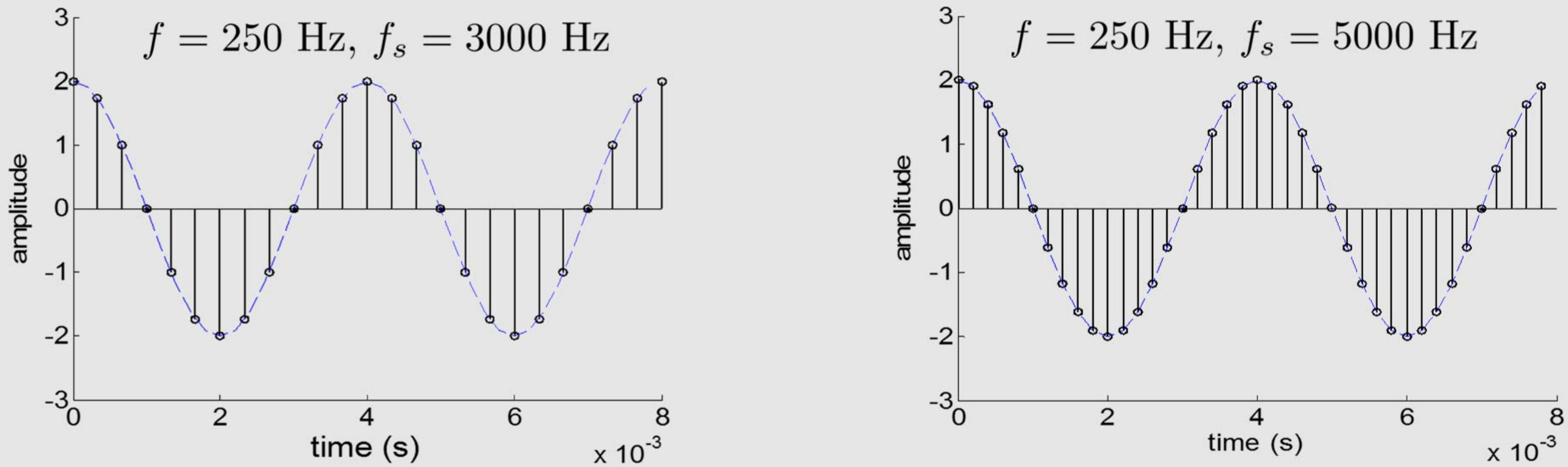
$$\begin{aligned} f_0 &: \text{digital frequency in cycles/sample} \\ \omega_0 &: \text{angular (digital) frequency in rad/sample} \end{aligned}$$

B) Sampling

- Sampling a continuous-time signal at regular interval results in a discrete-time signal.

- We often write
$$\begin{aligned} x[n] &= x_{\text{continuous}}(nT_s) \\ T_s &= 1/f_s \text{ is the sampling period in sec} \\ f_s &: \text{ sampling frequency in Hz} \end{aligned}$$

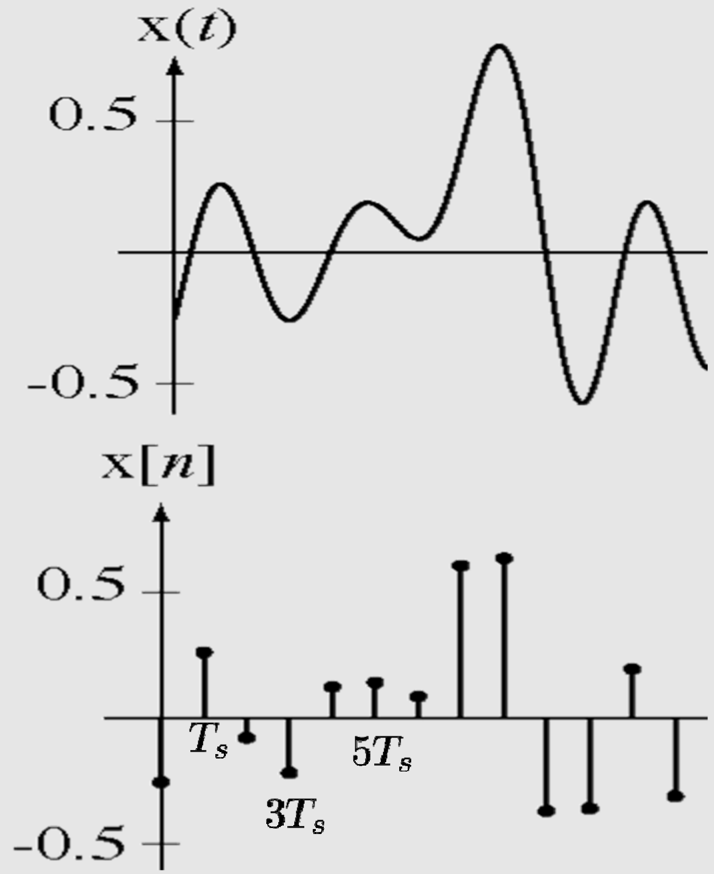
- A higher f_s implies that the analog signal is sampled more frequently.



NANYANG TECHNOLOGICAL UNIVERSITY

2.4 Discrete-Time Signal from Continuous Time Signal

$$\begin{aligned} x[n] &= x_{\text{continuous}}(nT_s) \\ T_s &= 1/f_s \text{ is the sampling period} \\ f_s &: \text{sampling frequency} \end{aligned}$$



- Therefore, samples are taken at regular time intervals

  $$\ldots, 0, T_s, 2T_s, \ldots$$

- This is equivalent to replacing the variable t in $x(t)$ by nT_s where n is the sample index.

2.4 Discrete-Time Signal from Continuous Time Signal

Example:

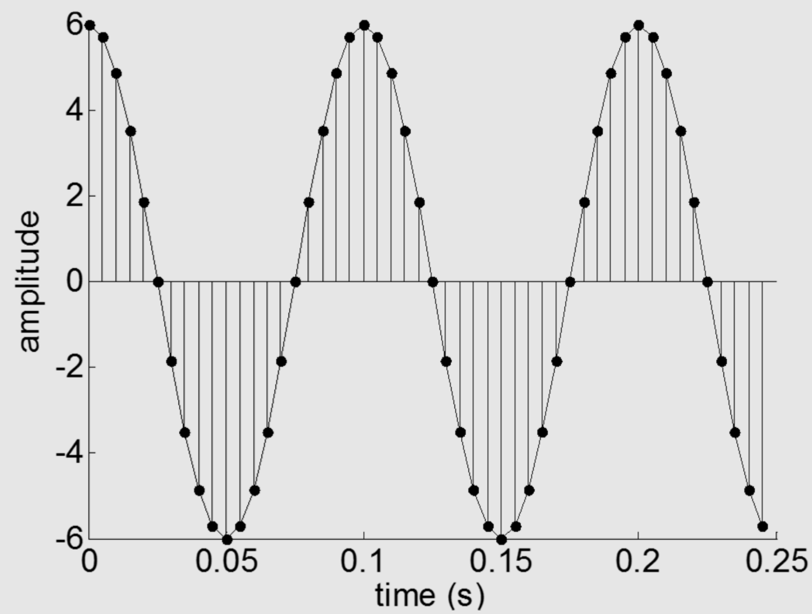
Consider an analog signal $x(t) = 6\cos(20\pi t)$. Given a sampling rate of $f_s = 200\ \mathrm{Hz}$, find the discrete representation of the signal.

A sampling rate of $f_s = 200\ \mathrm{Hz}$ corresponds to a sampling period of $T_s = 1/200 = 0.005\ \mathrm{sec.}$

This implies that we have a digital signal at sample index n every 0.005 s.

Sampling $x(t)$ at this period will result in

$$\begin{aligned} x[n] &= 6\cos(20\pi \times nT_s) \\ &= 6\cos(0.1\pi n) \end{aligned}$$



2.4 Discrete-Time Signal from Continuous Time Signal

- In the above example, the analog signal is $x(t) = 6\cos(20\pi t)$ and therefore

$$
\begin{aligned}
f &= 10 \text{ Hz} \\
\omega &= 20\pi \text{ rad/s}
\end{aligned}
$$

Once sampled, the digital signal $x[n] = 6\cos(0.1\pi n)$ where

$$
\begin{aligned}
f_0 &= 0.05 \text{ cycles/sample} \\
\omega_0 &= 0.1\pi \text{ rad/sample}
\end{aligned}
$$

The variable f_0 is sometimes known as the *normalized frequency* since, from the above, we can show that

$$
f_0 = \frac{f}{f_s} = \frac{10}{200} = 0.05
$$

2.4 Discrete-Time Signal from Continuous Time Signal

- From the above, since

$$\omega_0 = 2\pi f_0$$

$$f_0 = f / f_s$$

we will have

$$\omega_0 = 2\pi f / f_s$$

This means that the *angular frequency* of the digital signal is equivalent to the *normalized frequency* multiplied by a factor of 2π.

This page is intentionally left blank.

2.4 Discrete-Time Signal from Continuous Time Signal

C) Identical signals

- Two discrete-time sinusoids of different frequencies may be identical
- Consider two angular frequencies ω_0 and $\omega_0 + 2\pi$
- We can show that these two frequencies are identical by expressing

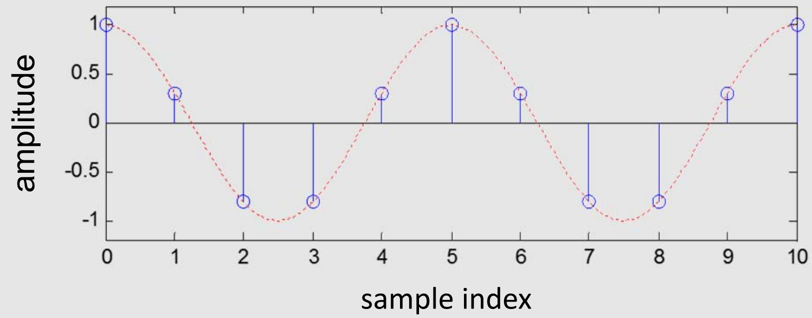
$$\begin{aligned} A_0 \cos[(\omega_0 + 2\pi)n + \phi] &= A_0 \cos[(\omega_0 n + \phi) + 2\pi n] \\ &= A_0 \cos(\omega_0 n + \phi)\cos(2\pi n) - A_0 \sin(\omega_0 n + \phi)\sin(2\pi n) \\ &= A_0 \cos(\omega_0 n + \phi) \end{aligned}$$

2.4 Discrete-Time Signal from Continuous Time Signal

- For example, consider the case where we have an analog signal with frequency $f = 200$ Hz. With a sampling rate of $f_s = 1000$ Hz,

$$\begin{aligned} \omega_0 &= 2\pi f / f_s \\ &= 0.4\pi \end{aligned}$$

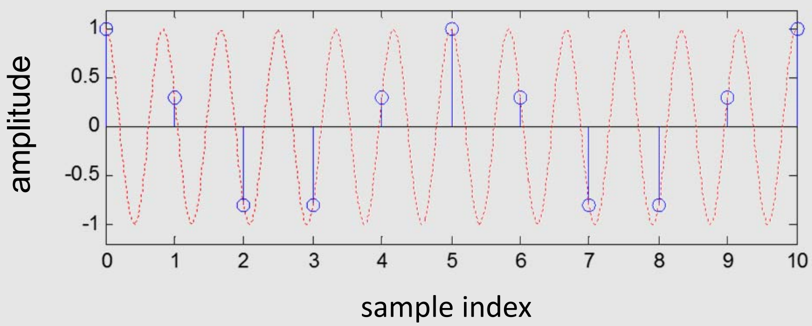
$$\cos(\omega_0 n) = \cos(0.4\pi n)$$



- Another analog signal of frequency $f = 1200$ Hz, with a sampling rate of $f_s = 1000$ Hz

$$\begin{aligned} \omega_0 &= 2\pi f / f_s \\ &= 2.4\pi \end{aligned}$$

$$\cos[(\omega_0 + 2\pi)n] = \cos(2.4\pi n)$$



- Therefore, analog signals of *different frequencies* can have the *same discrete signals*.

2.4 Discrete-Time Signal from Continuous Time Signal

D) Aliasing

- Consider an analog signal which contains 30 Hz and 170 Hz components, i.e.,

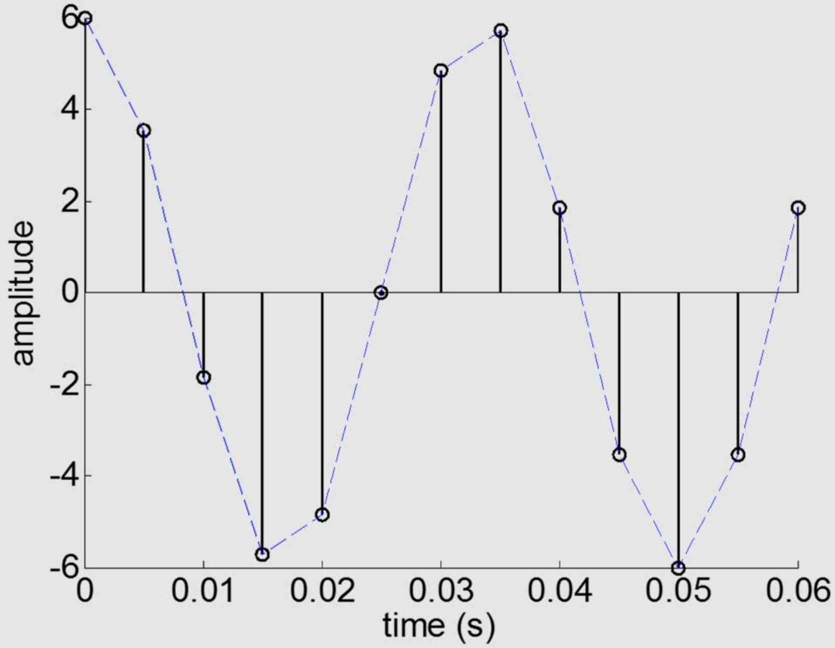
$$x(t) = 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t)$$

Digitizing this signal with a sampling frequency of $f_s = 200\ \mathrm{Hz}$

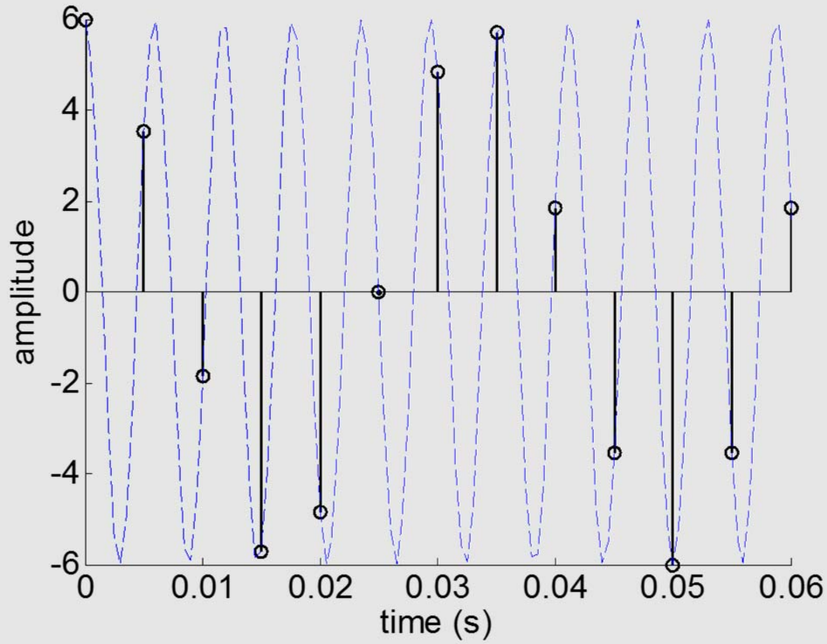
$$\begin{aligned} x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s) \\ &= 6\cos(0.3\pi n) + 6\cos(1.7\pi n) \\ &= 6\cos(0.3\pi n) + 6\cos((2\pi - 0.3\pi)n) \\ &= 6\cos(0.3\pi n) + 6\cos(0.3\pi n) \\ &= 12\cos(0.3\pi n) \end{aligned}$$

The above implies that the digital signal (generated by 2 analog signals) only contains <u>one</u> frequency $f_0 = 0.15$ which is a false representation of the analog signal !

2.4 Discrete-Time Signal from Continuous Time Signal



30 Hz analog signal sampled at 200 Hz



170 Hz analog signal sampled at 200 Hz

$$
\begin{aligned}
x(t) &= 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t) \\
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s) \\
&= 6\cos(0.3\pi n) + 6\cos(0.3\pi n)
\end{aligned}
$$

2.4 Discrete-Time Signal from Continuous Time Signal

$$
\begin{aligned}
x(t) &= 6\cos(2\times\pi\times 30t) + 6\cos(2\times\pi\times 170t) \\
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s) \\
&= 6\cos(0.3\pi n) + 6\cos(0.3\pi n)
\end{aligned}
$$

- The aliasing problem occurs because we are sampling at 200 Hz which is less than the Nyquist rate for the 170 Hz signal.

- If we are to sample at 500 Hz, i.e., more than twice the highest frequency, we can see that

$$
\begin{aligned}
x[n] &= 6\cos(60\pi n T_s) + 6\cos(340\pi n T_s) \\
&= 6\cos(0.12\pi n) + 6\cos(0.68\pi n)
\end{aligned}
$$

2.4 Discrete-Time Signal from Continuous Time Signal

- Note that when the analog signal is sampled without aliasing, i.e., the signal is sampled at least twice the frequency, we have

$$f_s \geq 2f$$

$$\Rightarrow f/f_s \leq 0.5$$

Since

$$\omega_0 = 2\pi f/f_s$$

we can therefore show that

$$\omega_0 \leq \pi$$

This implies that if the analog signal is to be faithfully represented, the *maximum* angular frequency of the digital signal is π.

2.4 Discrete-Time Signal from Continuous Time Signal

E) Periodicity of discrete sinusoids

- The period of discrete sinusoids is given by

$$N = 2\pi k f_s/\omega, \qquad k : \text{an integer}$$

The discrete signal will repeat itself after every $N = 2\pi k f_s/\omega$ samples.

- Rational: Consider an analog signal $x(t) = A\cos(2\pi f t)$. The digital signal is given by

$$x[n] = A\cos(2\pi f n T_s) = A\cos\left(2\pi \frac{f}{f_s} n\right) = A\cos\left(\frac{\omega}{f_s} n\right)$$

If $x[n]$ is to repeat itself, say every N samples, then $x[n] = x[n+N]$, i.e.,

$$x[n+N] = A\cos\left(\frac{\omega}{f_s}(n+N)\right) = A\cos\left(\frac{\omega}{f_s}n + \frac{\omega}{f_s}N\right)$$

Therefore $x[n] = x[n+N]$ can occur if $\omega N/f_s$ is a multiple of 2π, i.e.,

$$\frac{\omega N}{f_s} = 2\pi k$$

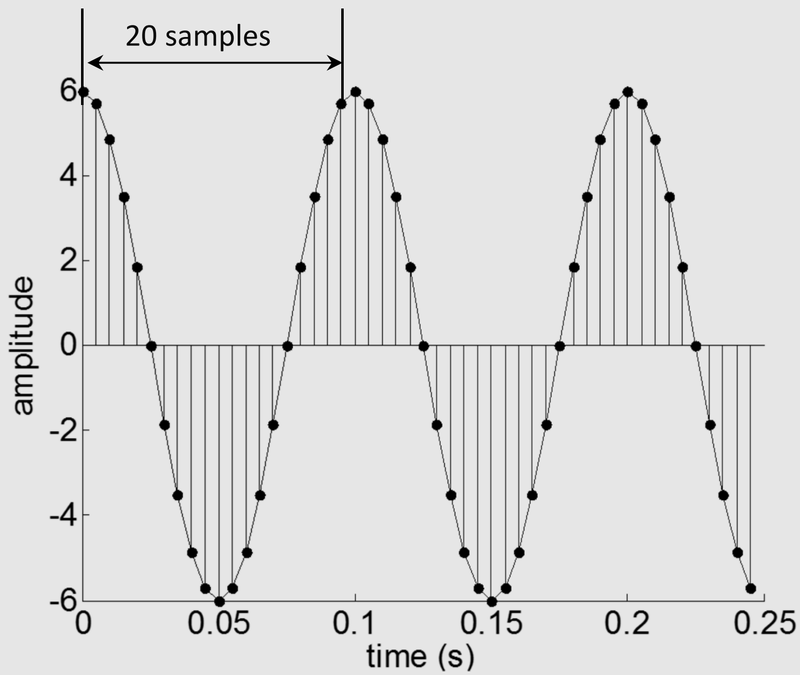
2.4 Discrete-Time Signal from Continuous Time Signal

Example:

Consider an analog signal given by $x(t) = 6\cos(20\pi t)$ and sampled at a sampling rate of $f_s = 200\ \mathrm{Hz}$.

For $k = 1$, the digital signal will repeat itself once every

$$
\begin{aligned}
N &= \frac{2\pi f_s}{\omega} \\
&= \frac{2\pi \times 200}{20\pi} \\
&= 20\ \text{samples}
\end{aligned}
$$



F) High/low frequencies in digital signals

- The interpretation of high and low frequencies is somewhat different for continuous-time and discrete-time sinusoids.

- For continuous-time signal,

$$\begin{aligned} x(t) &= A\cos(2\pi f t) \\ &= A\cos(\omega t) \end{aligned}$$

higher value of ω translates to higher frequency.

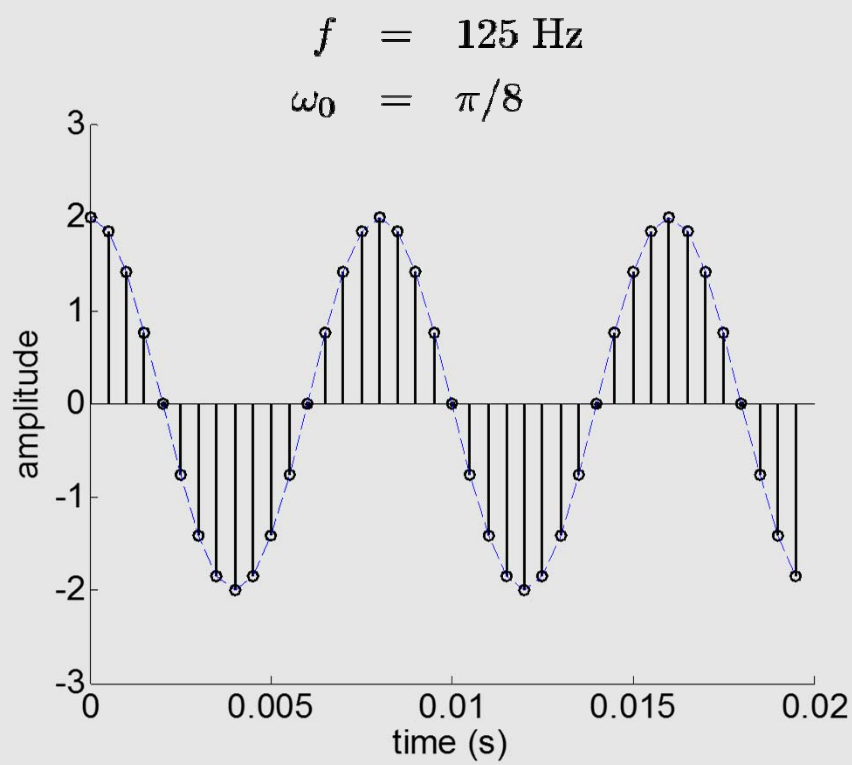
- For discrete-time signal,

$$\begin{aligned} x[n] &= A\cos(2\pi f_0 n) \\ &= A\cos(\omega_0 n) \end{aligned}$$

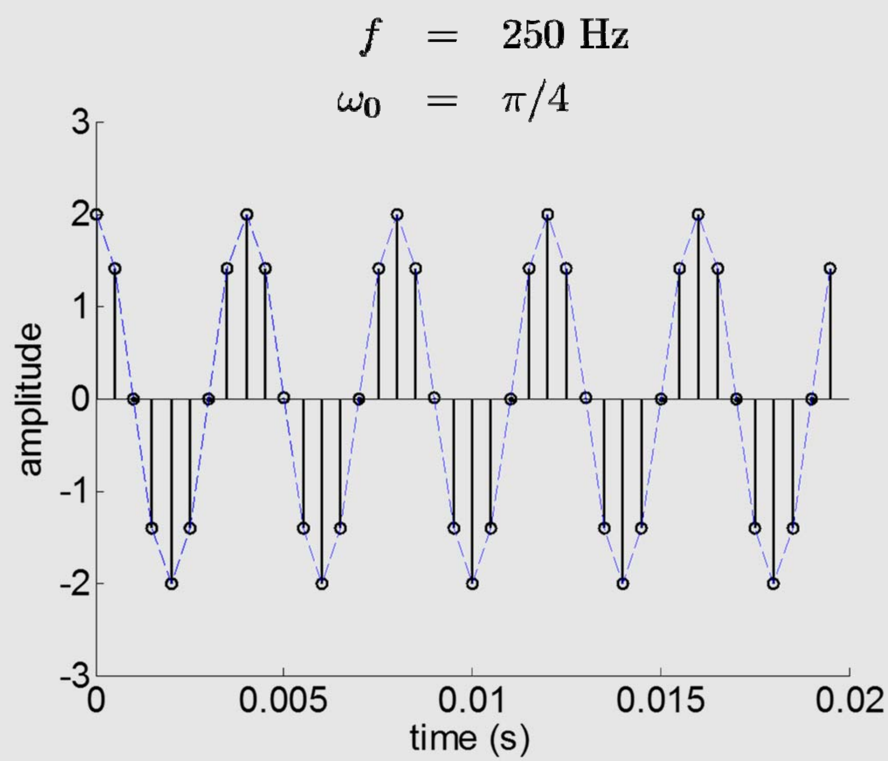
oscillation becomes *more* rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$
oscillation becomes *less* rapid for increasing ω_0 when $\pi \leq \omega_0 \leq 2\pi$

2.4 Discrete-Time Signal from Continuous Time Signal

$$f_s = 2000 \text{ Hz}$$



2.5 oscillations per 0.02 s

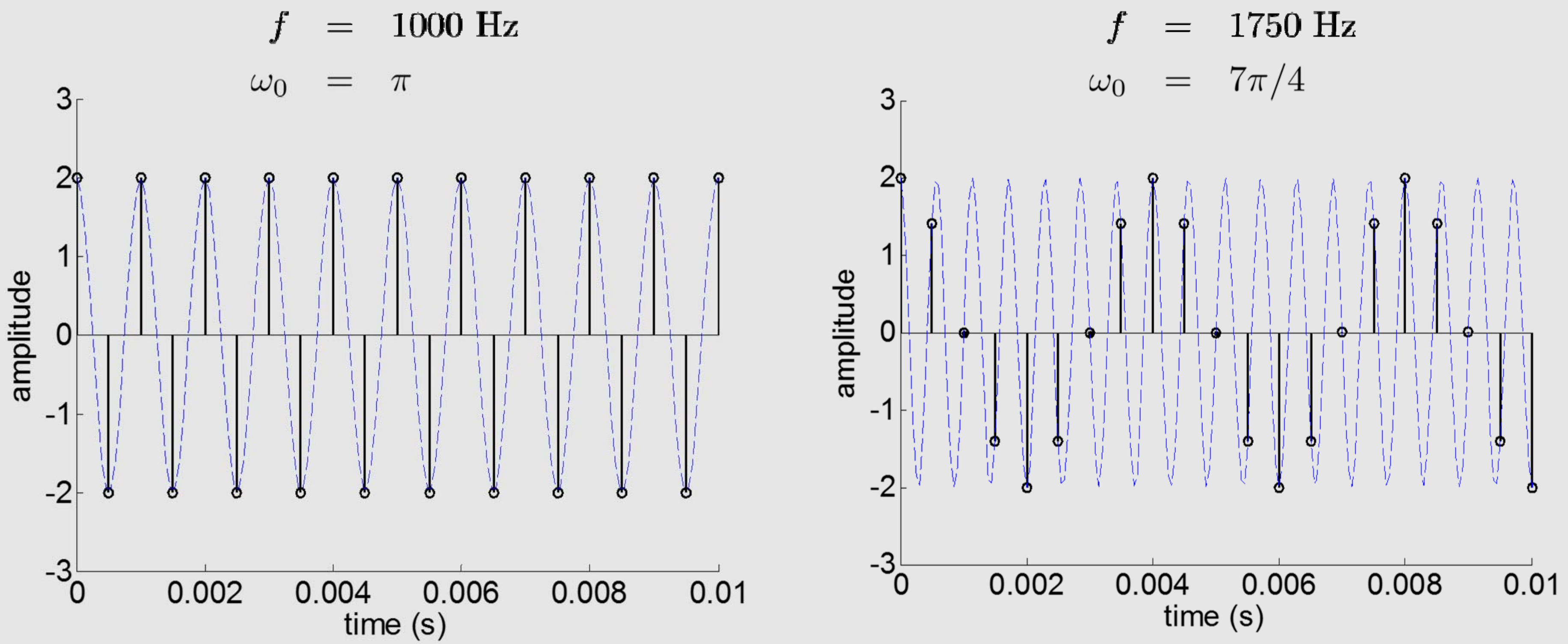


5 oscillations per 0.02 s

oscillation of discrete signal (black vertical lines) becomes *more* rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$

2.4 Discrete-Time Signal from Continuous Time Signal

$$f_s = 2000 \text{ Hz}$$



10 oscillations per 0.01 s

2.5 oscillations per 0.01 s

oscillation of discrete signal (black vertical lines) becomes *less* rapid for increasing ω_0 when $\pi \le \omega_0 \le 2\pi$

- A discrete-time signal can be expressed from a continuous-time signal by

$$x[n] = x_{\text{continuous}}(nT_s)$$

- The normalized frequency in the digital domain is given by

$$f_0 = f/f_s = \omega_0/2\pi$$

  and the maximum angular frequency of the digital signal corresponds to

$$\omega_0 \leq \pi$$

- The period of discrete sinusoid is given by $N = 2\pi k f_s/\omega$.

- For a discrete-time signal
  oscillation becomes more rapid for increasing ω_0 when $0 \leq \omega_0 \leq \pi$.
  oscillation becomes less rapid for increasing ω_0 when $\pi \leq \omega_0 \leq 2\pi$.

This page is intentionally left blank.