3GPP LTE : Cramer Rao Lower Bound on TOA

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1 Derivation: Fisher Information

$s(t)$ is the OFDM wave being transmitted by Reference Cell. τ is the time delay and channel is assumed to be 1. Received Samples can be given by the following :

$$y[n] = s(nT_s - \tau) + z[n],$$

where z[n] are iid AWGN noise samples with mean 0 and variance σ^2. Joint Conditional PDF of $\mathbf{y}|\tau$ can be given by the following:

$$p(\mathbf{y}|\tau) = \frac{1}{(\pi\sigma^2)^N}.exp(-\frac{1}{\sigma^2}\sum_{n=1}^{N}(s(nT_s-\tau)-y[n])^2)$$

And log likelihood will become :

$$ln(p(\mathbf{y}|\tau)) = -Nln(\pi\sigma^2) - \frac{1}{\sigma^2}\sum_{n=1}^{N}(s(nT_s-\tau)-y[n])^2$$

Fisher Information $\mathbf{J}(\tau)$ will then be given by:

$$\mathbf{J}(\tau) = \mathbf{E}\{-\frac{\mathbf{d^2}}{\mathbf{d^2}\tau}ln((p(\mathbf{y}|\tau))\}$$

Simplify the above to get :

$$\mathbf{J}(\tau) = \frac{2}{sigma^2}\sum_{n=1}^{N}\frac{\mathbf{d}}{\mathbf{d}\tau}s(nT_s-\tau).\frac{\mathbf{d}}{\mathbf{d}\tau}s^*(nT_s-\tau)$$

Since, $s(t) = \frac{1}{\sqrt{N}}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}S[k]e^{j2\pi k\Delta Ft}$, where ΔF is subcarrier spacing of OFDM system. Hence, $s(nT_s-\tau) = \frac{1}{\sqrt{N}}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}S[k]e^{j2\pi k\Delta F(nT_s-\tau)}$. Substitute this $s(nT_s-\tau)$ in the above equation to get :

$$\mathbf{J}(\tau) = \frac{2}{N.\sigma^2}\sum_{n=1}^{N}\{\frac{d}{d\tau}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}S[k]e^{j2\pi k\Delta F(nT_s-\tau)}\}.\{\frac{d}{d\tau}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}S[k]e^{j2\pi l\Delta F(nT_s-\tau)}\}$$

After carrying out differentiation and taking all terms which are not running in the sum out of the summation, we get the following :

$$\mathbf{J}(\tau) = \frac{2}{N\sigma^2}(-j2\pi\Delta F).(j2\pi\Delta F)\sum_{n=1}^{N}\{\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}kS[k]e^{j2\pi\Delta Fkn T_s}.\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}lS*[l]e^{-j2\pi\Delta FlnT_s}\}$$

Use orthogonality of Fourier Basis and assume $S[k]=1$, to get the following:

$$\mathbf{J}(\tau) = \frac{8\pi^2\Delta F^2}{N\sigma^2}\sum_{n=1}^{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}Nk^2 = \frac{8\pi^2\Delta F^2}{\sigma^2}\sum_{n=1}^{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}k^2 = \frac{8\pi^2\Delta F^2 N}{\sigma^2}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}k^2$$

Computing $\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}k^2$ by substituting $m = k+\frac{N}{2}$, gives the following :

$$\sum_{m=0}^{N-1}(m-\frac{N}{2})^2 = \frac{N^3}{12} + \frac{N}{6} \approx \frac{N^3}{12}$$

Finally, plug this approximated sum into Fisher Information sum to get the following :

$$\mathbf{J}(\tau) = \frac{8\pi^2 \Delta F^2 N}{\sigma^2} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k^2 \approx \frac{8\pi^2 \Delta F^2 N}{\sigma^2} \frac{N^3}{12},$$

We know that Bandwidth of OFDM is $\Delta F.N$, hence $(\Delta F.N)^2$ can be considered as Squared Bandwidth. Also We have considered $|S[k]|^2 = 1$, which will mean Pre-Correlation Signal Power is 1. If instead Pre-Correlation Signal Power was a constant P, then Post-Correlation signal power would have been $N.P$. And then the Fisher Information expression becomes :

$$\mathbf{J}(\tau) = \frac{2\pi^2(\Delta F.N)^2}{3}.\frac{N^2 P}{\sigma^2} = N.\frac{2\pi^2}{3}(\Delta F.N)^2 SNR_{corr}$$

2 CRLB : Expression

As, CRLB on uncertainty of the estimated parameter τ is lower bounded by reciprocal of Fisher Information, CRLB can be given by following:

$$var(\hat{\tau}) \geq \frac{1}{\mathbf{J}(\tau)},$$

$$or, var(\hat{\tau}) \geq \frac{3}{2N\pi^2} \frac{1}{(\Delta F.N)^2}.\frac{1}{SNR_{corr}}$$