3GPP LTE : Cramer Rao Lower Bound on TOA

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April 2020

1 Position reference Sequence Based CRLB

1.1 CRLB Analysis : General Delay Estimation in AWGN

Assume a signal $s(t)$ transmitted at time 0 till T_s from eNB and being received at UE from time τ_0 till $T_s + \tau_0$, T_s being symbol period. Received signal $x(t)$ can be expressed as follows:

$$x(t) = s(t - \tau_0) + w(t)$$

$w(t)$ is the White Gaussian Noise process with distribution $\sim \mathcal{N}(0, \sigma^2)$. Also, baseband signal $s(t)$ is band-limited to $f_{max} = B$.

Consider sampled version of $x(t)$ sampled at Nyquist Rate $2B = \frac{1}{\Delta}$. Sampled noise $w[n]$ will be independent. $x(n\Delta)$ will be non-zero only from time τ_0 to $\tau_0 + T_s$. Mathematically, $x[n\Delta]$ can be expressed as follows:

$$x[n\Delta] = s[n\Delta - \tau_0] + w[n],$$

Assuming Δ to be small enough such that τ_0 is some integral multiple of Δ, $\frac{\tau_0}{\Delta} = n_0$. Based on the final result of appendix **A**, Cramer-Rao Lower Bound for delay τ_0 can be given by the following expression:

$$var(\hat{\tau_0}) \geq \frac{\sigma^2}{\sum_{n=n_0}^{n_0+N-1} (\frac{\partial s[n\Delta - \tau_0]}{\partial \tau_0})^2} = \frac{\sigma^2}{\sum_{n=n_0}^{n_0+N-1} (\frac{\partial s(t)}{\partial t})|_{t=n\Delta-\tau_0}^2} = \frac{\sigma^2}{\sum_{n=0}^{N-1} (\frac{\partial s(t)}{\partial t})|_{t=n\Delta}^2} \tag{1}$$

The summation in the above integration can be approximated to be integration since Δ is assumed to be very small and $n_0\Delta = \tau_0$.

$$var(\hat{\tau_0}) \geq \frac{\sigma^2}{\frac{1}{\Delta}\int_0^{T_s} (\frac{\partial s(t)}{\partial t})^2 \partial t} = \frac{\Delta . N_0 . B}{\int_0^{T_s} (\frac{\partial s(t)}{\partial t})^2 \partial t}$$

$$var(\hat{\tau_0}) \geq \frac{N_0/2}{\int_0^{T_s} (\frac{\partial s(t)}{\partial t})^2 \partial t} \tag{2}$$

Observe that signal energy $\mathcal{E} = \int_0^{T_s} |s(t)|^2 \partial t$ and define a quantity $\bar{F}^2$ as follows:

$$\bar{F}^2 = \frac{\int_0^{T_s} (\frac{\partial s(t)}{\partial t})^2 \partial t}{\int_0^{T_s} |s(t)|^2 \partial t}$$

Now, we can write equation.2 using $\mathcal{E}$ and $\bar{F}^2$ as follows:

$$var(\hat{\tau_0}) \geq \frac{1}{\frac{\mathcal{E}}{N_0/2} \cdot \bar{F}^2} = \frac{1}{\textbf{SNR} \cdot \bar{F}^2} \tag{3}$$

This quantity $\bar{F}^2$ can be understood as Mean-Squared Bandwidth [1] of the transmitted signal $s(t)$. This is also called Gabor Bandwidth. Using Fourier Transform's properties $\bar{F}^2$ can be written as follows:

$$\bar{F}^2 = \frac{\int_{-\infty}^{\infty} (2\pi f)^2 |S(f)|^2 \partial f}{\int_{-\infty}^{\infty} |S(f)|^2 \partial f}$$

[1] Area under the curve f^2 weighted by transmitted signal's PSD $|S(f)|^2$ over $f \in (-\infty, \infty)$ will give squared bandwidth of $s(t)$. And, this squared bandwidth is divided by total signal power.

1.2 CRLB Derivation : PRS Based OFDM System

A transmitted signal $s[n]$ can be written as inverse discrete fourier transform summation as follows:

$$s[n] = \sqrt{\frac{2C}{N}} \sum_{k \in N_a} d_k exp(j2\pi nk/N)$$

N is the number of tones i.e. $12N_{RB}$, N_a is the number of active tones on which the PRS symbols are loaded, d_k are Gold Sequenced QAM symbols on k^{th} tone, and C is the total power in one OFDM Symbol of symbol period T_s, where $T_s = \frac{1}{\Delta f}$. C can be expressed in terms of total signal energy and symbol period as follows:

$$C = \frac{\sum_{k=0}^{N-1} |X[k\Delta f]|^2}{T_s}$$

And, hence One Symbol Energy $\mathcal{E}_s$ can be defined as: $\mathcal{E}_s = CT_s$. Also, we note that $X[k\Delta f] = d_k$.

Based on the analysis of CRLB for delay in previous section, we can write the CRLB for this PRS based OFDM system as follows:

$$var(\hat{\tau}) \geq \frac{\sigma^2}{\{\sum_{k=0}^{N-1} |X[k\Delta f]|^2\} \cdot \bar{F}^2} \tag{4}$$

For any OFDM System, Mean-Squared Bandwidth ($\bar{F}^2$) can be approximated by the following expression:

$$\bar{F}^2 = \frac{\sum_{k=0}^{N-1} (2\pi k\Delta f)^2 |X[k\Delta f]|^2}{\sum_{k=0}^{N-1} |X[k\Delta f]|^2} = \frac{\sum_{k=0}^{N-1} (2\pi k\Delta f)^2 |d_k|^2}{\sum_{k=0}^{N-1} |d_k|^2} \tag{5}$$

If we can assume that d_k will be one of the QAM symbols picked from the set: $\{\frac{1}{\sqrt{2}}(1+j), \frac{1}{\sqrt{2}}(1-j), \frac{1}{\sqrt{2}}(-1+j), \frac{1}{\sqrt{2}}(-1-j)\}$, then $|d_k|^2 = 1$ and hence equ.(4) can be simplified to the following:

$$\bar{F}^2 = \frac{(2\pi\Delta f)^2 \sum_{k=0}^{N-1} k^2}{N} \approx \frac{4\pi^2}{3}(N\Delta f)^2 \tag{6}$$

Using equ.(4) and equ.(6), we can write the CRLB for TOA for PRS based OFDM system as follows:

$$var(\hat{\tau}) \geq \frac{\sigma^2}{\frac{4\pi^2}{3}(N\Delta f)^2 \cdot \sum_{k=0}^{N-1} |X[k\Delta f]|^2}$$

$$var(\hat{\tau}) \geq \frac{\sigma^2}{\frac{4\pi^2}{3}(N\Delta f)^2 \cdot \sum_{k=0}^{N-1} |d_k|^2} = \frac{1}{\frac{4\pi^2}{3}(N\Delta f)^2 \cdot \frac{\sum_k |d_k|^2}{\sigma^2}}$$

Notice that the quantity $\frac{\sum_k |d_k|^2}{\sigma^2}$ is the Post-Correlation SNR. Because, when we correlate the received symbols with the Gold Sequence of PRS, what we will have as aggregate signal energy, is : $\sum_k |d_k|^2$, and we know that noise energy in the total OFDM Bandwidth $2B = 12N_{RB}\Delta f$ is nothing but $\sigma^2 = N_0/2.2B = N_0B$, where N_0 is double sided noise PSD. So, we can write $\mathbf{SNR}_{PostCorr} = \frac{\sum_k |d_k|^2}{\sigma^2}$ and then our final result can be given by:

$$var(\hat{\tau}) \geq \frac{1}{\frac{4\pi^2}{3}\ (N\Delta f)^2\ \mathbf{SNR}_{PostCorr}} \tag{7}$$

2 Alternative : CRLB for OFDM Based Systems

2.1 Derivation: Fisher Information

$s(t)$ is the OFDM wave being transmitted by Reference Cell. τ is the time delay and channel is assumed to be 1. Received Samples can be given by the following :

$$y[n] = s(nT_s - \tau) + z[n],$$

where z[n] are iid AWGN noise samples with mean 0 and variance σ^2. Joint Conditional PDF of $\mathbf{y}|\tau$ can be given by the following:

$$p(\mathbf{y}|\tau) = \frac{1}{(\pi\sigma^2)^N}.exp(-\frac{1}{\sigma^2}\sum_{n=1}^{N}(s(nT_s-\tau) - y[n])^2)$$

And log likelihood will become :

$$ln(p(\mathbf{y}|\tau)) = -Nln(\pi\sigma^2) - \frac{1}{\sigma^2}\sum_{n=1}^{N}(s(nT_s-\tau) - y[n])^2$$

Fisher Information $\mathbf{J}(\tau)$ will then be given by:

$$\mathbf{J}(\tau) = \mathbf{E}\{-\frac{\mathbf{d^2}}{\mathbf{d^2}\tau}ln((p(\mathbf{y}|\tau))\}$$

Simplify the above to get :

$$\mathbf{J}(\tau) = \frac{2}{sigma^2}\sum_{n=1}^{N}\frac{\mathbf{d}}{\mathbf{d}\tau}s(nT_s-\tau).\frac{\mathbf{d}}{\mathbf{d}\tau}s^*(nT_s-\tau)$$

Since, $s(t) = \frac{1}{\sqrt{N}}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S[k]e^{j2\pi k\Delta F t}$, where ΔF is subcarrier spacing of OFDM system. Hence, $s(nT_s - \tau) = \frac{1}{\sqrt{N}}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S[k]e^{j2\pi k\Delta F(nT_s-\tau)}$. Substitute this $s(nT_s-\tau)$ in the above equation to get :

$$\mathbf{J}(\tau) = \frac{2}{N.\sigma^2}\sum_{n=1}^{N}\{\frac{d}{d\tau}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S[k]e^{j2\pi k\Delta F(nT_s-\tau)}\}.\{\frac{d}{d\tau}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} S[k]e^{j2\pi l\Delta F(nT_s-\tau)}\}$$

After carrying out differentiation and taking all terms which are not running in the sum out of the summation, we get the following :

$$\mathbf{J}(\tau) = \frac{2}{N\sigma^2}(-j2\pi\Delta F).(j2\pi\Delta F)\sum_{n=1}^{N}\{\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} kS[k]e^{j2\pi\Delta F knT_s}.\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} lS*[l]e^{-j2\pi\Delta F lnT_s}\}$$

Use orthogonality of Fourier Basis and assume $S[k] = 1$, to get the following:

$$\mathbf{J}(\tau) = \frac{8\pi^2\Delta F^2}{N\sigma^2}\sum_{n=1}^{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} Nk^2 = \frac{8\pi^2\Delta F^2}{\sigma^2}\sum_{n=1}^{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k^2 = \frac{8\pi^2\Delta F^2 N}{\sigma^2}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k^2$$

Computing $\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k^2$ by substituting $m = k + \frac{N}{2}$, gives the following :

$$\sum_{m=0}^{N-1}(m - \frac{N}{2})^2 = \frac{N^3}{12} + \frac{N}{6} \approx \frac{N^3}{12}$$

Finally, plug this approximated sum into Fisher Information sum to get the following :

$$\mathbf{J}(\tau) = \frac{8\pi^2\Delta F^2 N}{\sigma^2}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k^2 \approx \frac{8\pi^2\Delta F^2 N}{\sigma^2}\frac{N^3}{12},$$

We know that Bandwidth of OFDM is $\Delta F.N$, hence $(\Delta F.N)^2$ can be considered as Squared Bandwidth. Also We have considered $|S[k]|^2 = 1$, which will mean Pre-Correlation Signal Power is 1. If instead Pre-Correlation Signal Power was a constant P, then Post-Correlation signal power would have been $N.P$. And then the Fisher Information expression becomes :

$$\mathbf{J}(\tau) = \frac{2\pi^2(\Delta F.N)^2}{3}.\frac{N^2P}{\sigma^2} = N.\frac{2\pi^2}{3}(\Delta F.N)^2 SNR_{corr}$$

2.2 CRLB : Expression

As, CRLB on uncertainty of the estimated parameter τ is lower bounded by reciprocal of Fisher Information, CRLB can be given by following:

$$var(\hat{\tau}) \geq \frac{1}{\mathbf{J}(\tau)},$$

$$or, var(\hat{\tau}) \geq \frac{3}{2N\pi^2}\frac{1}{(\Delta F.N)^2}.\frac{1}{SNR_{corr}}$$

A

CRLB : Signal($s[n;\theta]$) in White Gaussian Noise

Assume the observations $x[n]$ to be:

$$x[n] = s[n;\theta] + w[n],$$

$w[n]$ are iid noise samples distributed $\sim \mathcal{N}(0,\sigma^2)$. Joint PDF of N independent observations $p(\mathbf{x};\theta)$ and log-likelihood function will be given by:

$$p(\mathbf{x};\theta) = \frac{1}{(\sqrt{2\pi\sigma^2})^N} \cdot exp(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n;\theta])^2)$$

$$ln(p(\mathbf{x};\theta)) = -\frac{N}{2}ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n;\theta])^2$$

Differentiating with respect to θ, we get the following:

$$\frac{\partial ln(p(\mathbf{x};\theta))}{\partial\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial s[n;\theta]}{\partial\theta}(x[n]-s[n;\theta])$$

It can be verified that the regularity condition($\mathcal{E}\{\frac{\partial ln(p(\mathbf{x};\theta))}{\partial\theta}\} = 0, \forall\theta$), is satisfied. Differentiating again with respect to θ, we get the following:

$$\frac{\partial^2 ln(p(\mathbf{x};\theta))}{\partial^2\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(\frac{\partial^2 s[n;\theta]}{\partial^2\theta}(x[n]-s[n;\theta]) - (\frac{\partial s[n;\theta]}{\partial\theta})^2)$$

This leads to Fisher information $\mathbf{I}(\theta) = -\mathbf{E}\{\frac{\partial^2 ln(p(\mathbf{x};\theta))}{\partial^2\theta}\}$:

$$\mathbf{I}(\theta) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(\frac{\partial s[n;\theta]}{\partial\theta})^2$$

Cramer Rao Lower bound on estimation of unknown parameter θ is given by the reciprocal of Fisher information $\mathbf{I}(\theta)$, so in this case of general signal parametrized by unknown component θ is given as follows:

$$var(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1}(\frac{\partial s[n;\theta]}{\partial\theta})^2} \tag{8}$$

References

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Prentice Hall, 1993.

[2] F. Zanier and Massimo, *Achievable Localization Accuracy of the Positioning Reference Signal of 3GPP LTE.* IEEE, 2012.