$= \lim_{x \to 0}(f(x) - f(0))/(x-0) = a$. Consequently, $\lim_{n \to \infty}|U_n|/(1/n) = |a| \neq 0$. By the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ diverges absolutely since the harmonic series also does.

(3) We have determined that $f(0) = f'(0) = 0$ is necessary for convergence. We now assume that this condition holds and prove sufficiency. Take $0 < u < 1$ and consider the limit

$$\lim_{x \to 0^+} \frac{f(x)}{x^{1+u}} = \lim_{x \to 0^+} \frac{f'(x)}{(1+u)x^u} = \frac{1}{1+u} \lim_{x \to 0^+} \left(\frac{f'(x) - f'(0)}{x - 0} \right) x^{1-u} = \frac{f''(0)}{1+u} \lim_{x \to 0^+} x^{1-u} = 0,$$

where the first equality is an application of L'Hospital's rule. Therefore, $\lim_{n \to \infty}|U_n|/(1/n)^{1+u} = 0$ and again by the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ must converge absolutely since $\Sigma_{n=1}^{\infty} 1/n^{1+u}$ converges absolutely by the integral test.

Steps (1), (2), (3) complete the proof.

Perhaps you noticed in part (3) of the proof that the convergence did not depend critically on the existence of $f''(0)$. This is indeed the case and the existence of $f''(0)$ can be replaced by a weaker condition. We note that the condition cannot be completely removed since $\Sigma_{n=2}^{\infty} 1/n \ln n$, which is absolutely divergent by the integral test, has terms $f(1/n)$ where $f(x) = -x/\ln x$ (for $x > 0$ and zero otherwise); this function $f(x)$ has zero value and zero derivative at $x = 0$, but a non-existent second derivative.

The existence of $f''(0)$ in the differentiation test can be replaced, for example, by the existence of $\lim_{x \to 0^+} f'(x)/x^u$ or $d^2|x|^u f(x)/dx^2|_{x=0}$ for some $0 < u < 1$ (both conditions are implied by the existence of $f''(0)$ when $f(0) = f'(0) = 0$). Very minor modification of part (3) of the proof above is needed in these cases. Certain weaker conditions will also work; their discovery is left as a simple exercise.

It is also obvious that only the existence of $f'(0)$ is needed to conclude absolute divergence of a series using the test. Since divergence is seldom good news, I choose to leave the test in its simple symmetric form. Finally, one can apply the test with $f(1/n) = |U_n|$ instead of U_n. This covers complex series as well.

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References

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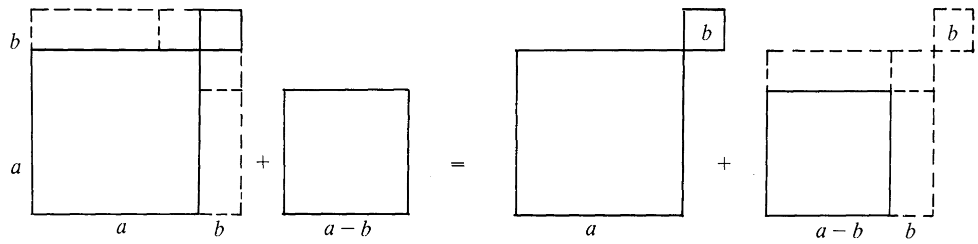
Proof without Words: Algebraic areas

$$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$$



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