$f(0)=0$, is it true that

$$\left(\int_0^1 f\,dx\right)^2 \le K\int_0^1 |f'|^2\,dx$$

for some constant K?

Beginning with (R), we have $\int_0^1 f\,dx = \int_0^1 (1-x)f'\,dx$. Therefore,

$$\begin{aligned}\left(\int_0^1 f\,dx\right)^2 &= \left(\int_0^1 (1-x)f'\,dx\right)^2\\ &\le \left(\int_0^1 (1-x)|f'|\,dx\right)^2\\ &\le \int_0^1 (1-x)^2\,dx \cdot \int_0^1 |f'|^2\,dx\\ &= \frac{1}{3}\int_0^1 |f'|^2\,dx,\end{aligned}$$

where the second inequality is an application of Schwarz's inequality. Thus, our answer is affirmative, with $K=1/3$. The interested reader might try to show that $K=1/3$ is best possible by showing that there exists a continuously differentiable function f on [0,1] with $f(0)=0$ and $(\int_0^1 f\,dx)^2 = (1/3)\int_0^1 |f'|^2\,dx$.

Behold! The Pythagorean Theorem via Mean Proportions

Michael Hardy, University of Minnesota, Minneapolis, MN

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2+b^2=c^2$$